Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 12, No. 3, 2020, Article ID IJIM-1016, 9 pages
DOR: http://dorl.net/dor/20.1001.1.20085621.2020.12.3.1.1
Researc Article

New Results on Ideals in MV-algebras

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Received Date: 2017-01-12 Revised Date: 2018-05-06 Accepted Date: 2018-08-27

Abstract

In the present paper, by considering the notion of ideals in MV-algebras, we study some kinds of ideals in MV-algebras and obtain some results on them. For example, we present definition of ultra ideal in MV-algebras, and we get some results on it. In fact, by definition of ultra ideals, we present new conditions to have prime ideals, positive implicative ideals and maximal ideals in MV-algebras. Also, we state some properties on contracted or extended ideals as useful examples of ideals in MV-algebras. Finally, we try to prove the Chines reminder theorem in MV-algebras.

Keywords : MV-algebra; Ideal; Ultra ideal; Chines reminder theorem; Pseudo-hoops.

1 Introduction

MV-algebras were defined by C. C. Chang [3, 4] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. In particular, emphasis has been put the ideal theory of MV-algebras [8, 11]. Hoo, Iseki and Tanaka introduced the notions of implicative and quasi-implicative ideals of MV-algebras [12, 13]. Many important properties can be derived from the fact, established by Chang that nontrivial MV-algebras are subdirect products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra. Recently, some reasearchers worked on MV-algebras and ideals in them (see [2, 10, 17, 18, 19]). For continuing of study of ideals in MV-algebras, we present definition of ultra ideal in MV-algebras and verify the relationship between it and some other ideals. Also, we introduce contraction and extension of an ideal in MV-algebras and we get related results.

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2 Preliminaries

In this section, we review some definitions and related lemmas and theorems in MV-algebras that we use in the next sections.

Definition 2.1 *[5] An MV-algebra is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that:*

$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $a \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
for every $a, b \in M$.
It is clear that $(M, \odot, 1)$ is an Abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice*. An MV-algebra M is a *Boolean* algebra if and only if the operation "$\oplus$" is idempotent, i.e., $x \oplus x = x$, for every $x \in M$. In MV-algebra M, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) there exists $c \in M$ such that $a \oplus c = b$, for every $a, b, c \in M$. For any two elements a, b of MV-algebra M, $a \leq b$ if and only if a, b satisfy in the above equivalent conditions $(i) - (iv)$. An ideal of MV-algebra M is a subset I of M, satisfying the following condition: $(I1)$ $0 \in I$, $(I2)$ $x \leq y$ and $y \in I$ imply that $x \in I$, $(I3)$ $x \oplus y \in I$, for every $x, y \in I$. Let I be an ideal of M and $I \neq M$ (we say I is a proper ideal of M) . Then (i) I is a prime ideal if and only if $x \ominus y \in I$ or $y \ominus x \in I$, for every $x, y \in M$. A proper ideal I of M is a maximal ideal of M if and only if no proper ideal of M strictly contains I. In MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv) $d(x, y) = d(x', y')$, (v) $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let I be an ideal of MV-algebra M. Then we denote $x \sim y$ ($x \equiv_I y$) if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in M\}$. Then $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an MV-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$. Let M and K be two MV-algebras. A mapping $f : M \to K$ is called an *MV-homomorphism* if $(H1)$ $f(0) = 0$, $(H2)$ $f(x \oplus y) = f(x) \oplus f(y)$ and $(H3)$ $f(x') = (f(x))'$, for every $x, y \in M$. If f is one to one (onto), then f is called an *MV-monomorphism* (*epimorphism*). If f is onto and one to one, then f is called an *MV-isomorphism*.(see [5])

Definition 2.2 *[6, 9] (i) An l-group is an algebra $(G, +, -, 0, \vee, \wedge)$, where the following properties hold:*

(a) $(G, +, -, 0)$ is a group,
(b) $(G, \vee, \wedge)$ is a lattice,
(c) $x \leq y$ implies that $x + a \leq y + a$, for any $x, y, a, b \in G$.
A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian l-groups with strong unit will be simply called lu-groups.
The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by $\mathcal{MV}$. The category whose objects are pairs (G, u), where G is an Abelian l-group and u is a strong unit of G and whose homomorphisms are l-group homomorphisms is denoted by $\mathcal{UG}$. The functor that establishes the categorial equivalence between $\mathcal{MV}$ and $\mathcal{UG}$ is

$$\Gamma : \mathcal{UG} \longrightarrow \mathcal{MV},$$

where $\Gamma(G, u) = [0, u]_G$, for every lu-group (G, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-group homomorphism h.

