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Research Article

Bernstein Multi-Scaling Operational Matrix Method for Nonlinear Matrix Differential Models of Second-Order

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Abstract

In The current paper presents an idea for solving a class of linear matrix differential equations of second order. To perform so, the operational matrix of the integration based on the Bernstein multi-scaling polynomials are used to reduce the main problem to system of matrix equations. Numerical experiments illustrate the applicably and efficiency of the propounded technique.

Keywords : Matrix differential equation; Bernstein multi-scaling polynomials; Operational matrix of integration.

1 Introduction

IN the presented work, we consider the following second-order matrix differential problem:

$$\begin{cases} Y''(x) = A(x)Y'(x) + B(x)Y(x) + C(x), \\ \qquad Y(a) = Y_a, Y'(a) = Y'_a. \end{cases} \tag{1.1}$$

Where$Y(x) \in R^{p\times q}$, is an unknown matrix, the matrices $Y_a, Y'_a \in R^{p\times q}$ $A(x), B(x) : [a,b] :\to R^{p\times p}$ and $C(x) : [a,b] :\to R^{p\times q}$ are given. In (1.1), we assume A, B, C satisfy in condition of existence and uniqueness of the solution. A great variety of phenomena in physics and engineering can be modelled in the form of matrix differential equations. Models of differential equations of second-order frequently appear in molecular dynamics, quantum mechanics and for scattering methods, where one solves scalar or vectorial problems with boundary value conditions [1, 2, 3, 4, 7, 14]. There are various ways to solve these equations. In recent research, the use of polynomials in solving equations is common [9, 12, 13]. Here, we use Bernstein multi-scaling polynomials (BMSPs) to find a solution of these equations. One of the advantages of this method is the ability to approximate piecewise continuous functions. At first, all of the functions in the problem (1.1) approximate in terms of the BMSPs of degree m with unknown coefficient. Then, we reach to a system of matrix equations. So, unknown coefficients will obtained by solving the matrix equations simply. The outline of this paper is arranged as follows: in Section 2, we presents a brief survey on some definitions and properties of the BMSPs which are needed for

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our researches. In section 3, it exposes that how the BMSPs can be implemented to change solving (1.1) into resolving matrix equations. Section 4 is assigned to illustrate some numerical experiments which show the accuracy of the proposed numerical approach for solving (1.1). Eventually, the paper is ended with a brief conclusion in Section 5.

2 An overview on BMSPs

Bernstein polynomials are one of the oldest and most famous polynomials. These polynomials have several properties that can be found in some of their features in [5, 8].

Definition 2.1 *Suppose m is a positive integer number, BPs of degree m on interval $[a,b]$ are defined as follows:*

$$B_{i,m}(x) = \binom{m}{i}\frac{(x-a)^i(b-x)^{m-i}}{(b-a)^m},\ \ 0 \leq i \leq m.$$

Also, $B_{i,m}(x) = 0$ if $i < 0$ or $i > m$. For convenience we consider $[a,b] = [0,1]$, namely $B_{i,m}(x) = \binom{m}{i}x^i(1-x)^{m-i},\ 0 \leq i \leq m.$

We denote Φ_m , an m-column vector as follows:

$$\Phi_m(x) = \begin{bmatrix} B_{0,m}(x) & B_{1,m}(x) & \cdots & B_{m,m}(x)\end{bmatrix}^T$$

Theorem 2.1 *[6] Suppose $H = l^2([a,b])$ is a Hilbert space with the inner product defined by $\langle f,g\rangle = \int_a^b f(t)g(t)dt$ and also, $Y = Span\{B_{0,m}(x), B_{1,m}(x), ..., B_{m,m}(x)\}$ be the span space by Bernsteins polynomials of degree m . Let f be an arbitrary element in H .*

Since Y is a finite dimensional and closed subspace, it is a complete subset of H . So, f has the unique best approximation out of Y such that y_o

$$\exists y_0 \in Y; \forall y \in Y : \| f - y_0 \|_2 \leqslant \| f - y \|_2 .$$

Therefore, there are the unique coefficients $\alpha_j, 0 \leq j \leq m.$ such that

$$f(t) \approx y_0(t) = \sum_{j=0}^{m} \alpha_j B_{j,m}(t) = \alpha^T.\Phi_m$$

where,$\alpha = \begin{bmatrix}\alpha_0 & \alpha_1 & \cdots & \alpha_m\end{bmatrix}^T$, can be obtained by

$$\alpha = \frac{\langle f(t), \Phi_m(t)\rangle}{\langle \Phi_m(t), \Phi_m(t)\rangle}$$

such that $\langle f, \Phi_m(t)\rangle = \int_a^b f(t)\Phi_m(t)dt.$
we denote $Q = \langle \Phi_m(t), \Phi_m(t)\rangle$ as dual matrix. Furthermore, it is easy to see

$$Q_{i,j} = \frac{\binom{m}{i-1}\binom{m}{j-1}}{(2m+1)\binom{2m}{i+j-2}}, i,j = 1, ..., m+1.$$

