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A Numerical Method For Solving Physiology Problems By Shifted Chebyshev Operational Matrix

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Abstract

In this study, a new numerical solution of singular nonlinear differential equations, stemming from biology and physiology problems, is proposed. The methodology is primarily based on the shifted Chebyshev polynomials operational matrix of derivative and collocation. Furthermore, the convergence analysis on the proposed method is carried out. To assess the accuracy and analysis of performance of the method, five numerical problems, based on the singular nonlinear differential equations, on different subjects, such as the human head, Oxygen diffusion in a spherical cell and Bessel differential equation, were solved. The numerical results were compared with other existed methods in tables for verification and further discussions.

Keywords : Differential equations; Shifted Chebyshev polynomials; Operational matrix of derivative; Convergence analysis; Physiology problems.

1 Introduction

The differential equations arise from various applications in fluid mechanics, biology, physics and engineering [1]-[16]. Such equations also appear in electromagnetic and electrodynamic, elasticity and dynamic contact, heat and mass transfer, fluid mechanic, acoustic, chemical and electrochemical processes, molecular physics, population, medicine and in many other fields [3]-[14]. For numerical solution of the differential equations, there are some well-known numerical methods [11]-[15].

In this paper, we consider the singular problems of the type

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x), \quad 0 < x \le 1, \tag{1.1}$$

subject to the conditions

$$\begin{cases} \alpha_1 y(0) + \beta_1 y'(0) = \gamma_1, \\ \alpha_2 y(1) + \beta_2 y'(1) = \gamma_2, \end{cases} \tag{1.2}$$

where $x = 0$ is a singular point in $p(x)$, also $p(x), q(x)$ and $g(x)$ are continuous functions on $(0, 1]$ and the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are real constants.

The Chebyshev polynomials have been in existence for over a hundred years and they have been used for solving many different problems [2].

In this paper, we propose a suitable way to approximate the solution of singular nonlinear differential equations with initial or boundary value problems on the interval $(0, L)$, by use of shifted Chebyshev collocation method based on the shifted Chebyshev operational matrix ot

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derivative. Also the convergence analysis of the proposed method is discussed in this paper.

This paper is organized as follows: In section 2, the shifted Chebyshev polynomials with their properties are introduced. In section 3, we derive an approximate formula for derivatives using shifted Chebyshev polynomials and estimate proposed formula. In section 4, we give the error analysis for the method. In section 5, the proposed method is applied to several examples. Finally Section 6 concludes the paper with some remarks.

2 Properties of Shifted Chebyshev Polynomials

The well known Chebyshev polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula [2, 4]:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 1, 2, \cdots \tag{2.3}$$

where $T_0(x) = 1$ and $T_1(x) = x$. The analytic form of the Chebyshev polynomial of degree n is given by :

$$T_n(x) = \frac{n}{2}\sum_{r=0}^{[\frac{n}{2}]}(-1)^r \frac{(n-r-1)!}{r!\,(n-2r)!}(2x)^{n-2r}. \tag{2.4}$$

In order to apply the Chebyshev polynomials in the interval $[0,1]$, we defined the so called shifted Chebyshev polynomials by introducing the change of variable $t = \frac{2x}{L} - 1$. Let the shifted Chebyshev polynomials $T_i(\frac{2x}{L} - 1)$ be denoted by $T_{L,i}(x)$. Then $T_{L,i}(x)$ can be generated by using the following recurrence relation:

$$T_{L,i+1}(x) = 2(\frac{2x}{L} - 1)T_{L,i}(x) - T_{L,i-1}(x), \quad i = 1, 2, \cdots \tag{2.5}$$

where $T_{L,0}(x) = 1$ and $T_{L,1}(x) = \frac{2x}{L} - 1$. The analytic from of the shifted Chebyshev polynomials $T_{L,i}(x)$ of degree i is given by

$$T_{L,i}(x) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!\,2^{2k}}{(i-k)!\,(2k)!\,L^k}x^k, \tag{2.6}$$

where $T_{L,i}(0) = (-1)^i$ and $T_{L,i}(L) = 1$. The orthogonality condition is

$$\int_0^L T_{L,k}(x)T_{L,j}(x)W_L(x)\,dx = h_k\delta_{jk}, \tag{2.7}$$

where $W_L(x) = \frac{1}{\sqrt{Lx-x^2}}$ and $h_k = \frac{\epsilon_k}{2}\pi$, with $\epsilon_0 = 2$, $\epsilon_i = 1, i \geq 1$. Any function $u(x)$, square integrable in $(0, L)$, may be expressed in terms of shifted Chebyshev polynomials as

$$u(x) = \Sigma_{j=0}^{\infty} a_j T_{L,j}(x), \tag{2.8}$$

where the coefficients a_j are given by

$$a_j = \frac{1}{h_j}\int_0^L u(x)T_{L,j}(x)W_L(x)\,dx, \; j = 0, 1, 2, \cdots \tag{2.9}$$