Lemma 2.1 *[5] Let M be an MV-algebra. Then $x \leq y$ implies that $x \odot z \leq y \odot z$ and $x \oplus z \leq y \oplus z$, for every $x, y, z \in M$.*

Definition 2.3 *[15] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:*

$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK2)$ $(x * (x * y)) * y = 0$,
$(BCK3)$ $x * x = 0$,
$(BCK4)$ $0 * x = 0$,
$(BCK5)$ if $x * y = y * x = 0$, then $x = y$, for all

$x, y, z \in X$.
The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order on X with 0 as least element. In BCK-algebra X, for any $x, y, z \in X$, we have $(BCK6)$ $(x * y) * z = (x * z) * y$.
Let $(X, *, 0)$ be a BCK-algebra. Subset $\emptyset \neq I \subseteq X$ is called an *ideal* of X, if $0 \in I$ and for any $x, y \in X$, $x * y \in I$ and $y \in I$, imply that $x \in I$. A nonempty subset I of X is said to be a *positive implicative ideal* if $0 \in I$ and $(x * y) * z \in I$, $y * z \in I$ imply that $x * z \in I$, for any $x, y, z \in X$. Furthermore, any positive implicative ideal must be an ideal. see [15]

Theorem 2.1 *[5] If $(M, \oplus, ', 0, 1)$ is an MV-algebra, then $(M, \ominus, 0)$ is a BCK-algebra.*

Corollary 2.1 *[5] (i) Every prime ideal I of an MV-algebra M is contained in a unique maximal ideal of M.*
(ii) Every proper ideal of an MV-algebra M is an intersection of prime ideals of M.

Lemma 2.2 *[5] Let M be an MV-algebra and $\emptyset \neq W \subseteq M$. If the generated ideal by W is denoted by $\prec W \succ$, then $\prec W \succ = \{x \in M : x \leq w_1 \oplus \cdots \oplus w_n,\ for\ some\ w_1, \cdots, w_n \in W\}$.*

Proposition 2.1 *[5] Let M, N be MV-algebras and J be a maximal ideal of N. Then for any homomorphism $h : M \longrightarrow N$, the inverse image $h^{-1}(J)$ is a maximal ideal of M.*

Lemma 2.3 *[5] Let M, N be two MV-algebras and $f : M \to N$ be an MV- homomorphism. Then the following properties hold:*
(i) $Ker(f)$ is an ideal of M,
(ii) if f is an MV-epimorphism, then $\frac{M}{Kerf} \cong N$,
(iii) $f(x) \leq f(y)$ iff $x \ominus y \in Ker(f)$,
(iv) f is injective iff $Ker(f) = \{0\}$.

Definition 2.4 *[6] A product MV-algebra (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $x + y$ is defined, then $x.z + y.z$ and $z.x + z.y$ are defined and $(x + y).z = x.z + y.z$, $z.(x + y) = z.x + z.y$, for every $x, y, z \in A$, where "+" is the partial addition on A. A unity for the product is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If A has a unity for product, then $e = 1$. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation.*

3 Some results on ideals

In this section, we verify some results on ideals.

Proposition 3.1 *Let M be an MV algebra and $I \subseteq M$. Then*
(1) I is an ideal of M if and only if the following holds:
(i) $0 \in I$,
(ii) $x \oplus y \in I$,
(iii) if $x \ominus y$, $y \in I$, then $x \in I$, for any $x, y \in M$.
(2) I is an ideal of M if and only if the following holds:
(i) $0 \in I$,
(ii) $x \oplus y \in I$,
(iii) if $z \ominus y$, $y \ominus x \in I$, then $z \ominus x \in I$, for any $x, y, z \in M$.

Proof. (1) ($\Rightarrow$) Let I be an ideal of M. Then (i) and (ii) are clear. Now, let $x \ominus y$, $y \in I$. Then by (ii) and $(MV7)$, $(y \ominus x) \oplus x = (x \ominus y) \oplus y \in I$. Since $x \leq (y \ominus x) \oplus x \in I$, we have $x \in I$.
($\Leftarrow$) Let (i), (ii) and (iii) be true. If $x \leq y$ and $y \in I$, then $x \ominus y = x \odot y' = 0 \in I$ and so by (iii), $x \in I$. Hence, I is an ideal of M.

(2) ($\Rightarrow$) Let I be an ideal of M. Then (i) and (ii) are clear. Now, let $z \ominus y$, $y \ominus x \in I$, for any $x, y, z \in M$. Then by Theorem 2.1 and $(BCK1)$, $((z \ominus x) \ominus (z \ominus y)) \ominus (y \ominus x) = 0$ and so by (1), $(z \ominus x) \in I$.
($\Leftarrow$) Let (i), (ii) and (iii) be true. If $x \leq y$ and $y \in I$, then $x \ominus y = x \odot y' = 0 \in I$. Since $y \ominus 0 = y \in I$, by (iii), $x = x \ominus 0 \in I$. Hence, I is an ideal of M.

Theorem 3.1 *Let J be an ideal of MV-algebra M and $a \in M$. Then*

$$\prec J \cup \{a\} \succ = \{x \in M : \exists n \in N, (x' \oplus na)' \in J\}.$$

Moreover, $\prec J \cup \{a\} \succ$ is the least ideal of M containing $J \cup \{a\}$.

