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Research Article

Nonlinear Fuzzy Volterra Integro-differential Equation of N-th Order: Analytic Solution and Existence and Uniqueness of Solution

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Abstract

Integro-differential equations play a fundamental role in various fields of applied mathematics. The solutions of many engineering problems in general and mechanics and physics in particular, lead to this kind of equations. This paper focuses on the fuzzy Volterra integro-differential equation of $n-$th order of the second-kind with nonlinear fuzzy kernel and initial values. This equation is transformed to a nonlinear fuzzy Volterra integral equation in multi-integrals by application of a certain analytic solution adapted on fuzzy $n-$th order derivation under generalized Hakuhara derivative. The derived integral equations are solvable, the solutions of which are unique under certain conditions. The existence and uniqueness of the solutions are investigated in a theorem and an upper boundary is found for solutions. An easily-followed algorithm is provided to illustrate the process. The application of the proposed method helps solving the equation on the basis of the Adomian decomposition method under generalized H-derivation. Comparison of the exact and approximated solutions shows the least error.

Keywords : General $n-$th order derivative; H-derivative; Fuzzy $n-$th Order Integro-differential Equation; Existence and Uniqueness theorem; Upper boundary of solution.

1 Introduction

Seikalla [25] introduced the H-derivative of the fuzzy number valued function that subsequently amplified the fuzziness by [8]. The strongly-generalized derivative was introduced in [7] that became the subject for discussion among mathematical researchers. This concept makes it possible to solve the problems of the H-derivative. The latter are crucial in solving fuzzy differential equations and fuzzy integro-differential equations.

The first-order equations under generalized H-derivation were initially studied by Bede et al. [7, 8]. Four case derivatives for fuzzy first-order derivative were introduced for this purpose. Two cases of which are always very important and the others are crucial to acquire switching point. These four case derivatives were presented to solve fuzzy differential equations. Chalco, applied the first two cases derivation, because the derivation in the other cases was constant [9].

Stefanini expanded the generalized Hakuhar difference and division of interval-valued functions in [26, 27]. Allahviranloo and Hooshangian have delved in depth and breadth with the fuzzy

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derivations of n-th order based on generalized Hakuhara derivatives. They have also elaborated for example, on fuzzy n-th order derivatives and their characteristics, properties and relationships [3].
Park et al. [19] studied the existence of solutions of fuzzy integro-differential in Banach spaces as follows:

$$\frac{dX(t)}{dt} = f(t, X(t), \int_{t_0}^{t} k(t, s, X(s))ds)$$

Park et al. [20] studied the conditions of existence and uniqueness of solutions of fuzzy Volterra-Fredholm integral equations. focused their investigation on the condition of existence of the solution of fuzzy integro-differential and fuzzy delay integro-differential under local condition. The existence and uniqueness of solutions of second-order fuzzy differential equations was covered by Allahviranloo et al. [2]. The existence and uniqueness of solution of fuzzy Volterra integro-differential equations of the second kind using strongly-generalized differentiability were discussed by Hajighasemi et al. [11]. Rahimi et al. proved some fixed point theorems for approximating the fuzzy solution of nonlinear fuzzy Volterra integro-differential equation and error bound was derived at [22]. Khezerloo and Hajighasemi proved the existence and uniqueness of solution of fuzzy Volterra equations of the second kind [15]. Zeinali et al. have critically investigated the existence and uniqueness results for the fuzzy integro-differential equations of first order [28]. Abu Arqub et al. on the other hand have surveyed the existence, uniqueness and other properties of solutions of a certain nonlinear fuzzy Volterra integro-differential equation under strongly-generalized differ- entiability [4]. Mosleh and Otadi have also focused their research on the existence of solution of fuzzy Fredholm integro-differential equations [17].
Researchers used the numerical solutions to solve fuzzy Fredholm or Fuzzy Volterra integro- differential equation such as Euler method, differential transform method, Laplace transform method, Adomian decomposition method, Variational iteration method and Homotopy perturbation method [5, 13, 16, 18, 23, 24].

Some relevant preliminaries have been reviewed briefly in section 2, followed by section 3 that provide some detailed explanation on the solution of fuzzy integro-differential equations of higher order. An example is solved for this purpose by the application of the Adomian decomposition method [1]. Some figures are presented to show the approximate solutions and their comparisons with the exact solutions. Section 4 covers the existence and uniqueness of solution of fuzzy higher order integro-differential equation and presents an algorithm. The general conclusions are covered in section 5.