Operational matrix is a matrix that works on basis like an operator. In [10, 11, 15] two different computational methods are presented for the operational matrices for the integration and product as follows:

(i)

$$\int_0^x \Phi_m(x)dx = M\Phi_m(x).$$

(ii)

$$\Phi_m^T C \Phi_m = \hat{C}^T\Phi_{2m}$$

Definition 2.2 *Assume $B_{i,m}(x)$ be th BPs of degreem on unit interval, Bernstein Multi-scaling polynomials on $[0,1)$ define as follow:*
$\psi_{i,j}(x) = \begin{cases} B_{i,m}(kx-j), & \frac{j}{k} \leqslant x < \frac{j+1}{k}, \\ 0, & otherwise. \end{cases}$ Where $k \geqslant 1$ is the number of partitions on $[0,1]$ and$i = 0, \cdots, m$ and also, $j = 0, \cdots, k-1$.
Now every function $f \in l^2([0,1])$ has the unique best approximation with respect to span space by BMSPs as follows:

$$f(x) = \sum_{j=0}^{k-1}\sum_{i=0}^{m} c_{i,j}\psi_{i,j} = C^T\Psi,$$

where

$$C^T = \begin{bmatrix} c_{0,0} & \cdots & c_{m,0} & \cdots & c_{0,k-1} & \cdots & c_{m,k-1}\end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} \psi_{0,0} & \cdots & \psi_{m,0} & \cdots & \psi_{0,k-1} & \cdots & \psi_{m,k-1}\end{bmatrix}^T$$

are two $k(m+1)$ column vectors.
The operational matrices for the integration $\bar{P}$, dual $\bar{Q}$ and product $\bar{C}$ are respectively given by

$$\int_0^x \Psi(t)dt = \bar{P}\Psi(x). \tag{2.2}$$

Table 1: Results on example 4.1.

x	$y_1(x)$	Exact solution	$y_2(x)$	Exact solution	$y_3(x)$	Exact solution	y˙4(x)	Exact solution
0.05	1.05172101	1.05127109	0.0	0.0	-0.0012920	-0.0012924	1.05127109	1.05127109
0.15	1.16183424	1.16183424	0.0	0.0	-0.0124408	-0.0124402	1.16183424	1.16183424
0.25	1.284024988	1.28402541	0.0	0.0	-0.036980	0.0369809	1.284024988	1.28402541
0.35	1.419064067	1.41906754	0.0	0.0	-0.0776011	-0.0776060	1.419064067	1.41906754
0.45	1.568315078	1.56831218	0.0	0.0	-0.0137419	-0.0137488	1.568315078	1.56831218
0.55	1.733205328	1.733253018	0.0	0.0	-0.220176	-0.2200361	1.733205328	1.733253018
0.65	1.915535467	1.915540829	0.0	0.0	-0.328976	-0. 3295607	1.915535467	1.915540829
0.75	2.117000134	2.117000017	0.0	0.0	-0.470739	-0.4707499	2.117000134	2.117000017
0.85	2.339698743	2.339646852	0.0	0.0	-0.6490590	-0.6490529	2.339698743	2.339646852
0.95	2.585740962	2.585709659	0.0	0.0	-0.8707165	-0.8707145	2.585740962	2.585709659

Table 2: Results on example 4.1.

x	$y_1(x)$	Exact solution	$y_2(x)$	Exact solution	$y_3(x)$	Exact solution	y˙4(x)	Exact solution
0.05	1.05172101	1.05127109	-0.0012920	-0.0012924	0.0	0.0	1.05127109	1.05127109
0.15	1.16183424	1.16183424	-0.0124408	-0.0124402	0.0	0.0	1.16183424	1.16183424
0.25	1.284024988	1.28402541	-0.036980	0.0369809	0.0	0.0	1.284024988	1.28402541
0.35	1.419064067	1.41906754	-0.0776011	-0.0776060	0.0	0.0	1.419064067	1.41906754
0.45	1.568315078	1.56831218	-0.0137419	-0.0137488	0.0	0.0	1.568315078	1.56831218
0.55	1.733205328	1.733253018	-0.220176	-0.2200361	0.0	0.0	1.733205328	1.733253018
0.65	1.915535467	1.915540829	-0.328976	-0. 3295607	0.0	0.0	1.915535467	1.915540829
0.75	2.117000134	2.117000017	-0.470739	-0.4707499	0.0	0.0	2.117000134	2.117000017
0.85	2.339698743	2.339646852	-0.6490590	-0.6490529	0.0	0.0	2.339698743	2.339646852
0.95	2.585740962	2.585709659	-0.8707165	-0.8707145	0.0	0.0	2.585740962	2.585709659

$$\int_0^x \Psi(t)\Psi^T(t)dt = \bar{Q} \tag{2.3}$$

$$\bar{C}\hat{\Psi}_p\hat{\Psi}_p^T = \hat{\Psi}_p^T\tilde{C} \tag{2.4}$$

The details of the obtaining of these matrices are given in [11].