Proof. Let $T = \{x \in M : \exists n \in N, (x' \oplus na)' \in J\}$. If $x \in \prec J \cup \{a\} \succ$, then by Lemma 2.2, there exist $b_1, \cdots, b_m \in J \cup \{a\}$ such that $x \leq b_1 \oplus b_2 \oplus \cdots \oplus b_m$ and so $x \odot (b_1 \oplus b_2 \oplus \cdots \oplus b_m)' = 0$. It means that $((x \ominus b_1) \ominus b_2) \ominus \cdots) \ominus b_m = 0 \in J$. We consider two cases. Let $b_i \neq a$, for any $1 \leq i \leq m$. Then by Theorem 2.1 and $(BCK6)$, $(((x \ominus a) \ominus b_1) \ominus b_2) \ominus \cdots) \ominus b_m = (((x \ominus b_1) \ominus b_2) \ominus \cdots) \ominus b_m) \ominus a = 0 \ominus a = 0 \in J$. Since $b_1, \cdots, b_m \in J$, we have $x \ominus a \in J$ and so $x \in T$. If there exists $b_i = a$, for some $1 \leq i \leq m$, then by renumbering, there exist $n, k \in N$ and $n, k < m$ such that $(((x' \oplus na)' \ominus b_1) \ominus \cdots) \ominus b_k = 0 \in J$. It results that $(x' \oplus na)' \in J$ and so $x \in T$. Now, let $x \in T$. Then there exists $n \in N$ such that $(x' \oplus na)' \in J$. Let $u = (x' \oplus na)'$. Then $u \in J$ and $(x' \oplus na)' \ominus u = u \ominus u = 0$. Hence, $x \in \prec J \cup \{a\} \succ$. Finally, we will show that $\prec J \cup \{a\} \succ$ is the least ideal of M containing $J \cup \{a\}$. Let C be an ideal of M containing $J \cup \{a\}$. We must show that $\prec J \cup \{a\} \succ \subseteq C$. Let $x \in \prec J \cup \{a\} \succ$. Then there is $n \in N$ such that $(x' \oplus na)' \in J \subseteq C$. Since $a \in C$, we have $(x' \oplus na)' \oplus na \in C$. Now, by $(MV4)$, we have $x \leq (x \oplus (na)')' \oplus x = (x' \oplus na)' \oplus na \in C$. It results that $x \in C$. Therefore, $\prec J \cup \{a\} \succ \subseteq C$.

Proposition 3.2 *Let $a, b \in M$ and J be an ideal of M. Then $\prec J \cup \{a\} \succ \cap \prec J \cup \{b\} \succ \subseteq \prec J \cup \{a \oplus b\} \succ$.*

Proof. Let $x \in \prec J \cup \{a\} \succ \cap \prec J \cup \{b\} \succ$. Then by Theorem 3.1, there exist $m, n \in N$ such that $(x' \oplus na)' \in J$ and $(x' \oplus mb)' \in J$. Let $u = (x' \oplus na)'$ and $v = (x' \oplus ma)'$. By Theorem 2.1 and $(BCK6)$, we have

$$\begin{aligned} (((x \ominus u) \ominus v)' \oplus na)' &= ((x \ominus u) \ominus v) \ominus na \\ &= ((x \ominus u) \ominus na) \ominus v \\ &= ((x \ominus na) \ominus u) \ominus v \\ &= ((x' \oplus na)' \ominus u) \ominus v \\ &= (u \ominus u) \ominus v = 0. \end{aligned}$$

Similarly, we have $((x \ominus u) \ominus v)' \oplus mb)' = ((x' \oplus mb)' \ominus v) \ominus u = (v \ominus v) \ominus u = 0$. Let $t = (x \ominus u) \ominus v$. We have $a \leq a \oplus b$. Then by Lemma 2.1, $t \ominus (a \oplus b) \leq t \ominus a$ and $(t \ominus (a \oplus b)) \ominus (a \oplus b) \leq (t \ominus a) \ominus (a \oplus b) = (t \ominus (a \oplus b)) \ominus a \leq (t \ominus a) \ominus a$. Hence, $(t \ominus (a \oplus b)) \ominus (a \oplus b) \leq (t \ominus a) \ominus a$. Similarly, it results that $(t' \oplus n(a \oplus b))' \leq (t' \oplus na)' = 0$ and so $(((x \ominus u) \ominus v)' \oplus n(a \oplus b))' = 0$. It is easy to show that $((x' \oplus n(a \oplus b))' \ominus u) \ominus v = 0$. Since $u, v \in J$, by Proposition 3.1(1), we get $(x' \oplus n(a \oplus b))' \in J$ and so by Theorem 3.1, $x \in \prec J \cup \{a \oplus b\} \succ$. Therefore, $\prec J \cup \{a\} \succ \cap \prec J \cup \{b\} \succ \subseteq \prec J \cup \{a \oplus b\} \succ$.

Notation: In general, the converse of Proposition 3.2, is not true.