2 Basic Concepts

The basic definitions of a fuzzy number are given as follows:

Definition 2.1 *[19] A fuzzy number is a fuzzy set like* $u : \mathbb{R} \rightarrow [0, 1]$ *which satisfies:*

1. *u is an upper semi-continuous function,*
2. $u(x) = 0$ *outside some interval [a,d],*
3. *There are real numbers* b, c *such as* $a \leq b \leq c \leq d$ *and*
   - *3.1* $u(x)$ *is a monotonic increasing function on [a, b],*
   - *3.2* $u(x)$ *is a monotonic decreasing function on [c, d],*
   - *3.3* $u(x) = 1$ *for all* $x \in [b, c]$.

Remark 2.1 [21] Get $\mathbb{R}_F$ is denoted the class of fuzzy subsets of real axis, $u(r) = (\underline{u}(r), \overline{u}(r))$ and $v(r) = (\underline{v}(r), \overline{v}(r))$ and $r \in [0, 1]$. The metric structure is given by Hausdorff distance satisfying the following properties:

$$D(u(r), v(r)) =$$

$$Max\{sup_{0\leq r\leq 1}|\underline{u}(r) - \underline{v}(r)|, sup|\overline{u}(r) - \overline{v}(r)|\}$$

$(\mathbb{R}_F, D)$ is a complete metric space and following properties are well known:

$$D(u + w, v + w) = D(u, v), \forall u, v, w \in \mathbb{R}_F$$

$$D(ku, kv) = |k|D(u, v), \forall u, v \in \mathbb{R}_F, \forall k \in \mathbb{R}$$

$D(u+v, w+e) \leq D(u, w) + D(v, e),$

$\forall u, v, w, e \in \mathbb{R}_F$

Definition 2.2 *[21] Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called the H-differential of x, y and it is denoted $x \ominus y$.*

Definition 2.3 *[9] Let $F : I \to \mathbb{R}_F$ and $t_0 \in I$. We say that F is differentiable at t_0 if there is $F'(t_0) \in \mathbb{R}_F$ such that either*

(I) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0+h) \ominus F(t_0)$ and $F(t_0) \ominus F(t_0-h)$ exist and the following limits
$lim_{h\searrow 0}\frac{F(t_0+h)\ominus F(t_0)}{h} = lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0-h)}{h}$
or
(II) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0+h)$ and $F(t_0-h) \ominus F(t_0)$ exist and the following limits
$lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0+h)}{-h} = lim_{h\searrow 0}\frac{F(t_0-h)\ominus F(t_0)}{-h}$
or
(III) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0+h) \ominus F(t_0)$ and $F(t_0-h) \ominus F(t_0)$ exist and the following limits
$lim_{h\searrow 0}\frac{F(t_0+h)\ominus F(t_0)}{h} = lim_{h\searrow 0}\frac{F(t_0-h)\ominus F(t_0)}{-h}$
or
(IV) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0+h)$ and $F(t_0) \ominus F(t_0-h)$ exist and the following limits
$lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0+h)}{-h} = lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0-h)}{h}$

Definition 2.4 *[12] Let $F : I \to \mathbb{R}_F$ be a set-valued function. A point $t_0 \in I$ is said to be a switching point for the differentiability of F, if in any neighborhood T of t_0 there exist points $t_1 < t_0 < t_2$ such that:*

Type 1: F is differentiable at t_1 in the sense (I) of definition (2.4) while it is not differentiable in the sense (II) of definition (2.4) and F is differentiable at t_2 in the sense (II) of definition (2.4) while it is not differentiable in the sense (I) of definition (2.4).
or
Type 2: F is differentiable at t_1 in the sense (II) of definition (2.4) while it is not differentiable in the sense (I) of definition (2.4) and F is differentiable at t_2 in the sense (I) of definition (2.4) while it is not differentiable in the sense (II) of definition (2.4).

Theorem 2.1 *[12] Let $F : I \to \mathbb{R}_F$ be differentiable on each $t \in I$ in the sense (III) or (IV) in definition (2.4). Then $F'(t) \in \mathbb{R}$ for all $t \in I$.*

Definition 2.5 *[3] Let $F : I \to \mathbb{R}_F$ and $t_0 \in I$. We can say that F is differentiable of $n-$ordered at t_0 if there is $F^{(n-1)}(t_0) \in \mathbb{R}_F$ such that either*

(I) For $h > 0$ sufficiently close to 0, for all $n \in \mathbb{N}$, the H-differences $F^{(n-1)}(t_0+h) \ominus F^{(n-1)}(t_0)$ and $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0-h)$ exist , and the following limits

