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Application of Laguerre Polynomials For Solving Infinite Boundary Integro-Differential Equations

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Abstract

In this study, an efficient method is presented for solving infinite boundary integro-differential equations (IBI-DE) of the second kind with degenerate kernel in terms of Laguerre polynomials. Properties of these polynomials and operational matrix of integration are first presented. These properties are then used to transform the integral equation to a matrix equation which corresponds to a linear system of algebraic equations with unknown Laguerre coefficients. We prove the convergence analysis of method applied to the solution integro-differential equations. Finally, numerical examples illustrate the efficiency and accuracy of the method.

Keywords : Infinite boundary integro-differential equations; Laguerre polynomials; Operational matrix; Linear sets.

1 Introduction

The integral equation is called infinite boundary integro-differential equation if one or both of it's limits are improper. Many problems of theoretical physics, electromagnetics, scattering problems, boundary integral equations [8, 10, 5, 12] leads to (IBI-DE) of the second kind of the form

$$\begin{cases} u'(x) - (\vartheta\, u)(x) = f(x), \\ u(0) = u_0, \end{cases} \qquad (1.1)$$

where

$$(\vartheta\, u)(x) := \lambda \int_0^\infty e^{-t} k(x,t) u(t) dt,\ x \in \mathbb{R}_+. \qquad (1.2)$$

In Eq. (1.1), λ is parameter and $f(x)$ is continuous function and the kernel $k(x,t)$ might has singularity in the region $D = \{(x,t) : 0 \le x, t < \infty\}$, and $u(x)$ is the unknown function which to be determined. Many researchers have developed the approximate method to solve infinite boundary integral equation using Galerkin and Collocation methods with Laguerre and Hermite polynomials as a bases function [2, 6, 3]. Moreover there are several numerical methods for solving Eq. (1.1) when the limit of integration is finite. For example sine-cosine wavelets [11], A sinc collocation method [4], CAS wavelet operational matrix of integration [1], An application of walsh functions [7], Legendre polynomial and block-pulse func-

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tions [9]. However, method of solution for Eq. (1.1) is too rear in the literature. Our aim in this paper is to obtain the analytical-numerical solutions by using the Laguerre polynomials for the (IBI-DE). The layout of this paper is organized as follows: In Section 2, we introduce some necessary definitions and give some relevant properties of the Laguerre polynomials and approximate the function $f(x)$ and also the kernel function $k(x,t)$ by these polynomials and related operational matrices. Section 3, is devoted to present a computational method for solving Eq. (1.1) utilizing Laguerre polynomials and approximate the unknown function $u(x)$. Convergence theorem of the Laguerre polynomials bases is presented in Section 4. Section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Finally, we conclude the article in Section 6.

2 Preliminary Notes

In this section, for the reader's convenience, we present some necessary definitions and properties of the Laguerre polynomials which are used further in this article.

Definition 2.1 *The Laguerre polynomials of nth-degree are defined on the interval $[0, \infty)$ as [2, 6]:*

$$L_n(x) = \frac{1}{n!} e^x \partial_x^n (x^n e^{-x}), \quad n = 0, 1, \dots. \quad (2.3)$$

They satisfy the equation

$$\partial_x(x e^{-x} \partial_x L_n(x)) + n e^{-x} L_n(x) = 0, \quad x \in \mathbb{R}_+. \quad (2.4)$$

Laguerre polynomials are important in approximation theory and numerical analysis and in some quadrature rules based on this polynomials such as Gauss-Laguerre rule that appears in the theory of numerical integration. Laguerre polynomials are orthogonal with respect to the weight function $w(x) = e^{-x}$ and the following properties:

(i) $\parallel L_n(x) \parallel = 1, \quad n = 0, 1, 2, \dots,$

(ii) $\int_0^x L_n(t)\, dt = L_n(x) - L_{n+1}(x),$

(iii) $\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = 0, \quad m \neq n,$

(iv) $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x), \quad n = 1, 2, \dots.$