Example 3.1 *Let $M = \{0, 1, 2, 3\}$ and operation "$\oplus$" is defined on M as follows:*

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

If $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$, then $(M, \oplus, 0, 3)$ is an MV-algebra and $I = \{0, 1\}$ is an ideal of M. It is easy to show that $\prec I \cup \{1 \oplus 2\} \succ = \prec I \cup \{3\} \succ = \{x : \exists n \in N, (x' \oplus n3)' \in I\} = \{0, 1, 2, 3\}$, $\prec I \cup \{1\} \succ = \{0, 1\}$ and $\prec I \cup \{2\} \succ = \{0, 1, 2, 3\}$. It results that $\prec I \cup \{1 \oplus 2\} \succ \nsubseteq \prec I \cup \{1\} \succ \cap \prec I \cup \{2\} \succ$.

4 Ultra ideals

In this section, we present definition of ultra ideals in MV-algebras. Then we verify some properties about them, and we obtain the relationship between ultra ideals and some other ideals.

Definition 4.1 *Let M be an MV-algebra and I be a non trivial ideal of M. Then I is called an ultra ideal of M if for every $x \in M$, $x \in I$ if and only if $x' \notin I$.*

Example 4.1 *Let $M = \{0, 1, 2, 3, 4\}$ and the operation "$\oplus$" on M is defined as follows:*

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	2	2	2	4	4
3	3	3	4	3	4
4	4	4	4	4	4

If $0' = 4$, $1' = 4$, $2' = 3$, $3' = 2$ and $4' = 0$, then $(M, \oplus, 0, 4)$ is an MV-algebra and $I = \{0, 1, 2\}$, $J = \{0, 1, 3\}$ and $K = \{0, 1\}$ are ideals of M. It is easy to show that I, J are ultra ideals of M. Since $2' = 3 \notin K$ and $2 \notin K$, K is not an ultra ideal of M.

Theorem 4.1 *Let I be an ultra ideal of MV-algebra M, J be a proper ideal of M and $I \subseteq J$. Then J is an ultra ideal of M, too.*

Proof. Let $x \in J$. If $x' \in J$, then by (I_3), $1 = x \oplus x' \in J$, which is a contradiction. Now, let $x' \notin J$. If $x \notin J$, then $x \notin I$ and so $x' \in I \subseteq J$, which is a contradiction.

By Theorem 2.1, in MV-algebra $(M, \oplus, ', 0, 1)$, if I is an ideal of BCK-algebra $(M, \ominus, 0)$ and it satisfies in (I_3), then I is an ideal of MV-algebra $(M, \oplus, ', 0, 1)$, too. Hence, in this case, definition of positive implicative ideals in BCK-algebras can be translated to MV-algebras. Then we can present the definition of positive implicative ideals in MV-algebras as follows:
let M be an MV-algebra and $\emptyset \neq I \subseteq M$. Then I is called a *positive implicative* ideal of M if the following hold: (i) $0 \in I$, (ii) $x \oplus y \in I$, (iii) if $(x \ominus y) \ominus z \in I$ and $y \ominus z \in I$, then $x \ominus z \in I$, for any $x, y, z \in M$. Also, in this field, all of proved theorems of ideals in BCK-algebras are true in MV-algebras.

Example 4.2 (i) *Let $M = \{0, 1, 2\}$ and operation $\oplus$ be defined by*

$\oplus$	0	1	2
0	0	1	2
1	1	1	2
2	0	2	2

If $0' = 2$, $1' = 1$ and $2' = 0$, then $(M, \oplus, ', 0, 2)$ is an MV-algebra. It is easy to show that $I = \{0, 1\}$ is a positive implicative ideal of M.
(ii) *In Example 4.1, K is a positive implicative ideal of M.*
(iii) *Let $M_2(\mathbb{R})$ be the ring of square matrixes of order 2 with real elements and let 0 be the matrix with all elements 0. It is easy to see that $M_2(\mathbb{R})$ is an l-group. If $v = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then $(M_2(\mathbb{R}), v)$ is an lu-group and so $M = \Gamma(M_2(\mathbb{R}), v)$ is an MV-algebra. It is easy to see that $I(M) = \{\{0\}, M\}$, where $I(M)$ is the set of ideals of M. It is easy to see that $\{0\}$ is not a positive implicative ideal of M.*

In following, we verify the relationship between ultra ideals and positive implicative (prime) ideals.