In practice, only the first $(N+1)$-terms shifted Chebyshev polynomials are considered. Hence, if we write

$$u(x) \simeq \Sigma_{j=0}^{N} c_j T_{L,j}(x) = C^T\phi(x), \tag{2.10}$$

where shifted Chebyshev polynomials coefficient vector C and the shifted Chebyshev polynomials vector $\phi(x)$ are given by

$$C^T = [c_0, c_1, \ldots, c_N], \tag{2.11}$$

$$\phi(x) = [T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,N}(x)]^T, \tag{2.12}$$

then the derivative of the vector $\phi(x)$ can be expressed by [4]

$$\frac{d\phi(x)}{dx} = D^1\phi(x), \tag{2.13}$$

where D^1 is the $(N+1) \times (N+1)$ operational matrix of derivative given by

$$D^1 = (d_{ij}) = \begin{cases} \frac{4i}{\epsilon_j L}, & \text{i=j+k,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

If N is odd $k = 1, 3, 5, \cdots, N$, if N is even $k = 1, 3, 5, \cdots, N-1$, for example for even N, we have D^1 as follows:

$$\frac{2}{L}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 6 & 0 & \cdots & 0 & 0 \\ 0 & 8 & 0 & 8 & \cdots & 0 & 0 \\ 5 & 0 & 10 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ N-1 & 0 & 2N-2 & 0 & \cdots & 0 & 0 \\ 0 & 2N & 0 & 2N & \cdots & 2N & 0 \end{pmatrix} \tag{2.15}$$

By using Eq. (2.3), it is clear that

$$\frac{d^n\phi(x)}{dx^n} = (D^1)^n\phi(x), \tag{2.16}$$

where $n \in N$ and the superscript, in $D^{(1)}$, denotes matrix power thus

$$D^{(n)} = (D^{(1)})^n, \qquad n = 1, 2, \cdots \tag{2.17}$$

3 Implementation of Shifted Chebyshev Polynomials Method on Physiology Problems

In this section we use shifted Chebyshev vector and its operational matrix of derivative to solve nonlinear singular boundary value problem of the form Eqs. (2)-(3), we approximate $y(x)$ and $g(x)$ by Chebyshev polynomials as

$$y(x) \simeq C^T\phi(x), \tag{3.18}$$

$$g(x) \simeq G^T\phi(x), \tag{3.19}$$

we have

$$y'(x) \simeq C^T D^1\phi(x), \tag{3.20}$$

$$y''(x) \simeq C^T (D^1)^2\phi(x), \tag{3.21}$$

by employing Eqs. (3.18)-(2.13) in Eq. (2) we have

$$\begin{aligned} C^T(D')^2\phi(x) + p(x)C^T D'\phi(x) + \\ q(x)(C^T\phi(x)) = G^T\phi(x). \end{aligned} \tag{3.22}$$

Also by using Eqs,(3),(3.18) and (3.20) we have

$$\alpha_1 C^T\phi(0) + \beta_1 C^T D^1\phi(0) = \gamma 1, \tag{3.23}$$

$$\alpha_2 C^T\phi(1) + \beta_2 C^T D^1\phi(1) = \gamma 2. \tag{3.24}$$

Equations (3.23) and (3.24) give 2 linear equations. Since the total unknowns for vector C in Eq. (3.22) are $N+1$, we collocate that in $N-1$ roots of Chebyshev polynomials as:

$$x_p = \cos\frac{p\pi}{2N}, \qquad p = 1, \ldots, N-1. \tag{3.25}$$

Then we have Eq. (3.22) as following system of nonlinear equations

$$\begin{aligned} C^T(D')^2\phi(x_p) + p(x_p)C^T D'\phi(x_p) + \\ q(x_p)(C^T\phi(x_p)) = G^T\phi(x_p), \\ p = 1, 2\cdots, N-1. \end{aligned} \tag{3.26}$$

