Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 10, No. 4, 2018 Article ID IJIM-0849, 10 pages
Research Article
Science and Research Branch (IAU)

The Combined Reproducing Kernel Method and Taylor Series for Handling Fractional Differential Equations

A. Alvandi *, M. Paripour †‡

Received Date: 2016-03-28 Revised Date: 2017-07-18 Accepted Date: 2018-04-21

Abstract

This paper presents the numerical solution for a class of fractional differential equations. The fractional derivatives are described in the Caputo [1] sense. We developed a reproducing kernel method (RKM) to solve fractional differential equations in reproducing kernel Hilbert space. This method cannot be used directly to solve these equations, so an equivalent transformation is made by using Taylor series. Some numerical examples are studied to demonstrate the accuracy of the given method.

Keywords : Kernel; Reproducing kernel; Fractional differential equation; Fractional derivative.

1 Introduction

FRactional calculus has various applications in the nonlinear oscillation of earthquakes [2], fluid-dynamic traffic model [3], the transient analysis of viscoelastically damped structures [4, 5], continuum and statistical mechanics [6], colored noise [7], solid mechanics [8], economics [9], bioengineering [10, 11], anomalous transport [12], and dynamics of interfaces between nanoparticles and substrates [13]. The approximate solutions to the fractional differential equations were given during the last decades, such as method of Collocation [14], a new operational matrix method [15], Laplace transform method [16], Adomian decomposition method [17], variational iteration method [18, 19], homotopy perturbation method [20], and other methods occurred in [21, 22, 23, 24].

Reproducing kernel method is practical to solve singular or nonsingular integral equations in the reproducing kernel Hilbert space which was presented by Zaremba in 1908. For example, singular integral equations with cosecant kernel [25], Fredholm integral equation of the first kind [25], singular Fredhom integro-differential equations with weakly singularity [27], a class of third-order differential equations [28], nonlinear Fredholm-Volterra integro-differential equations [29], integro-differential equations of fractional order [30], singularly perturbed turning point problems [31], Wiener-Hopf equations of the second kind [32], nonlinear Abel's integral equations with weakly singular kernel [33]. For further see [34, 35, 36, 37].

In this paper, we introduce the use of RKM to solve fractional differential equations of the form:

$$\begin{aligned} a_1(x)D^{\alpha_1}u(x) + a_2(x)D^{\alpha_2}u(x) + \cdots \\ +a_q(x)D^{\alpha_q}u(x) + a_{q+1}u(x) = f(x) \end{aligned} \tag{1.1}$$

*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran.

†Corresponding author. paripour@hut.ac.ir, Tel:+98(918)3165994.

‡Department of Computer Engineering and Information Technology, Hamedan University of Technology, Hamedan, Iran.

with initial conditions

$$u^{(i)}(0) = d_i, i = 0, 1, \cdots, \gamma, \tag{1.2}$$

where $a_j(x)$, for $j = 1, 2, \cdots, q+1$ are continuous real valued functions and also $\gamma - 1 < \alpha_1 \leq \gamma, \gamma \in \mathbb{N}, 0 < \alpha_q < \alpha_{q-1} < \cdots < \alpha_1$, $D^{\alpha_i}, i = 1, 2, \cdots, q$.

denote the Caputo fractional derivative of order α_i, $f(x) \in C[0,1]$ is a given function defined and $u(x)$ is an unknown function to be determined.

This paper is organized in six sections including the introduction. In the next section, Some preliminaries in fractional calculus are presented. Reproducing kernel space is defined in Section 3. Transformation of Eq. (1.1) is introduced in Section 4. We report our numerical findings and demonstrate the accuracy of the new numerical scheme by considering some examples in Section 5. The last section is a brief conclusion.

2 Some preliminaries in fractional calculus

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition 2.1 *A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, +\infty)$ and it is said to be in the space C_μ^r iff $f^{(r)} \in C_\mu$ for $r \in \mathbb{N}$.*

Definition 2.2 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as:*

$$\begin{cases} {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(t) \mathrm{d}t, \\ \alpha > 0, t > 0, \\ {}_0I_t^0 f(t) = f(t). \end{cases} \tag{2.3}$$

Definition 2.3 *The fractional derivative of $f(t)$ in the Caputo sense is defined as follows:*

$$\begin{gathered} {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} \frac{d^\gamma}{d\tau^\gamma} f(\tau) \mathrm{d}\tau, \\ \gamma - 1 < \alpha < \gamma, \gamma \in \mathbb{N}, f \in C_{-1}^r, \end{gathered} \tag{2.4}$$

where $\alpha > 0$ is the order of the derivative and γ is the smallest integer greater than α. For the Caputo derivative, we have

$$D^\alpha C = 0 \qquad (C \text{ is a constant}) \tag{2.5}$$

$$D^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N} \cup \{0\} \\ & \text{and } \beta \geq \lceil \alpha \rceil \text{or} \beta \notin \mathbb{N} \text{ and} \beta > \lfloor \alpha \rfloor. \end{cases} \tag{2.6}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α, and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order.