Theorem 4.2 *Let I be an ultra ideal of M. Then*
(i) *I is a positive implicative ideal of M,*
(ii) *I is a prime ideal of M.*

Proof. (i) Let $(z \ominus y) \ominus x \in I$, $y \ominus x \in I$, where $x, y, z \in M$. We must show that $z \ominus x \in I$. Let $z \ominus x \notin I$. Then $(z \ominus x)' \in I$. Since $x \ominus (z \ominus x)' = x \odot (z \ominus x) = x \odot (z \odot x') = 0 \in I$, we get $x \in I$. Now, since $y \ominus x$, $x \in I$, we have $y \in I$. On the other hand, by Theorem 2.1 and $(BCK6)$, since $(z \ominus x) \ominus y = (z \ominus y) \ominus x \in I$, we have $z \ominus x \in I$, which is a contradiction. Therefore, I is a positive implicative ideal of M.
(ii) If I is not a prime ideal of M, then there exist $x, y \in M$ such that $x \ominus y \notin I$ and $y \ominus x \notin I$. Since I is an ultra ideal of M, we have $(x \ominus y)' \in I$ and $(y \ominus x)' \in I$. Then $1 = (x' \oplus y) \oplus (y' \oplus x) = (x \ominus y)' \oplus (y \ominus x)' \in I$, which is a contradiction. Therefore, I is a prime ideal of M.

Example 4.3 (i) *In Example 4.1, K is a positive implicative ideal, but it is not an ultra ideal.*
(ii) *In example 4.2 (i), $\{0\}$ is a prime ideal of M, but it is not an ultra ideal of M.*
(iii) *In example 4.2 (iii), $\{0\}$ is neither a positive implicative ideal of M nor an ultra ideal of M. Also, $\{0\}$ is not a prime ideal of M.*

Definition 4.2 *Let M be an MV-algebra. $B \subseteq M$ is said to have the finite union property if $a_1 \oplus a_2 \oplus \cdots \oplus a_n \neq 1$, for any $a_1, \cdots, a_n \in B$ and $a_i \neq 1$, where $1 \leq i \leq n$.*

Example 4.4 *In Example 4.1, $B = \{0, 1, 2\}$ has finite union property, but $C = \{2, 3\}$ has not finite union property (note that $2 \oplus 3 = 4$).*

Theorem 4.3 *Let M be an MV-algebra, $B \subseteq A$ and $1 \notin B$. Then $\prec B \succ$ is a proper ideal of M if and only if B has the finite union property.*

Proof. $(\Rightarrow)$ Let $\prec B \succ$ be a proper ideal of M and B has not the finite union property. Then there exist $a_1, \cdots, a_n \in B$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1$. By Lemma 2.2, $1 \in \prec B \succ$ and so $\prec B \succ = M$, which is a contradiction.
$(\Leftarrow)$ Let B has the finite union property and $\prec B \succ = M$. Then $1 \in \prec B \succ$ and so by Lemma 2.2, there exist $a_1, \cdots, a_n \in B$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n \geq 1$, which is a contradiction.

Note. It is easy to see that every non trivial ideal of an MV-algebra has the finite union property. The proof is similar to the proof of Theorem 4.3 ($\Rightarrow$).

Lemma 4.1 *Let M be an MV-algebra, $x \in M$ and I be an ideal of M such that I have the finite union property. If $x \notin I$ and $x' \notin I$, then $I \cup \{x\}$ has the finite union property.*

Proof. Let $B = I \cup \{x\}$. We will show that $b_1 \oplus b_2 \cdots \oplus b_n \neq 1$, for any $b_1, \cdots, b_n \in B$ and $b_i \neq 1$. If $b_1, \cdots, b_n \in I$, then the proof is clear. If W. O. L. G, $b_1 = x$ and $b_1 \oplus b_2 \cdots \oplus b_n = 1$, for some $b_1, \cdots, b_n \in B$, then $(x' \odot (b_2 \oplus \cdots \oplus b_n)')' = x \oplus b_2 \cdots \oplus b_n = 1$ and so
$x' \ominus (b_2 \oplus \cdots \oplus b_n) = x' \odot (b_2 \oplus \cdots \oplus b_n)' = 0 \in I$. Since $b_2 \oplus \cdots \oplus b_n \in I$, by Proposition 3.1(1), we have $x' \in I$, which is a contradiction. Therefore, $b_1 \oplus b_2 \cdots \oplus b_n \neq 1$, for any $b_1, \cdots, b_n \in B$ and so $I \cup \{x\}$ has the finite union property.

Theorem 4.4 *Let M be an MV-algebra and $I \subseteq M$. Then I is an ultra ideal of M if and only if I is a non trivial maximal ideal of M.*

Proof. ($\Rightarrow$) Let I be an ultra ideal of M, and I is not a maximal ideal of M. Then there exists a proper ideal J of M such that $I \subsetneq J$ and so there exists $x \in J$ such that $x \notin I$. It results that $x' \in I$ and so $x' \in J$. Since $1 \ominus x = 1 \odot x' = (0 \oplus x')' = x' \in J$ and $x \in J$, we get $1 \in J$, which is a contradiction.
($\Leftarrow$) Let I be a maximal ideal of M. If $x \in I$ and $x' \in I$, for some $x \in M$, then $1 \in I$, which is a contradiction. Hence, $x \in I$ implies that $x' \notin I$. Now, let there exists $x \in A$ such that $x' \notin I$ and $x \notin I$. Consider $B = I \cup \{x\}$. Then by Lemma 4.1, B has the finite union property. Hence, by Theorem 4.3, $\prec B \succ$ is a proper ideal of M, which is a contradiction. Because, $I \subset \prec B \succ \subset M$ and I is a maximal ideal of M. Hence, $x' \notin I$ implies that $x \in I$. Therefore, I is an ultra ideal of M.