Now the resulting Eqs. (3.23), (3.24) and (3.26) generate a system of $N+1$ nonlinear equations which can be solved using Newton's iterative method. So we have the approximate solution of Eq. (2) with the initial conditions (3) by Chebyshev polynomials as:

$$y_m(x) = \sum_{j=0}^{N} c_j T_{L,j}(x) = C^T\phi(x). \tag{3.27}$$

4 Error Analysis

Theorem 4.1 *(Chebyshev truncation theorem) The error in approximating $y(x)$ by the sum of its first N terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$y_N(x) = \sum_{j=0}^{N} c_j T_{L,j}(x) = C^T\phi(x), \tag{4.28}$$

then

$$|E_T(N)| = |y(x) - y_N(x)| \leq \sum_{k=m+1}^{\infty} |c_k|, \tag{4.29}$$

for all $y(x)$, all N, and all $x \in [-1, 1]$.
Proof. See [26]. □

Error estimate are, most of the time, obtained for functional spaces weighed with the Chebyshev weight, in the $H_w^p(-1,1)$-norm, is found to satisfy

$$\|y - y_N\|_{H_w^p(-1,1)} \leq CN^{-\frac{1}{2}+2p-m}\|u\|_{H_w^m(-1,1)}. \tag{4.30}$$

for $p \geq 1$, if $y \in H_w^m(-1,1)$ for some $m \geq 1$. The constant C is independent of N. The space $H_w^p(-1,1)$ is the weighted Sobolev space of order p whose norm is defined by

$$\|y\|_{H_w^p(-1,1)} = \left(\sum_{k=0}^{p}\int_{-1}^{1}|y^k(x)|^2 w(x)\,dx\right)^{\frac{1}{2}}. \tag{4.31}$$

The functions are approximated with the collocation Chebyshev method. The first one is defined by $y_\alpha(x) = x^\alpha$, $0 \leq x < 1$, $y_\alpha(x) = 0$, $-1 \leq x < 0$, where $\alpha > 0$.

This function exhibits a singular behaviour at the center of the interval $(-1, 1)$ and belong to $H_w^m(-1,1)$ with $m < \alpha + \frac{1}{2}$. Let $e_N(x)$ the difference between the function $y_\alpha(x)$ and its interpolation polynomials $y_N(x)$, namely,

Table 1: Approximate and exact solutions for Example 5.1.

x_i	Present method N=12	Method in [20]	Method in [17]	Method in [12]
0.0	0.367517	0.367518	0.367516	0.367516
0.1	0.366362	0.366362	0.366362	0.366362
0.2	0.362894	0.362895	0.362894	0.362894
0.3	0.357098	0.357097	0.357097	0.357097
0.4	0.348948	0.348948	0.348948	0.348948
0.5	0.338412	0.338412	0.338412	0.338412
0.6	0.325444	0.325443	0.325443	0.325443
0.7	0.309986	0.309986	0.309986	0.309986
0.8	0.291971	0.291971	0.291971	0.291971
0.9	0.271371	0.271317	0.271317	0.271310
1.0	0.247928	0.247927	0.247927	0.247927

Table 2: Approximate and exact solutions for Example 5.2.

x_i	Present method N=10	Present method N=13	Method [7] M=10	Exact solution
0.1	0.683211	0.683197	0.68319682	0.68319685
0.2	0.653939	0.653927	0.65392655	0.65392646
0.3	0.606981	0.606970	0.60696936	0.60696948
0.4	0.544738	0.544727	0.54472710	0.54472718
0.5	0.470013	0.470004	0.47000366	0.47000362
0.6	0.385560	0.385663	0.38566250	0.38566248
0.7	0.294377	0.294371	0.29437106	0.29437106
0.8	0.198455	0.198451	0.19845088	0.19845093
0.9	0.099822	0.099820	0.09982033	0.09982033
1.0	2.48702×10^{-17}	-7.10152×10^{-18}	0	0

Table 3: Approximate and exact solutions for Example 5.3.