In [1], the following properties for $f \in C_\mu$ and $\mu \geq -1$ have been proved

$$\begin{cases} 1)\ {}_0I_t^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{\alpha+k}, k \in \mathbb{N} \cup 0, t > 0, \\ 2)\ {}_0^C(D_t^\alpha)_0 I_t^\alpha f(t) = f(t),\ \alpha > 0, f(t) \in C[0,1], \\ 3)\ {}_0^C D_t^\beta f(t) = {}_0I_t^{\alpha-\beta}\, {}_0^C D_t^\alpha f(t), \qquad \alpha, \beta > 0, \\ 4)\ {}_0I_t^\alpha\, {}_0^C D_t^\alpha f(t) = f(t) - \sum_{j=0}^{\gamma-1} \frac{1}{j!} t^j (D_t^j f)(0), \\ f(t) \in C^\gamma[0,1], \gamma - 1 < \alpha \leq \gamma, \end{cases} \tag{2.7}$$

if $0 < \alpha \leq 1$ and $f(t) \in C^1[0,1]$ then ${}_0I_t^\alpha\, {}_0^C D_t^\alpha f(t) = f(t) - f(0)$.

3 Reproducing kernel space

First, we construct the closet subspace ${}^oW^m[0,1]$ of the reproducing kernel space $W^m[0,1]$ by imposing condition $u(0) = 0, u'(0) = 0, \cdots, u^{(\gamma)}(0) = 0$.

Definition 3.1 ${}^oW^m[0,1] = \{u(x) | u^{(m-1)}(x)$ *is an absolutely continuous real value function,* $u^{(m)}(x) \in L^2[0,1]$,

$$u(0) = 0, u'(0) = 0, \cdots, u^{(\gamma)}(0) = 0\}.$$

The inner product and norm in ${}^oW^m[0,1]$ are given respectively by $\langle u, v \rangle =$

$\sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)\,\mathrm{d}x$, and

$$\|u\|_m = \sqrt{\langle u, u\rangle_m}, \qquad u, v \in {}^oW^m[0,1].$$

According to [37], the space ${}^oW^m[0,1]$ is a reproducing kernel Hilbert space. There exists $R_y(x) \in {}^oW^m[0,1]$, for any $u(y) \in {}^oW^m[0,1]$ and each fixed $x \in [0,1]$, $y \in [0,1]$, such that $\langle u(y), R_x(y)\rangle = u(x)$. The reproducing kernel $R_y(x)$ can be denoted by

$$R_y(x) = \begin{cases} R_1(x,y) = \sum_{i=1}^{2m} c_i(y)x^{i-1}\, y \le x, \\ \\ R_2(x,y) = \sum_{i=1}^{2m} d_i(y)x^{i-1},\ y > x, \end{cases} \tag{3.8}$$

where coefficients $c_i(y), d_i(y)$, $(i = 1,2,\cdots,2m)$, could be obtained by solving the following equations

$$\frac{\partial^i R_y(x)}{\partial x^i}|_{x=y+0} = \frac{\partial^i R_y(x)}{\partial x^i}|_{x=y-0}, \quad i = 0,1,2,\cdots,2m-2, \tag{3.9}$$

$$(-1)^m(\frac{\partial^{2m-1}R_y(x)}{\partial x^{2m-1}}|_{x=y+0} - \frac{\partial^{2m-1}R_y(x)}{\partial x^{2m-1}}|_{x=y-0}) = 1,$$

$$\begin{cases} \frac{\partial^i R_y(0)}{\partial x^i} - (-1)^{m-i-1}\frac{\partial^{2m-i-1}R_y(0)}{\partial x^{2m-i-1}} = 0, \\ \\ i = \gamma+1, \gamma+2, \cdots, m-1, \\ \\ \frac{\partial^{2m-i-1}R_y(1)}{\partial x^{2m-i-1}} = 0,\ i = 0,1,\cdots,m-1, \\ \\ R_y^{(i)}(0) = 0,\ i = 0,1,\cdots\gamma. \end{cases} \tag{3.10}$$

4 The analytical solution

4.1 Transformation of Eq. (1.1)

In this section, we convert Eq. (1.1) into an equivalent equation, which is easily solved by using RKM.

With the Taylor series expansion of $u^{(\gamma)}(t)$ based on expanding about the given point x belonging to the interval $[0,1]$, we have the Taylor series approximation of $u^{(\gamma)}(t)$ in the following form

$$u^{(\gamma)}(t) = u^{(\gamma)}(x) + (t-x)u^{(\gamma+1)}(x) + \frac{(t-x)^2}{2!}u^{(\gamma+2)}(x) + \cdots + \frac{(t-x)^n}{n!}u^{(\gamma+n)}(x) + \frac{(t-x)^{n+1}}{(n+1)!}u^{(\gamma+n+1)}(\zeta_{x,t}),$$

where $\zeta_{x,t}$ is between x and t. We use the truncated Taylor series and substitute in fractional derivative of $u(x)$,

$$ {}_0^C D_x^\alpha u(x) = \frac{1}{\Gamma(\gamma-\alpha)}\int_0^x (x-t)^{\gamma-\alpha-1}\frac{d^\gamma}{dt^\gamma}u(t)\mathrm{d}t, \quad \gamma - 1 < \alpha < \gamma, \gamma \in \mathbb{N}. \tag{4.11}$$