Lemma 4.2 *Let M be an MV-algebra and $I \subseteq M$. If I has the finite union property, then there exists an ultra ideal B of M such that $I \subseteq B$.*

Proof. Let $E = \{B : I \subseteq B$, *where B is a proper ideal of M*$\}$. Since I has the finite union property, by Theorem 4.3, $\prec I \succ$ is a proper ideal of M. Since $I \subseteq \prec I \succ$, we have $\prec I \succ \in E$ and so $E \neq \emptyset$. Let $F = \{B_i\}_{i \in \mathbb{N}}$ be a chain in E and $B_1 = \bigcup_{i \in \mathbb{N}} B_i$. Since B_1 is an upper bound of F in E and B_1 is an ideal of M, $B_1 \in E$. Hence, by Zorn's lemma, E has a maximal element B and so by Theorem 4.4, B is an ultra ideal of M such that $I \subseteq B$.

Theorem 4.5 *Any proper ideal in MV-algebra M, contained at least one ultra ideal.*

Proof. Let I be a proper ideal of M. Since $I = \prec I \succ$, by Theorem 4.3, I has the finite union property and so by Lemma 4.2, there exists an ultra ideal B of M such that $I \subseteq B$.

5 Contraction and Extension of ideals in MV-algebras

In this section, we verify some properties on contracted or extended ideals as useful examples of ideals in MV-algebras. Also, we try to prove the Chines reminder theorem in MV-algebras.
Remark: Let M, N be MV-algebras, $f : M \longrightarrow N$ be an MV-homomorphism, $I \subseteq M$ and J be an ideal of N. Then we set $f^{-1}(J) = J^c$ and $\prec f(I) \succ = I^e$. It is clear that J^c (contraction of J) is an ideal of M and I^e (extension of I) is an ideal of N.

Theorem 5.1 *Let M, N be MV-algebras, $f : M \longrightarrow N$ be an MV-homomorphism, I be an ideal of M and J be an ideal of N. Then*
(i) $I \subseteq I^{ec}$,
(ii) $J^{ce} \subseteq J$,
(iii) $J^c = J^{cec}$,
(iv) $I^e = I^{ece}$,
(v) If $K = \{I : I$ is an ideal of M and $I^{ec} = I\}$, $E = \{J : J$ is an ideal of N and $J^{ce} = J\}$, $K' = \{J^c : J$ is an ideal of $N\}$ and $E' = \{I^e : I$ is an ideal of $M\}$, then $K = K'$, $E = E'$ and there exists an isomorphism $\Phi : K \longrightarrow E$.

Proof.
(i) The proof is clear.
(ii) Let $y \in J^{ce} = \prec f(J^c) \succ$. Then there exist $t_1, t_2, \cdots, t_k \in J^c$ such that $y \leq f(t_1) \oplus \cdots \oplus f(t_k)$ and so $y \leq f(f^{-1}(a_1)) \oplus \cdots \oplus f(f^{-1}(a_k))$, where $f(t_i) = a_i \in J$, for any $1 \leq i \leq k$. It results that $y \leq a_1 \oplus \cdots \oplus a_k$ and so $y \in \prec J \succ = J$. Hence,

$J^{ce} \subseteq J$.
(iii) By (i), we have $J^c \subseteq J^{cec}$. We show that $J^{cec} \subseteq J^c$, i.e., $f^{-1}(\prec f(f^{-1}(J)) \succ) \subseteq f^{-1}(J)$. Let $x \in J^{cec}$. Then $f(x) \in \prec f(f^{-1}(J)) \succ$ and so $f(x) \leq f(t_1) \oplus \cdots \oplus f(t_k)$, where $t_i \in f^{-1}(J)$, for any $1 \leq i \leq k$. It results that $f(x) \leq a_1 \oplus \cdots \oplus a_k$, where $f(t_i) = a_i \in J$, for any $1 \leq i \leq k$ and so $f(x) \in \prec J \succ = J$. It means that $x \in J^c$. Therefore, $J^c = J^{cec}$.
(iv) By (ii), $I^{ece} \subseteq I^e$. Let $y \in I^e$. Then there exist $a_1, \cdots, a_k \in I$ such that $y \leq f(a_1) \oplus \cdots \oplus f(a_k)$. Since $a_i \in I$, by (i), we have $a_i \in I^{ec}$. It means that $y \in I^{ece}$ and so $I^e \subseteq I^{ece}$.
(v) The proof is routine.

In Theorem 5.1, it is not necessary that $I = I^{ec}$ or $J^{ce} = J$, where I is an ideal of M and J be an ideal of N.