2.1 Function Approximation

An arbitrary function $f(x) \in L_w^2[0, \infty)$ may be expanded in terms of Laguerre polynomials as follows:

$$f(x) = \sum_{i=0}^\infty f_i L_i(x). \quad (2.5)$$

If the infinite series in (2.5) is truncated up to n term, then (2.5) can be written as

$$f(x) \simeq \sum_{i=0}^n f_i L_i(x) = F^T L_x, \quad (2.6)$$

where $f_0, f_1, f_2, \dots, f_n$ are arbitrary coefficients, F and L_x are $(n+1) \times 1$ vectors given by $F = [f_0, \ f_1, \ f_2, \dots, \ f_n]^T$ and $L_x = [L_0(x), \ L_1(x), \ L_2(x), \dots, \ L_n(x)]^T$. But F^T can be obtained by

$$F^T < L_x, L_x > = < f, L_x >, \quad (2.7)$$

where

$$< f, L_x > = \int_0^\infty w(x) f(x) L_x^T dx, \quad (2.8)$$

where $w(x)$ is the weight function e^{-x}, and $< L_x, L_x >$ is a $(n+1) \times (n+1)$ matrix which is said the dual matrix of L_x denoted by Q and will be introduced in the following. Therefore

$$Q = < L_x, L_x > = \int_0^\infty e^{-x} L_x L_x^T dx = I, \quad (2.9)$$

where I is an identity matrix, and then

$$F^T = \int_0^\infty w(x) f(x) L_x^T dx. \quad (2.10)$$

Similarly, a function of two variables, $k(x,t) \in L_w^2([0,\infty) \times [0,\infty))$ may be approximated as follows:

$$k(x,t) \simeq \sum_{i=0}^n \sum_{j=0}^n k_{i,j} L_i(x) L_j(t) = L_x^T K L_t, \quad (2.11)$$

where

$$K = \begin{bmatrix} k_{0,0} & k_{0,1} & k_{0,2} & \cdots & k_{0,n} \\ k_{1,0} & k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n,0} & k_{n,1} & k_{n,2} & \cdots & k_{n,n} \end{bmatrix}. \quad (2.12)$$

K is a $(n + 1) \times (n + 1)$ matrix and the $k_{i,j}$ for $i, j = 0, 1, \ldots, n$ elements are given by

$$k_{i,j} = \frac{< L_i(x), < k(x,t), L_j(t) >>}{< L_i(x), L_i(x) >< L_j(t), L_j(t) >}, \quad (2.13)$$

where $< ., . >$ denotes the inner product. Due to (2.9), we can see

$$K = < L_x, < k(x,t), L_t >> . \quad (2.14)$$

2.2 Operational matrix of integration

Theorem 2.1 *The integration of the Laguerre vector L_x defined in Eq. (2.6) can be expressed as:*

$$\int_0^x L_t dt \simeq PL_x, \quad (2.15)$$

where P is the $(n+1) \times (n+1)$ operational matrix for integration as follows:

$$P = \begin{bmatrix} \Omega(0,0) & \Omega(0,1) & \cdots & \Omega(0,n) \\ \Omega(1,0) & \Omega(1,1) & \cdots & \Omega(1,n) \\ \vdots & \vdots & \vdots & \vdots \\ \Omega(i,0) & \Omega(i,1) & \cdots & \Omega(i,n) \\ \vdots & \vdots & \vdots & \vdots \\ \Omega(n,0) & \Omega(n,1) & \cdots & \Omega(n,n) \end{bmatrix}, \quad (2.16)$$

where

$$\Omega(i,j) =$$

$$\sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{k+r} i! \, j! \, \Gamma(k+r+2)}{(i-k)! \, (j-r)! \, (k+1)! \, k! \, (r!)^2}. \quad (2.17)$$

Proof. The analytic form of the Laguerre polynomials $L_i(x)$ of degree i is given as follows:

$$L_i(x) = \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)! \, (k!)^2} x^k, \quad (2.18)$$

where $L_0(x) = 1$. Using Eq. (2.18), and since the integration is a linear operation, we get the following:

$$\int_0^x L_i(t) dt = \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)! \, (k!)^2} \int_0^x t^k dt$$

$$= \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)! \, (k+1)! \, (k!)} x^{k+1}. \quad (2.19)$$