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h} =$$

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h}$$

or
(II) For $h > 0$ sufficiently close to 0, for all $n \in \mathbb{N}$, the H-differences $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0+h)$ and $F^{(n-1)}(t_0-h) \ominus F^{(n-1)}(t_0)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0+h)}{-h} =$$

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0-h)\ominus F^{(n-1)}(t_0)}{-h}$$

or
(III) For $h > 0$ sufficiently close to 0, for all $n \in \mathbb{N}$, the H-differences $F^{(n-1)}(t_0+h) \ominus F^{(n-1)}(t_0)$ and $F^{(n-1)}(t_0-h) \ominus F^{(n-1)}(t_0)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h} =$$

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0-h)\ominus F^{(n-1)}(t_0)}{-h}$$

or
(IV) For $h > 0$ sufficiently close to 0, for all $n \in \mathbb{N}$, the H-differences $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0+h)$ and $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0-h)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0+h)}{-h} =$$

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h}$$

3 Analytic Solution of Nonlinear fuzzy integro-differential equation of $n-$th order

Fuzzy integro-differential equation of nth order is defined by following:

$$x^{(n)}(t)=f(t,x(t),x'(t),x''(t),...,x^{(n-1)}(t))+\int_{t_0}^{t}g(t,s,x(s),x'(s),x''(s),...,x^{(n-1)}(s))ds$$

where $x(t)$ is a fuzzy function of t, $f(t,x(t),x'(t),x''(t),...,x^{(n-1)}(t))$ is a fuzzy-valued function, the fuzzy variables $x'(t),x''(t),...,x^{(n-1)}(t)$ are defined derivatives of $x(t),x'(t),...,x^{(n-2)}(t)$ respectively and the kernel $g(t,x(s),x'(s),...,x^{(n-1)}(s))$ is a nonlinear function. Given initial values $x(t_0)=k_0,x'(t_0)=k_1,...,x^{(n-1)}(t_0)=k_{n-1}$, are used for fuzzy integro-differential problem of $n-$th order is obtained as follows:

$$\begin{cases} x^{(n)}(t)=f(t,x(t),x'(t),...,x^{(n-1)}(t))+ \\ \int_{t_0}^{t}g(t,s,x(s),x'(s),...,x^{(n-1)}(s))ds, \\ x(t_0)=k_0, \\ x'(t_0)=k_1, \\ \vdots \\ x^{(n-1)}(t_0)=k_{n-1} \end{cases} \tag{3.1}$$

for all $n\in\mathbb{N}$. The analytic solution of Eq. (3.1) is acquired and proved by following theorem:

Theorem 3.1 *Let $t_0\in[a,b]$, and assume that $f:[a,b]\times\underbrace{\mathbb{R}_F\times\mathbb{R}_F\times...\times\mathbb{R}_F}_{n-1}\to\mathbb{R}_F$ is continuous. A mapping $x:[a,b]\to\mathbb{R}_F$ is a solution to the initial value problem Eq. (3.1) if and only if x and x',..., $x^{(n-1)}$ are continuous and satisfy the following integral equation for all $t\in[a,b]$:*

$$x(t)=k_0+c_1(k_1(t-t_0)+c_2(\frac{k_2}{2!}(t-t_0)^2$$
$$+...+c_{n-1}(\frac{k_{n-1}}{(n-1)!}(t-t_0)^{(n-1)}+$$
$$c_n(\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n}f(s,x(s),...,x^{(n)}(s))ds...ds+$$
$$\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n+1}g(t,s,x(s),x'(s),...,x^{(n-1)}(s))ds...ds)))...). \tag{3.2}$$

where

$$c_i=\begin{cases}1 & x^{(i)}(t) is\ \ (I)-differentiable,\\ \ominus(-1) & x^{(i)}(t) is\ \ (II)-differentiable.\end{cases}$$

for all $i=1,2,...,n$.