x_i	Present method N=4	Method [7] k=0, M=4	Exact solution
0.0	-1.32775×10^{-16}	-0.281	0
0.1	0.001	0.100	0.001
0.2	0.008	0.271	0.008
0.3	0.027	0.292	0.027
0.4	0.064	0.224	0.064
0.5	0.125	0.125	0.125
0.6	0.216	0.056	0.216
0.7	0.343	0.077	0.343
0.8	0.512	0.249	0.512
0.9	0.729	0.630	0.729
1.0	1	0	1

$$e_N(x) = y_\alpha(x) - y_N(x) \tag{4.32}$$

Equation(4.30), give the following estimates

$$\|e_N\| L^2_w(-1,1) \leq C_1 N^{-\frac{1}{2}-\alpha}, \tag{4.33}$$

$$\|e_N\| L^\infty(-1,1) \leq C_2 N^{-\alpha}, \tag{4.34}$$

$$\|e_N\| H^1_w(-1,1) \leq C_3 N^{\frac{1}{2}-\alpha}. \tag{4.35}$$

where C_1, C_2 and C_3 are positive constants independent of N. The continuous $L^2_w(-1,1)$ and $H^1_w(-1,1)$ norms are calculated by evaluating the value of the interpolating polynomial $u_N(x)$ and its derivative $y'_N(x)$ on the $M+1$ Gauss-Lobatto points $x_j = cos\frac{\pi j}{M}$, $j = 0, 1, \cdots M$ with M much larger than N. Then the integrals are evaluated by means of the Gauss-Lobatto based on the points x_j, $j = 0, 1, \cdots M$. In an analogous

Table 4: Approximate and exact solutions for Example 5.4.

x_i	Present method with N=11	Method [21] with M=3, K=2	Method [7] with M=10, K=0	Exact solution of $J_0(x)$
0.1	0.997501562	0.997502	0.997501562	0.997501562
0.2	0.990024972	0.990024	0.990024972	0.990024972
0.3	0.977626246	0.977625	0.977626246	0.977626246
0.4	0.960398226	0.960396	0.960398226	0.960398226
0.5	0.938469807	0.938468	0.938469807	0.938469807
0.6	0.912004863	0.912004	0.912004863	0.912004863
0.7	0.881200888	0.881200	0.881200888	0.881200888
0.8	0.846287352	0.846285	0.846287352	0.846287352
0.9	0.807523798	0.807524	0.807523798	0.807523798
1.0	0.765197686	0.765197	0.765197686	0.765197686

Table 5: Approximate and exact solutions for Example 5.5.

x_i	Present method N=7	Method [6] N=13	Method [13] N=14	Method [9] N=20
0.0	0.828483	0.828483	0.828432	0.828483
0.1	0.829706	0.829706	0.829706	0.829706
0.2	0.833375	0.833374	0.833374	0.833374
0.3	0.839489	0.839489	0.839489	0.839473
0.4	0.848053	0.848052	0.848052	0.848052
0.5	0.859065	0.859064	0.859064	0.859064
0.6	0.872528	0.872528	0.872528	0.872528
0.7	0.888445	0.888445	0.888445	0.888445
0.8	0.906819	0.906818	0.906818	0.906818
0.9	0.927951	0.927950	0.927650	0.927650
1.0	0.950946	0.950945	0.950957	0.950945

way, the $L^\infty(1,1)$-norm is calculated by taking the maximum of $e_N(x)$ on the above $M+1$ Gauss-Lobatto points. Numerical estimates of the order of the error, namely,$e_N = O(N^{-q})$.

Theorem 4.2 *Suppose* $X = C[0,1]$ *and* Y *be Banach spaces with the norm* $\|z\|= max|z(x)|, x \in X$. *Let* $N : X \to Y$ *which satisfies the Lipschitz condition*

$$\|y(x_1) - y(x_2)\|\leq \beta\|x_1 - x_2\|, \forall x_1, x_2, 0 \leq \beta < 1. \tag{4.36}$$

If we assume the $\|y_0\|< \infty$, then the sequence $s_n = C + N(s_{n-1})$, converges to the exact solution y.

Proof: See [23].

Now we have to prove that the result is true for $n = k + 1$.

Hence the result is true for all values of n. We complete the proof by showing that s_n is a Cauchy sequence on the Banach space X.