Now, by approximating x^{k+1} by the $n + 1$ terms of the Laguerre series, we have the following:

$$x^{k+1} = \sum_{j=0}^{n} b_j L_j(x), \quad (2.20)$$

where b_j is given from Eq. (2.10) with $f(x) = x^{k+1}$, that is,

$$b_j = \sum_{r=0}^{j} \frac{(-1)^r j! \, \Gamma(k+r+2)}{(j-r)! \, (r!)^2}, \quad j = 0, \ldots, n. \quad (2.21)$$

In virtue of Eqs. (2.19) and (2.20), we get the following:

$$\int_0^x L_i(t) dt = \sum_{j=0}^{n} \Omega(i,j) L_j(x), \quad (2.22)$$

where

$$\Omega(i,j) = \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{k+r} i! \, j! \, \Gamma(k+r+2)}{(i-k)! \, (j-r)! \, (k+1)! \, k! \, (r!)^2}. \quad (2.23)$$

Accordingly, Eq. (2.22) can be written in a vector form as follows:

$$\int_0^x L_i(t) dt \simeq [\Omega(i,0), \Omega(i,1), ..., \Omega(i,n)] L_x. \quad (2.24)$$

Eq. (2.24) leads to the desired result.

3 Infinite Boundary Integro Differential Equations

In this section, we consider (IBI-DE) of the second kind in Eq. (1.1) and approximate to solution by means of finite Laguerre series defined in Eq. (2.6). The aim is to find Laguerre coefficients, we approximate functions $f(x)$, $k(x,t)$ and $u'(x)$ with respect to Laguerre polynomials as mentioned in the previous section as follows:

$$f(x) \simeq F^T L_x, \quad u'(x) \simeq C'^T L_x,$$

$$u(0) \simeq C_0^T L_x, \quad k(x,t) \simeq L_x^T K L_t, \quad (3.25)$$

where L_x is defined in Eq. (2.6), the vectors F^T, C'^T, and matrix K are Laguerre coefficients of $f(x)$, $u'(x)$, and $k(x,t)$, respectively. Then

$$u(x) = \int_0^x u'(t) dt + u(0)$$

$$\simeq \int_0^x C'^T L_t dt + C_0^T L_x$$

$$\simeq C'^T P L_x + C_0^T L_x = (C'^T P + C_0^T) L_x, \quad (3.26)$$

where P is a $(n + 1) \times (n + 1)$ matrix given in (2.16). With substituting the approximations Eqs. (3.25) and (3.26) into equation (1.1), we have:

$$L_x^T C' =$$

$$L_x^T F + \lambda \int_0^\infty e^{-t} L_x^T K L_t L_t^T (p^T C' + C_0) dt$$

$$= L_x^T F + \lambda L_x^T K \int_0^\infty e^{-t} L_t L_t^T dt (p^T C' + C_0)$$

$$= L_x^T F + \lambda L_x^T K (p^T C' + C_0), \quad (3.27)$$

then we have the following linear system:

$$(I - \lambda K P^T) C' = F + \lambda K C_0. \quad (3.28)$$

Eq. (3.28) is a linear system of algebraic equations that can be easily solved by direct or iterative methods. In equation (3.28), if $D(\lambda) = |I - \lambda K P^T| \neq 0$ we get

$$C' = (I - \lambda K P^T)^{-1} (F + \lambda K C_0), \quad \lambda \neq 0. \quad (3.29)$$

We can find the vector C', so

$$C^T = C'^T P + C_0^T \implies u(x) \simeq C^T L_x. \quad (3.30)$$

Remark 3.1 $D(\lambda)$ *is a polynomial in λ of degree at most $n+1$, $D(\lambda)$ is not identically zero, since when $\lambda=0$, $D(\lambda) = 1$.*

4   Convergence Analysis

In this section, we discuss the theoretical analysis of convergence of our approach. We assume throughout this paper, all functions are continuously differential finitely or infinitely.