Proof. Since f is continuous, then f is integrable. It is clear that by integrating from Eq. (3.1) over $[t_0,t]$, the following equation is obtained:

$$x^{(n-1)}(t)=$$
$$k_{n-1}+c_{n-1}(\int_{t_0}^{t}f(t,x(t),x'(t),...,x^{(n-1)}(t)ds$$
$$+\int_{t_0}^{t}\int_{t_0}^{t}g(t,x(s),x'(s),...,x^{(n-1)}(s))dsds)), \tag{3.3}$$

now we integrate from Eq. (3.3), then we have:

$$x^{(n-2)}(t)=k_{n-2}+c_{n-2}(k_{n-1}(t-t_0)$$
$$+c_{n-1}(\int_{t_0}^{t}\int_{t_0}^{t}f(t,x(t),x'(t),...,x^{(n-1)}(t))dsds$$
$$+\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}g(t,x(s),x'(s),...,x^{(n-1)}(s))dsdsds)), \tag{3.4}$$

by recurrence, the following equation is gained:

$$x'(t)=k_1+c_1(k_2(t-t_0)+c_2(k_3\frac{(t-t_0)^2}{2}$$
$$+...+c_{n-2}(k_n\frac{(t-t_0)^{n-1}}{(n-1)!}$$
$$+c_{n-1}(\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n-1}f(t,x(t),x'(t),...,x^{(n-2)}(t))+$$
$$\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n}g(t,x(s),x'(s),...,x^{(n-2)}(s))dsds...ds)))), \tag{3.5}$$

Consequently, the solution of $x(t)$ by integration of Eq. (3.4) over $[t_0, t]$ equivalently:

$$x(t) = k_0 + c_0(k_1(t-t_0) + c_1(k_2\frac{(t-t_0)}{2} +$$

$$... + c_{n-1}(k_n\frac{(t-t_0)^n}{(n)!}$$

$$+c_n(\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} f(t, x(t), x'(t), ..., x^{(n-1)}(t))$$

$$+\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n+1} g(t, x(s), x'(s), ..., x^{(n-1)}(s))ds...ds)))).$$

thus the proof is complete. Indeed, the solution is gained by integrating n times from Eq. (3.1) over $[t_0, t]$. □

Example 3.1 *Let following fuzzy integro-differential equation with initial values:*

$$\begin{cases} x''(t) = [-r^2t^3, -(2-r)^2t^3] + \int_0^t (x^2(s))ds \\ x(0) = [\alpha-1, 1-\alpha], \\ x'(0) = [\alpha, 2-\alpha] \end{cases}$$

The exact solution is $[3tr, 3t(2-r)]$ Then the solution of this equation by applying theorem (3.1) is:

$$x(t) = [\alpha-1, 1-\alpha] + c_1([\alpha, 2-\alpha]t$$

$$+c_2((\int_0^t\int_0^t [-r^2t^3, -(2-r)^2t^3]dsds$$

$$+\int_0^t\int_0^t\int_0^t (x^2(s))dsdsds))$$

case (1): Let $x(t)$ and $x'(t)$ be (I)-differentiable, the solution by theorem (3.1) is obtained in the following:

$$x(t) = [\alpha-1, 1-\alpha] + [\alpha, 2-\alpha]t$$

$$+\int_0^t\int_0^t [-r^2t^3, -(2-r)^2t^3]dsds$$

$$+\int_0^t\int_0^t\int_0^t (x^2(s))dsdsds$$

case (2): Let $x(t)$ and $x'(t)$ be (II)-differentiable, the solution by theorem (3.1) is gained in the following interval equation:

$$x(t) = [\alpha-1, 1-\alpha] + \ominus(-1)[\alpha, 2-\alpha]t$$

$$+\int_0^t\int_0^t [-r^2t^3, -(2-r)^2t^3]dsds$$

$$+\int_0^t\int_0^t\int_0^t (x^2(s))dsdsds$$

case (3): Let $x(t)$ be (I)-differentiable and $x'(t)$ be (II)-differentiable, the solution by theorem (3.1) is in the following interval equation:

$$x(t) = [\alpha-1, 1-\alpha] + [\alpha, 2-\alpha]t$$

$$+\ominus(-1)\int_0^t\int_0^t [-r^2t^3, -(2-r)^2t^3]dsds$$

$$\ominus(-1)\int_0^t\int_0^t\int_0^t (x^2(s))dsdsds$$

case (4): Let $x(t)$ be (II)-differentiable and $x'(t)$ be (I)-differentiable, the solution by theorem (3.1) is gained in the following equation:

$$x(t) = [\alpha-1, 1-\alpha] \ominus (-1)[\alpha, 2-\alpha]t$$

$$+\ominus(-1)\int_0^t\int_0^t [-r^2t^3, -(2-r)^2t^3]dsds$$

$$\ominus(-1)\int_0^t\int_0^t\int_0^t (x^2(s))dsdsds$$

These four cases are solved by the Adomian decomposition method, the solutions of which are shown in Fig. 1, Fig. 2, Fig. 3 and Fig. 4 by maple programming. Some figures are presented to illustrate the comparison between the exact solution and the approximate solution Fig. 5, Fig. 6, Fig. 7 and Fig. 8. As can be seen, the solutions are fuzzy number over the domain in four cases.