Example 5.1 (i) *In example 3.1, let $f : M \longrightarrow M$ be zero homomorphism. Consider $I = \{0\}$ that is an ideal of M. We have $I^e = \prec f(I) \succ = \{0\}$ and $I^{ec} = M$. Then $I \neq I^{ec}$.*
(ii) *In Example 3.1, let $f : M \longrightarrow M$ be defined by $f(0) = f(1) = 0$ and $f(2) = f(3) = 3$. It is easy to see that f is an MV-homomorphism. Consider $J = \{0\}$ that is an ideal of M. We have $J^c = \{0, 1\}$ and $J^{ce} = \prec \{0, 1\} \succ = \{x \in M : x \leq w_1 \oplus \cdots \oplus w_n,$ for some $w_1, \cdots, w_n \in \{0, 1\}\} = \{0, 1\}$. Hence $J^{ce} \neq J$*

Definition 5.1 *[10] Let I be an ideal of MV-algebra M. Then we set $rad(I) = \bigcap_{I \subseteq m} m$, where m is any maximal ideal of M. Moreover, if there is not any maximal ideal of M containing I, then we let $rad(I) = M$.*

Notation: By Corollary 2.1, any proper ideal of a PMV-algebra is contained in a maximal ideal (note that every PMV-algebra is an MV-algebra).

Theorem 5.2 *Let M, N be MV-algebras, I_1, I_2, I be ideals of M, J_1, J_2, J be ideals of N and $f : M \longrightarrow N$ be an MV-homomorphism. Then*
(i) $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$,
(ii) $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$,
(iii) $(I_1 \oplus I_2)^e \subseteq (f(I_1) \oplus f(I_2))^e$, *where* $I_1 \oplus I_2 = \{a \oplus b : a \in I_1,\ b \in I_2\}$,
(iv) $(rad(I))^e \subseteq rad(I^e)$,
(v) $rad(J^c) \subseteq (rad(J))^c$.

Proof. (i) Let $y \in (I_1 \cap I_2)^e$. Then by Lemma 2.2, there exist $a_1, \cdots, a_k \in I_1 \cap I_2$ such that $y \leq f(a_1) \oplus \cdots \oplus f(a_k)$. Since $a_i \in I_1$ and $a_i \in I_2$, we have $f(a_i) \in f(I_1)$ and $f(a_i) \in f(I_2)$, for any $1 \leq i \leq n$. It results that $y \in I_1^e \cap I_2^e$.
(ii) The proof is routine.
(iii) Let $y \in (I_1 \oplus I_2)^e$. Then by Lemma 2.2, there exist $a_i \oplus b_i \in I_1 \oplus I_2$, for any $1 \leq i \leq n$ such that $y \leq f(a_1 \oplus b_1) \oplus \cdots \oplus f(a_n \oplus b_n) = f(a_1) \oplus f(b_1) \oplus \cdots \oplus f(a_n) \oplus f(b_n)$. It results that $y \in (f(I_1) \oplus f(I_2))^e$.
(iv) Let $y \in (rad(I))^e = \prec f(\bigcap_{I \subseteq K} K \succ$, where K is every maximal ideal of M. Then there exist $a_1, \cdots, a_k \in \bigcap_{I \subseteq K} K$ such that $y \leq f(a_1) \oplus \cdots \oplus f(a_n)$. We must show that $y \in \bigcap_{\prec f(I) \succ \subseteq L} L$, where L is any maximal ideal of N containing $\prec f(I) \succ$. We have $a_i \in K$, for any maximal ideal of M containing I. Then $f(a_i) \in f(K) \subseteq \prec f(K) \succ$. Let $\prec f(K) \succ \neq N$. Then by above Notation, $f(a_i) \in n$, where L is a maximal ideal of N containing $\prec f(K) \succ$ (if $\prec f(K) \succ = N$, then there is no maximal ideal of N containing $\prec f(K) \succ$ and so by definition of 5.1, we consider $L = N$). On the other hand, $I \subseteq K$ implies that $\prec f(I) \succ \subseteq \prec f(K) \succ \subseteq L$. It results that $f(a_i) \in \bigcap_{\prec f(I) \succ \subseteq L} L = rad(I^e)$.
(v) Let $x \in (rad(J))^c = f^{-1}(\bigcap_{J \subseteq L} L)$, where L is any maximal ideal of N. Then $f(x) \in \bigcap_{J \subseteq L} L \subseteq L$ and so $x \in f^{-1}(L) = L^c$. It results that $x \in \bigcap_{J \subseteq L} L^c = \bigcap_{J^c \subseteq L^c} L^c$ and so by Proposition 2.1, $x \in rad(J^c)$. Hence, $(rad(J))^c \subseteq rad(J^c)$.