Theorem 4.1 *Let $u(x) \in L_w^2[0, \infty)$(a Hilbert space), $u_n(x) = \sum_{i=0}^n c_i L_i(x)$ be the best approximation polynomials of $u(x)$, then series solution $u_n(x)$ converges towards $u(x)$.*

Proof. Suppose

$$u(x) \simeq u_n(x) = \sum_{i=0}^n c_i L_i(x), \quad (4.31)$$

where

$$c_i = \int_0^\infty e^{-x} u(x) L_i(x) dx = <u(x), L_i(x)> . \quad (4.32)$$

Then Eq. (4.31) can be written as

$$u(x) \simeq \sum_{i=0}^n <u(x), L_i(x)> L_i(x). \quad (4.33)$$

Now let

$$\alpha_i = <u(x), L_i(x)> . \quad (4.34)$$

Define the sequence of partial sums $\{S_m\}$ of $(\alpha_i L_i(x))$, let S_m and S_n be arbitrary partial sums with $m \geq n$. We are going to prove that $\{S_m\}$ is a Cauchy sequence in Hilbert space. Let

$$S_m = \sum_{i=1}^m \alpha_i L_i(x). \quad (4.35)$$

Now

$$<u(x), S_m> = <u(x), \sum_{i=1}^m \alpha_i L_i(x)>$$

$$= \sum_{i=1}^m \bar{\alpha}_i <u(x), L_i(x)> = \sum_{i=1}^m |\alpha_i|^2. \quad (4.36)$$

We must show that

$$\|S_m - S_n\|^2 = \sum_{i=n+1}^m |\alpha_i|^2 \quad for \quad m > n, \quad (4.37)$$

so, for showing this we can write

$$\|S_m - S_n\|^2 = \|\sum_{i=n+1}^m \alpha_i L_i(x)\|^2$$

$$= <\sum_{i=n+1}^m \alpha_i L_i(x), \sum_{j=n+1}^m \alpha_j L_j(x)>$$

$$= \sum_{i=n+1}^m \sum_{j=n+1}^m \alpha_i \bar{\alpha}_j <L_i(x), L_j(x)>$$

$$= \sum_{i=n+1}^m \alpha_i \bar{\alpha}_i = \sum_{i=n+1}^m |\alpha_i|^2. \quad (4.38)$$

Table 1: Absolute error for Example 5.2

i	x_i	$n = 10$	$n = 15$
0	0.0	$4.8830e - 004$	$1.5279e - 005$
1	0.1	$4.7608e - 005$	$1.8500e - 006$
2	0.2	$1.6432e - 004$	$6.7000e - 006$
3	0.3	$2.2533e - 004$	$5.3077e - 006$
4	0.4	$1.9443e - 004$	$1.5373e - 006$
5	0.5	$1.1518e - 004$	$2.3194e - 006$
6	0.6	$1.8679e - 005$	$5.0754e - 006$
7	0.7	$7.3937e - 005$	$6.2874e - 006$
8	0.8	$1.4936e - 004$	$5.9841e - 006$
9	0.9	$2.0031e - 004$	$4.4645e - 006$
10	1.0	$2.2408e - 004$	$2.1536e - 006$

From Bessel's inequality, we have $\sum_{i=n+1}^{m}|\alpha_i|^2$ is convergent and hence

$$\|S_m - S_n\|^2 \to 0 \quad as \quad m, n \to \infty,$$

$$(i.e) \quad \|S_m - S_n\| \to 0. \qquad (4.39)$$

And $\{S_m\}$ is a Cauchy sequence and it converges to say '*s*'. We assert that $u(x) = s$.
Infact,

$$< s - u(x), L_i(x) >$$
$$= < s, L_i(x) > - < u(x), L_i(x) >$$
$$= < \lim_{m \to \infty} S_m, L_i(x) > -\alpha_i$$
$$= \lim_{m \to \infty} < S_m, L_i(x) > -\alpha_i$$
$$= \alpha_i - \alpha_i = 0. \qquad (4.40)$$

Hence $u(x) = s$, and $u_n(x)$ converges to $u(x)$, which completes the proof. As the convergence has been proved, consistency and stability are ensured automatically.