For every $m, n \in N, m \leq n$, we have
$\|s_n - s_m\|= \|(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \dots (s_{m+1} - s_m)\|$
$\leq \|s_n - s_{n-1}\|+\|s_{n-1} - s_{n-2}\|+ \dots + \|s_{m+1} - s_m\|$
$\leq \beta^{n-1}\|y_0\|+\beta^{n-2}\|y_0\|+ \dots + \beta^{m-+1}\|y_0\|+\beta^m\|y_0\|$
$\leq \|y_0\|\beta^m(1 + \beta + \beta^2 + \dots + \beta^{n-1-m})$
$\leq \|y_0\|\beta^m(\frac{1-\beta^{n-m}}{1-\beta})$.
since $0 < \beta < 1$, $1 - \beta^{n-m} < 1$ and $\|y_0\|< \infty$,

$$\|s_n - s_m\|\leq \|y_0\|\frac{\beta^m}{1-\beta}. \tag{4.37}$$

Taking limit as $n, m \to \infty$,

$$\lim_{n,m\to\infty}\|s_n - s_m\|= 0. \tag{4.38}$$

Therefore, s_n is a Cauchy sequence in the Banach space X. This implies that the series solution $y_m(x) = \sum_{j=0}^{N} c_j T_{L,j}(x) = C^T\phi(x)$ by the present method is convergent to exact solution y . □

5 Numerical examples

To illustrate the effectiveness of the proposed methods in the present paper, several test examples are carried out in this section.

Example 5.1 *Consider this problem that is coincided by heat conduction model of the human head,*

$$y''(x) + \frac{2}{x}y'(x) = -e^{-y}. \tag{5.39}$$

we consider the solution this problem with condition as follows:

$$y'(0) = 0, \qquad y(1) + y'(1) = 0. \tag{5.40}$$

In this example, we do not have exact solution. We solved this equation by presented method and compared our results by method of [12, 17, 20]. The results can be seen in Table 1.

Example 5.2 *Consider the singular boundary value problem for* $0 \leq x \leq 1$ *as:*

$$y''(x)+\frac{0.5}{x}y'(x) = e^{y(x)}(0.5-e^{y(x)}), \tag{5.41}$$

$$y(0) = \ln(2), \qquad y(1) = 0. \tag{5.42}$$

which has the exact solution $y(x) = \ln(\frac{2}{x^2+1})$*. We applying the method with* $N = 10, N = 13$ *and compared the results by Wavelet method results on paper [7]. The numerical results can be seen in Table 2.*

Example 5.3 *Consider the singular boundary value problem, which has been considered in [7] for* $x \in (0, 1]$ *as:*

$$y''(x)+e^{\frac{1}{x}}y'(x)+y(x) = 6x+x^3+3x^2e^{\frac{1}{x}}, \tag{5.43}$$

$$y(0) = 0, \qquad y(1) = 1. \tag{5.44}$$

The exact solution of this problem is

$$y(x) = x^3. \tag{5.45}$$

We solve this problem by applying the presented method with $N = 4$ and compared the results by results of method [7] in Table 3. As its clear from the table present method has exact solutions by small number of basis and has very better results than previous method.

Example 5.4 *Consider the Bessel differential equation of order zero [21, 7]*

$$xy''(x) + y'(x) + xy(x) = 0, \;\; x \in (0, 1] \tag{5.46}$$

$$y(0) = 1, \qquad y'(0) = 0. \tag{5.47}$$

A solution known as the **Bessel** function of the first kind of order of zero denoted by $J_0(x)$ is

$$J_0(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{(q!)^2}(\frac{x}{2})^{2q}. \tag{5.48}$$

Table 4 compares the $y(x)$ obtained by the proposed method in this paper and the method of [21] and [7].

Example 5.5 *Consider the following oxygen diffusion problem*

$$y''(x) + \frac{2}{x}y'(x) = \frac{0.76129y}{y + 0.03119}, \tag{5.49}$$

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5. \tag{5.50}$$

As this problem is a real world problem we don't have its exact answer, because of this we compare different numerical method answers for this example [6, 13, 9] that are presented in Table 5.

6 Conclusion

In this paper, we implemented an efficient numerical method for solving the singular nonlinear differential equations. The properties of the Chebyshev polynomials matrix of derivative are used to reduce the differential equations to a system of algebraic equations. The convergence analysis of the proposed method is introduced. From illustrative examples, it can be seen that the proposed numerical approach can obtain very accurate and satisfactory results and has better results analogy to other existed methods.

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