3 Implementation of the procedure

Let us approximate each of the entries of $Y''(x)$ in (1.1), on by the BMSPs. That is,

$$Y''(x) = \hat{\Psi}_p^T\bar{Y} \tag{3.5}$$

Where $\bar{Y}$, is a $p(m+1)\times q$ unknown matrix and $\hat{\Psi}_p = \Psi \otimes I_p$. Also, the the notation $\otimes$ stands for the well-known Kronecker product, I_p is the identity matrix of order p . Also, using Eq. (2.2) implies:

$$Y'(x) = \hat{\Psi}_p^T\hat{P}_p^T\bar{Y} + \hat{Y}_p^T\bar{Y}_0' \tag{3.6}$$

$$Y(x) = \hat{\Psi}_p^T(\hat{P}_p^T)^2\bar{Y} + \hat{\Psi}_p^T\hat{P}_p^T\bar{Y_0}' + \hat{\Psi}_p^T\bar{Y}_0 \tag{3.7}$$

Where, $\bar{Y_0}'$ and are two known $p(m+1)\times q$ matrix and $\hat{P}_p = \bar{P}\otimes I_p$. In a similar way for matrix functions $A(x), B(x), C(x)$ we can see:

$$A(x) = \bar{A}\hat{\Psi}_p,\ B(x) = \bar{B}\hat{\Psi}_p,\ C(x) = \hat{\Psi}_p^T\bar{C}, \tag{3.8}$$

Where$\bar{A}$, $\bar{B}$ are two $p\times p(m+1)$ matrices and $\bar{C}$ is a $p\times q(m+1)$ matrix. The substitution of (3.5) to (3.8) in (1.1) concludes:

$$\begin{aligned}\hat{\Psi}_p^T\bar{Y} &= \bar{A}\hat{\Psi}_p\left(\hat{\Psi}_p^T\hat{P}_p^T\bar{Y} + \hat{\Psi}_p^T\bar{Y_0}'\right) \\ &+ \bar{B}\hat{\Psi}_p\left(\hat{\Psi}_p^T(\hat{P}_p^T)^2\bar{Y} + \hat{\Psi}_p^T\hat{P}_p^T\bar{Y_0}' + \hat{\Psi}_p^T\bar{Y}_0\right) \\ &+ \hat{\Psi}_p^T\bar{C}.\end{aligned} \tag{3.9}$$

Now (2.4) implies:

$$\begin{aligned}\bar{A}\hat{\Psi}_p\hat{\Psi}_p^T &= \hat{\Psi}_p^T\tilde{A}, \\ \bar{B}\hat{\Psi}_p\hat{\Psi}_p^T &= \hat{\Psi}_p^T\tilde{B}.\end{aligned}$$

So, (3.9) changes to:

$$\begin{aligned}\hat{\Psi}_p^T\bar{Y} &= \hat{\Psi}_p^T\tilde{A}\left(\hat{P}_p^T\bar{Y} + \bar{Y_0}'\right) \\ &+ \hat{\Psi}_p^T\tilde{B}\left((\hat{P}_p^T)^2\bar{Y} + \hat{P}_p^T\bar{Y_0}' + \bar{Y_0}\right) \\ &+ \hat{\Psi}_p^T\bar{C}.\end{aligned}$$

Consequently, the final system is:

$$\begin{aligned}\bar{Y} &= \tilde{A}\left(\hat{P}_p^T\bar{Y} + \bar{Y_0}'\right) \\ &+ \tilde{B}\left((\hat{P}_p^T)^2\bar{Y} + \hat{P}_p^T\bar{Y_0}' + \bar{Y_0}\right) + \bar{C}.\end{aligned}$$

4 Numerical Examples

In this section, two numerical examples are examined to illustrate the efficiency of proposed method and the presented theoretical results. All of the numerical experiments are performed using maple 18 and E.E.S. 11. In the following tables, exact solution denoted by $Y(X) = \begin{bmatrix} y_1(x) & y_2(x) \\ y_3(x) & y_4(x) \end{bmatrix}$.

Also, we use absolute error for every entry of solution matrix. Also, all example computed with $m = 3$ and $k = 4$.

Example 4.1 *Consider the following equation*

$$\begin{cases} Y''(x) + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} Y'(x) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} Y(x) = 0 \\ Y(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Y'(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{cases}$$

where $Y(x) = \begin{bmatrix} e^x & 0 \\ -1 + e^x - xe^x & e^x \end{bmatrix}$ is the exact solution. Table 1 shows results of example 4.1.

Example 4.2 *Consider Eq. (1.1) where:*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, Y(0) = Y'(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The exact solution is :
$Y(x) = \begin{bmatrix} e^x & -1 + e^x - xe^x \\ 0 & e^x \end{bmatrix}$.
The results of example 4.2 are presented in table 2.

5 Conclusion

The properties of the BMSPs and their operational matrix have been utilized to numerically solve a class of the second order matrix differential problems. The proposed method converts the main problem to a linear matrix equations. Numerical examples have illustrated to demonstrate the efficiency and applicably of our offered approach. Finally, we mention that the proposed technique can be extended for more complicated types of matrix differential equations.

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