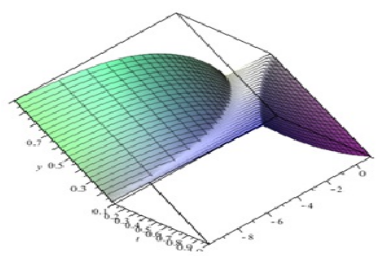

Figure 1: Case (1) in Example (3.1)

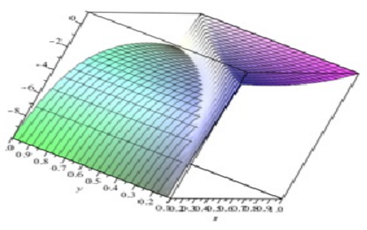

Figure 2: Case (2) in Example (3.1)

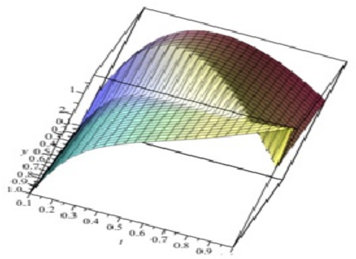

Figure 3: Case (3) in Example (3.1)

4 Existence and uniqueness of solution of fuzzy integro-differential equation

In this section, the existence and uniqueness of the solution of fuzzy integro-differential equation of nth order have been studied:

Theorem 4.1 *Let*

$$f:[t_0,t]\underbrace{\times\mathbb{R}_F\times...\times\mathbb{R}_F}_{n-1}\to\mathbb{R}_F$$

is continuous and suppose that there exist $M_1,M_2,...,M_n>0$ *such that*

$$D(f(t,s,x_1,x_2,...,x_n),f(t,s,y_1,y_2,...,y_n))\\ \leq\sum M_iD(x_i,y_i).$$

for all $t\in[a,b]$, $x_i,y_i\in\mathbb{R}_F, i=1,2,...,n$ and there exist $m_1,m_2...m_n>0$ which

$$D(\int_{t_0}^{t}g(t,s,x(s),x'(s),x''(s),...,x^{(n-1)}(s)),\\ \int_{t_0}^{t}g(t,s,y,y'(s),y''(s),...,y^{(n-1)}(s)))\\ \leq\sum m_iD(x_i,y_i),$$

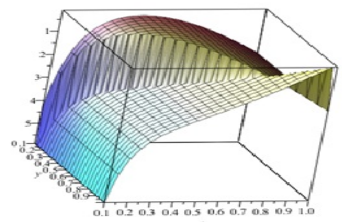

Figure 4: Case (4) in Example (3.1)

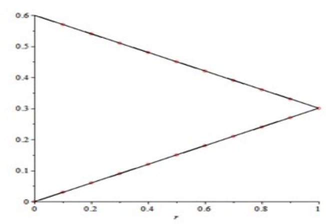

Figure 5: Compare between the exact solution and obtained solution with t=0.1 in Example (3.1), case (1)

there is $r_i>0$ such that $D(k_i,\tilde{0})\leq r_i$ and $Max(M_1,...,M_{n-1},m_1,...,m_{n-1})\leq M$. Then the initial value problem (1) has a unique solution on $[t_0,t]$, for all $t\in[a,b]$ in each sense of differentiability and the following successive iterations:

$$\begin{cases} x_1^{(n)}(t)=f(t,s,x_0'(s),x_0'',...,x_0^{(n-1)}),\\ x_i^{(n)}(t)=f(t,s,x_{i-1},x_{i-1}',...,x_{i-1}^{(n-1)})+\\ \int_{t_0}^{t}g(t,s,x_{i-1},x_{i-1}',...,x_{i-1}^{(n-1)}),\\ foralli=2,3,... \end{cases}\tag{4.6}$$

where Eq. (4.6) is uniformly convergent to $x(t)$ on $[t_0,t]$.