Lemma 5.1 *Let A be a PMV-algebra. Then $\sum_{i \in I} A = A \oplus A \oplus \cdots \oplus A$ is a PMV-algebra.*

Proof. We define $\{a_i\}_{i=1}^n \oplus \{b_i\}_{i=1}^n = \{a_i \oplus b_i\}_{i=1}^n$, $\{a_i\}_{i=1}^n . \{b_i\}_{i=1}^n = \{a_i . b_i\}_{i=1}^n$ and $(\{a_i\}_{i=1}^n)' = \{a_i'\}_{i=1}^n$, for every $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \in \sum_{i=1}^n A$. It is easy to show that $(\sum_{i=1}^n A, \oplus, ', ., \{0\})$ is a PMV-algebra.

Theorem 5.3 *Let M be an MV-algebra and $I_1, \cdots, I_n$ be ideals of M. Then there exists an MV-homomorphism $\Phi : M \longrightarrow \frac{M}{I_1} \bigoplus \cdots \bigoplus \frac{M}{I_n}$ such that*
(i) *Φ is an MV-monomorphism if and only if $\bigcap_{i=1}^n I_i = \{0\}$,*
(ii) *if Φ is onto, then $\prec I_i \bigoplus I_j \succ = M$, for any $1 \leq i, j \leq n$, where $I_i \bigoplus I_j = \{\alpha \oplus \beta : \alpha \in I_i, \beta \in I_j\}$.*

(iii) if $y = (\frac{x_1}{I_1}, \cdots, \frac{x_n}{I_n}) \in \sum_{i=1}^{n} \frac{M}{I_i}$ *implies that* $x_i \ominus \bigwedge_{i=1}^{n} x_i \in I_i$*, then* ϕ *is onto.*

Proof. By Lemma 5.1, $\sum_{i=1}^{n} \frac{M}{I_i}$ is an MV-algebra (note that every PMV-algebra is an MV-algebra). We define $\Phi(a) = (\frac{a}{I_1}, \cdots, \frac{a}{I_n})$, for any $a \in M$. It is clear that $\Phi(0) = 0$. It is easy to show that $\Phi(a \oplus b) = \Phi(a) \oplus \Phi(b)$, for any $a, b \in M$. We have $\Phi(a') = (\frac{a'}{I_1}, \cdots, \frac{a'}{I_n}) = (\phi(a))'$. Hence, ϕ is an MV-homomorphism.
(i) Let ϕ be an MV-monomorphism. Then by Lemma 2.3(iv), $Ker(\phi) = \{0\}$. If $a \in \bigcap_{i=1}^{n} I_i$, then $a \in I_i$ and so $d(a, 0) = a \in I_i$, for any $1 \leq i \leq n$. It means that $\frac{a}{I_i} = \frac{0}{I_i}$ and so $\Phi(a) = (\frac{a}{I_1}, \cdots, \frac{a}{I_n}) = (\frac{0}{I_1}, \cdots, \frac{0}{I_n}) = 0$. Hence, $a \in Ker(\phi) = \{0\}$. It results that $\bigcap_{i=1}^{n} I_i = \{0\}$. Similarly, if $\bigcap_{i=1}^{n} I_i = \{0\}$, then $Ker(\phi) = \{0\}$ and so ϕ is an MV-monomorphism.
(ii) Let ϕ be an MV-epimorphism. We show that $\prec I_1 \oplus I_2 \succ = M$. Since $0, 1 \in M$, we have $(\frac{1}{I_1}, \frac{0}{I_2} \cdots, \frac{0}{I_n}) \in \sum_{i=1}^{n} \frac{M}{I_i}$. Since ϕ is onto, there exists $x \in M$ such that
$\phi(x) = (\frac{x}{I_1}, \cdots, \frac{x}{I_n}) = (\frac{1}{I_1}, \frac{0}{I_2} \cdots, \frac{0}{I_n})$. It results that $x' = d(1, x) \in I_1$, $x = d(0, x) \in I_2$ and so $1 = x' \oplus x \in I_1 \bigoplus I_2$. It means that $\prec I_1 \oplus I_2 \succ = M$. Similarly, we can show that $\prec I_i \oplus I_j \succ = M$, for any $1 \leq i, j \leq n$.
(iii) Let $y = (\frac{x_1}{I_1}, \cdots, \frac{x_n}{I_n}) \in \sum_{i=1}^{n} \frac{M}{I_i}$. Then we consider $x = \bigwedge_{i=1}^{n} x_i$. Since $x \leq x_i \in I_i$, we have $x \in I_i$. Since $d(x, x_i) = (x \ominus x_i) \oplus (x_i \ominus x) = 0 \oplus (x_i \ominus x) \in I_i$, we have $\frac{x}{I_i} = \frac{x_i}{I_i}$, for any $1 \leq i \leq n$. It means that $\phi(x) = (\frac{x}{I_1}, \cdots, \frac{x}{I_n}) = (\frac{x_1}{I_1}, \cdots, \frac{x_n}{I_n}) = y$. Therefore, ϕ is an MV-epimorphism.

6 Conclusion

We obtained some new results in ideals theory and opened new fields to anyone that is interested to studying and development of ideals in MV-algebras.

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