Corollary 4.1 *If the exact solution of the integral equation (1.1) is a polynomial of degree n, then the proposed method with Laguerre polynomials up to n degree will obtain in the real solution.*

5 Numerical Examples

In this section, we applied the method presented in this paper for solving integral equation (1.1) and solved two examples. For each example we find the approximate solutions using different degree of Laguerre polynomials. The computations associated with the examples were performed in a personal computer using Matlab.

Example 5.1 *Consider the following infinite boundary integro-differential equation of second kind (Constructed):*

$$u'(x) = 3x^2 + sin(x) - \frac{1}{2}(4 + cos(x))$$

$$+ \int_0^\infty e^{-t} sin(x - t)u(t)dt, \quad u(0) = 1. \quad (5.41)$$

Exact solution of this problem is $u(x) = x^3 - 2x + 1$. If we apply the technique described in the section 3 with $n = 3$, then the approximate solution can be written as follows:

$$u(x) \simeq u_n(x) = \sum_{i=0}^{3} c_i L_i(x) = C^T L_x, \quad (5.42)$$

where

$$C = [c_0, \ c_1, \ c_2, \ c_3]^T,$$

$$L_x = [L_0(x), \ L_1(x), \ L_2(x), \ L_3(x)]^T. \quad (5.43)$$

And hence, from Eqs. (2.10), (2.16) and (2.12), we find the matrices

$$F = \begin{bmatrix} 17/4 \\ -49/4 \\ 45/8 \\ -1/4 \end{bmatrix}, \ P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 1/4 & 1/4 & 1/8 \\ -1/4 & 0 & 1/8 & 1/8 \\ -1/4 & -1/8 & 0 & 1/16 \\ -1/8 & -1/8 & -1/16 & 0 \end{bmatrix}. \quad (5.44)$$

Next, we substitute these matrices into equation (3.29) and then simplify to obtain

$$\begin{bmatrix} c_0' \\ c_1' \\ c_2' \\ c_3' \end{bmatrix} =$$

$$\begin{bmatrix} 309/389 & -6/3389 & 34/389 & 40/389 \\ -68/389 & 345/3389 & -10/389 & 34/389 \\ -28/389 & -41/3389 & 362/389 & 14/389 \\ 6/389 & -19/3389 & -22/389 & 386/389 \end{bmatrix}.$$

$$\begin{bmatrix} 17/4 \\ -25/2 \\ 43/8 \\ -3/8 \end{bmatrix}. \qquad (5.45)$$

By solving the linear system (5.45)), we have the following:

$$c_0' = 4, \ c_1' = -12, \ c_2' = 6, \ c_3' = 0. \qquad (5.46)$$

By substituting the obtained coefficients in (3.30) the solution of (5.41) becomes

$$u(x) \simeq 5L_0(x) - 16L_1(x) + 18L_2(x) - 6L_3(x) \quad (5.47)$$

or

$$u(x) \simeq x^3 - 2x + 1, \qquad (5.48)$$

which is the exact solution. Also, if we choose $n \geq 4$, we get the same approximate solution as obtained in equation (5.48). Numerical results will not be presented since the exact solution is obtained.

Example 5.2 *As the second example, consider the following infinite boundary integro-differential equation (Constructed):*

$$u'(x) = \frac{1}{5}(-5 + 2sin(x) - cos(x))e^{-x}$$

$$+ \int_0^\infty e^{-t-x}sin(t-x)u(t)dt, \quad u(0) = 1. \quad (5.49)$$

With the exact solution $u(x) = e^{-x}$.

Table 1 shows the absolute values of error $|e|= |u_{exact.}(x) - u_{app.}(x)|$ for $n = 10$, and $n = 15$ using the present method in equally divided interval $[0, 1]$.

6   Conclusion

Finding exact solutions for infinite boundary integro-differential equations of the second kind is often difficult, and therefore approximating these solutions is very important. In this study we develop an efficient and accurate method for solving (IBI-DE) of the second kind. The properties of Laguerre polynomials are used to reduce the problem into solution of a system of algebraic equations whose matrix is sparse. Error analysis shows that the approximation becomes more accurate when n is increased. So for better results, using a large n is recommended.

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