Proof. First, it is proved that $x_n(t)$ is bounded. Then is prove that $x_i(t)$ is reached as a sequence of bounded functions on $[t_0,t]$. Next it is proved that $x_n(t)$ for all $n\in\mathbb{N}$ are continuous on $[t_0,t]$. For $t_0\leq t_1\leq t_2\leq t$, let:

$$D(x_i(t),\tilde{0})\leq D(k_0,\tilde{0})+c_1(D(k_1,\tilde{0})(t-t_0)\\ +c_2(D(k_2,\tilde{0})\frac{(t-t_0)^2}{2!}+...\\ +c_{n-1}(D(k_{n-1},\tilde{0})\frac{(t-t_0)^{(n-1)}}{(n-1)!}$$

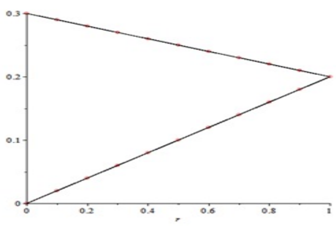

Figure 6: Compare between the exact solution and obtained solution with t=0.1 in Example (3.1), case (2)

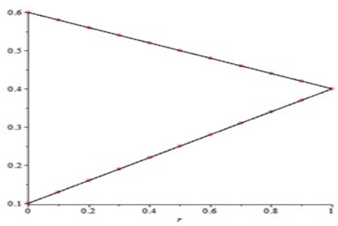

Figure 7: Compare between the exact solution and obtained solution with t=0.1 in Example (3.1), case (3)

$$+c_n(\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(f(s,x_{i-1}(s),x'_{i-1}(s),...,x_{i-1}^{(n-1)}(s)))$$

$$,\tilde{0})ds...ds+\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n+1} D(g(t,s,x_{i-1}(s),x'_{i-1}(s),...,$$

$$x_i^{(n-1)}(s),\tilde{0})ds...ds))...)) \leq r_0+c_1r_1(t-t_0)$$

$$+c_1c_2r_2\frac{(t-t_0)^2}{2!}+...+c_1c_2...c_{n-1}r_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}$$

$$+c_1c_2...c_n\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(f(s,x_{i-1}(s)$$

$$,...,x_{i-1}^{(n-1)}(s)),\tilde{0})ds...ds$$

$$+c_1c_2...c_n\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n+1} D(g(t,s,x_{i-1}(s),x'_{i-1}(s),...,$$

$$x_i^{(n-1)}(s)),\tilde{0})ds...ds \leq r_0+c_1r_1(t-t_0)+$$

$$c_1c_2r_2\frac{(t-t_0)^2}{2!}+...$$

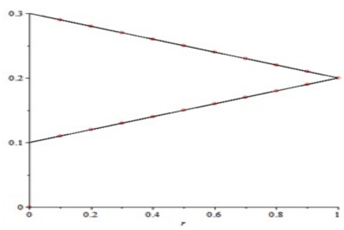

Figure 8: Compare between the exact solution and obtained solution with t=0.1 in Example (3.1), case (4)

$$+c_1c_2...c_{n-1}r_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}$$

$$+c_1c_2...c_n\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n}\sum M_iD(x_{i-1},\tilde{0})ds...ds+$$

$$c_1c_2...c_n\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n}\sum m_iD(x_i,\tilde{0})ds...ds$$

$$\leq Max(r_0,...,r_{n-1},M_0,...,M_{n-1},$$

$$m_0,...,m_{n-1})\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(x_{i-1},\tilde{0})ds...ds$$

It is concluded that x_i is bounded.

Now we prove that $x_i^{(n)}$ is continuous:

$$D(x_i^{(n)}(t_1),x_i^{(n)}(t_2)) \leq D(f(t_1,s,x_{i-1},$$

$$x'_{i-1},...,x_{i-1}^{(n-1)})+\int_{t_0}^{t_1} g(t_1,s,x_{i-1}$$

$$,x'_{i-1},...,x_{i-1}^{(n-1)})ds, f(t_2,s,x_{i-1},x'_{i-1},...,$$

$$x_{i-1}^{(n-1)})+\int_{t_0}^{t_2} g(t_2,s,x_{i-1},x'_{i-1},...,$$

$$x_{i-1}^{(n-1)})ds) \leq D(f(t_1,s,x_{i-1},x'_{i-1},...,$$

$$x_{i-1}^{(n-1)}), f(t_2,s,x_{i-1},x'_{i-1},...,x_{i-1}^{(n-1)}))$$

$$+D(\int_{t_0}^{t_1} g(t_1,s,x_{i-1},x'_{i-1},...,x^{(n-1)_{i-1}})ds$$

$$,\int_{t_0}^{t_2} g(t_2,s,x_{i-1},x'_{i-1},...,x_{i-1}^{(n-1)})ds)$$

$$\leq D(f(t_1, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)})$$

$$, f(t_2, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)}))$$

$$+D(\int_{t_0}^{t_1} g(t_1, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)} ds)$$

$$, \int_{t_0}^{t_1} g(t_2, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)}) ds)$$

$$+D(\int_{t_1}^{t_2} g(t_2, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)}) ds, 0)$$

$$\leq D(f(t_1, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)})$$

$$, f(t_2, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)}))$$

$$+ \int_{t_0}^{t_1} D(g(t_1, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)})$$

$$, g(t_2, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)})) ds$$

$$+ \int_{t_1}^{t_2} D(g(t_2, s, x_{i-1}, x'_{i-1}, ..., x_{i-1}^{(n-1)}), 0)$$

Thus it is given:

$$D(x_i^{(n)}(t_1), x_i^{(n)}(t_2)) \to 0$$

as $t_1 \to t_2$.
Therefore the sequence $x_i^{(n)}(t)$ is continuous on $[t_0, t]$.

Here for proving that x_i is continuous, we get:

$$D(x_{i+1}(t), x_i(t))$$

$$\leq D(\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} f(s, x_i(s), ..., x_i^{(n-1)}(s))) ds...dsds$$

$$+ \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} g(t, s, x_i(s), ..., x_i^{(n-1)}(s))) ds...ds,$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} f(s, x_{i-1}(s), ..., x_{i-1}^{(n-1)}(s))) ds...dsds+$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} g(t, s, x_{i-1}(s), ..., x_{i-1}^{(n-1)}(s))) ds...ds)$$

$$\leq \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(f(s, x_i(s), ..., x_i^{(n-1)}(s))+$$

$$\int_{t_0}^{t} g(t, s, x_i(s), ..., x_i^{(n-1)}(s)) ds,$$

$$f(s, x_{i-1}(s), ..., x_{i-1}^{(n-1)}(s))$$

$$+ \int_{t_0}^{t} g(t, s, x_{i-1}(s), ..., x_{i-1}^{(n-1)}(s)) ds) ds...dsds$$

$$\leq \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} (D(f(s, x_i(s), ..., x_i^{(n-1)}(s))$$

$$, f(s, x_{i-1}(s), ..., x_{i-1}^{(n-1)}(s)))$$

$$+ \int_{t_0}^{t} D(g(t, s, x_i(s), ..., x_i^{(n-1)}(s)$$

$$, g(t, s, x_{i-1}(s), x'_{i-1}(s)), ..., x_{i-1}^{(n-1)}(s))) ds)...ds$$

$$\leq \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} \sum_{i=0}^{n} M_i D(x_i(s), x_{i-1}(s)) ds...ds$$

$$+ \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} \sum_{i=0}^{n} m_i D(x_i(s), x_{i-1}(s)) ds...ds$$

$$\leq Max(M_1, ..., M_{n-1}, m_1, ..., m_{n-1})$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(x_i(s), x_{i-1}(s)) ds...ds$$

Thereupon if

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(x_i(s), x_{i-1}(s)) ds...ds \leq S$$

which $S \in \mathbb{N}$, then it is written as follows:

$$D(x_{i+1}(t), x_i(t)) \leq MS$$

Now for $n = 1$, it is given:

$$D(x_2(t), x_1(t)) \leq \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(f(s, x_1(s),$$

$$x_1'(s), ..., x_1^{(n)}(s)), f(s, x_0(s), x_0'(s),$$

$$..., x_0^{(n)}(s)))ds...dsds + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} D(g(t, s, x_1(s),$$

$$x_1'(s), ..., x_1^{(n)}(s)))ds...ds, 0) \leq$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} \sum_{i=0}^{n-1} M_i D(x_1, x_0)ds...ds+$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} \sum_{i=0}^{n-1} m_i D(x_1, 0)ds...ds$$

$$\leq Max(M_1, ..., M_{n-1}) \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n}$$

$$D(x_1, x_0)ds...ds + Max(m_1, ..., m_{n-1})$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(x_1, 0)ds...ds.$$

Now if

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} D(x_1, x_0)ds...ds \leq P$$

and

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} D(x_1, 0)ds...ds \leq U$$

where $P, U \in \mathbb{N}$, then we get:

$$D(x_2(t), x_1(t)) \leq Max(M_1, ..., M_{n-1})P$$

$$+Max(m_1, ..., m_{n-1})U$$

To prove that the solution of fuzzy integro-differential of nth order equation is unique, let $y(t)$ is a continuous solution of Eq. (3.1) on $[t_0, t]$. Then

$$D(y(t), x_i(t))$$

$$= D(\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} f(s, y(s), ..., y^{(n)}(s)))ds...ds$$

$$+ \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} g(t, s, y(s), ..., y^{(n)}(s)))ds...ds,$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} f(s, x_{i-1}(s), x_{i-1}'(s),$$

$$..., x_{i-1}^{(n)}(s)))ds...dsds + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1}$$

$$g(t, s, x_{i-1}(s), x_{i-1}'(s), ..., x_{i-1}^{(n)}(s))ds...ds)$$

$$\leq \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(f(s, y(s), y'(s), ..., y^{(n)}(s))+$$

$$\int_{t_0}^{t} g(t, s, y(s), ..., y^{(n)}(s))ds, f(s, x_{i-1}(s),$$

$$x_{i-1}'(s), ..., x_{i-1}^{(n)}(s)) + \int_{t_0}^{t} g(t, s$$

$$, x_{i-1}(s), x_{i-1}'(s), ..., x_{i-1}^{(n)}(s)))ds...ds \leq$$

$$\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(f(s, y(s), ..., y^{(n)}(s)), f(s, x_i(s),$$

$$..., x_i^{(n)}(s))) + \int_{t_0}^{t} D(g(t, s, y(s), y'(s),$$

$$..., y^{(n)}(s)), g(t, s, x_{i-1}(s), x_{i-1}'(s), ...$$

$$, x_{i-1}^{(n)}(s)))ds...ds \leq \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} D(f(s, y(s),$$

$$y'(s), ..., y^{(n)}(s)), f(s, x_{i-1}(s), x_{i-1}'(s), ...$$

$$, x_{i-1}^{(n)}(s))) + \underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} D(g(t, s, y(s), y'(s)$$

$$, ..., y^{(n)}(s)), g(t, s, x_{i-1}(s), x_{i-1}'(s), ...,$$

$$x_{i-1}^{(n)}(s)))ds...ds \leq \underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n}\sum_{i=0}^{n-1} M_i D(y(s)$$

$$,x_{i-1}(s))ds...ds + \underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n}\sum_{i=0}^{n-1} m_i D(x(s),$$

$$x_{i-1}(s))ds...ds \leq Max(M_1,...,M_n,m_1,$$

$$...,m_n)\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(y(s),x_{i-1}(s))ds...ds.$$

Now it can be given:

$D(y(t),x_i(t)) \leq$

$$M\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(y(s),x_{i-1}(s))ds...ds \leq$$

$$M^2\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(y(s),x_{i-2}(s))ds...ds$$

$$\leq ... \leq M^i\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(y(s),x_0(s))ds...ds.$$

Accordingly if

$$\underbrace{\int_{t_0}^{t}...\int_{t_0}^{t}}_{n} D(y(s),x_0(s))ds...ds \leq K$$

where

$$K \in \mathbb{N}$$

then

$$D(y(t),x_i(t)) \leq M^i K.$$

Since $M < 1$ then $lim_{i\to\infty} x_i(t) = y(t) = x(t)$. Hence the theorem is completely proved. □

Algorithm:

step 1: For $i = 1$ set

$$x_i^{(n)}(t) = f(t,s,x_{i-1},x'_{i-1},...,x_{i-1}^{(n-1)});$$

step 2: Calculate x_i from Eq. (3.1);

step 3: Obtain that $D(x_i,0)$ is bounded;

step 4: Prove that x_i is continuous;

step 5: Obtain that $D(x_{i+1}(t),x_i(t))$ is bounded;

step 6: Let $i+1 = i$ and set

$$x_i^{(n)}(t) = f(t,s,x_{i-1},x'_{i-1},...,x_{i-1}^{(n-1)})$$

$$+\int_{t_0}^{t} g(t,s,x_{i-1},x'_{i-1},...,x_{i-1}^{(n-1)})ds$$

then go to step 2.

5 Conclusion

The paper studied analytic solution of nonlinear fuzzy Volterra integro-differential of n-th order of the second kind under generalized derivation. This analytic solution was a nonlinear fuzzy Volterra integral equation with fuzzy nonlinear kernels. Adomian method was applied to solve this analytic solution. An example was used to compare the exact solution with the analytic solution. A least error was found and the approximated results were found to be fuzzy numbers. Moreover, upper bound on the solution of fuzzy n-th order integro-differential is introduced and the existence and uniqueness of its solution were investigated. The results of this method can be generalized as a credible model for getting approximate solution with a certainty the solutions exist and are unique. An algorithm was also proposed to vividly show the existence and uniqueness of the solution. The computations in this paper were performed by the application of the Maple 18 .

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