Then, we have

$$ {}_0^C D_x^\alpha u(x) = \frac{1}{\Gamma(\gamma-\alpha)}\int_0^x (x-t)^{\gamma-\alpha-1} \sum_{K=0}^{n}\frac{(-1)^K u^{(\gamma+K)}(x)(x-t)^K}{K!}\mathrm{d}t = \sum_{K=0}^{n}\frac{(-1)^K u^{(\gamma+K)}(x)}{\Gamma(\gamma-\alpha)K!}\int_0^x (x-t)^{\gamma+K-\alpha-1}\mathrm{d}t, \tag{4.12}$$

where $u^{(0)}(x) = u(x)$, and for the two cases $K = 0$ and $K > 0$, $\int_0^x (x-t)^{\gamma+K-\alpha-1}\mathrm{d}t$ is computable.

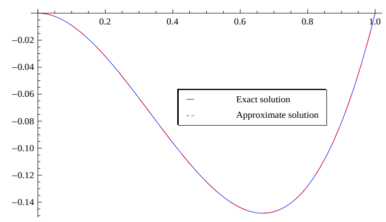

Figure 1: The comparisons between numerical and exact solution for m = 8 and $\gamma = 1$ for Example 5.1.

4.2 Solution of Eq. (1.1)

In this section, we shall give the exact and approximate solution of Eq. (1.1) in a reproducing kernel space ${}^oW^m[0,1]$. We assume that Eq. (1.1) has a unique solution. To deal with the system, we consider Eq. (1.1) as

$$\mathbb{L}(u) = a_1(x)D^{\alpha_1}u(x) + a_2(x)D^{\alpha_2}u(x) + \cdots + a_q(x)D^{\alpha_q}u(x) + a_{q+1}u(x), \tag{4.13}$$

Table 1: Numerical results of Example 5.1.

Node	$\lvert u_N(x) - u(x)\rvert$	$\lvert u_N(x) - u(x)\rvert$	$\lvert u_N(x) - u(x)\rvert$
	$W^6[0,1]$	$W^7[0,1]$	$W^8[0,1]$
0	0	0	0
0.1	2.39620E-10	3.40281E-12	2.31398E-13
0.2	6.10843E-9	5.88242E-11	1.35855E-12
0.3	3.14653E-8	2.11194E-10	3.47779E-12
0.4	7.64493E-8	3.03003E-10	4.19513E-12
0.5	1.14036E-7	7.10780E-11	2.38995E-12
0.6	8.82634E-8	1.51490E-9	2.31165E-11
0.7	8.20796E-8	4.75600E-9	6.68909E-11
0.8	4.97484E-7	1.05654E-8	1.40545E-10
0.9	1.27101E-6	1.96875E-8	2.52866E-10
1	2.52153E-6	3.27755E-8	4.08365E-10

Table 2: Numerical results of Ex. 5.2.

Node	$\lvert u_N(x) - u(x)\rvert$	$\lvert u_N(x) - u(x)\rvert$	$\lvert u_N(x) - u(x)\rvert$
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$
0	0	0	0
0.1	2.52709E-12	1.19622E-12	1.86295E-13
0.2	1.05300E-11	1.39666E-13	4.22162E-14
0.3	9.06302E-11	1.29017E-11	1.89376E-13
0.4	3.68997E-10	1.24541E-11	7.11486E-13
0.5	7.35732E-10	3.60521E-11	7.77822E-13
0.6	9.80372E-10	1.35855E-10	1.47532E-12
0.7	8.42491E-10	2.48874E-10	2.80775E-13
0.8	6.77430E-11	3.13591E-10	4.75972E-12
0. 9	1.54901E-9	2.64031E-10	9.08093E-12
1	4.13576E-9	4.48061E-11	6.62046E-12

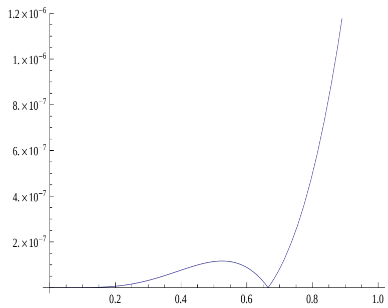

Figure 2: The absolute errors in spaces $W^6[0,1]$ for Example 5.1.

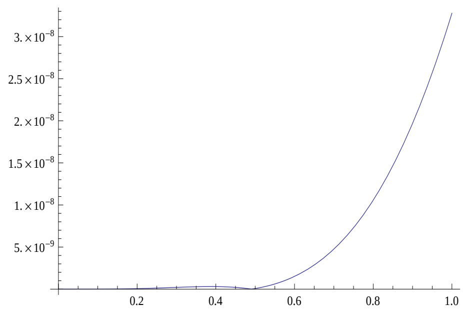

Figure 3: The absolute errors in spaces $W^7[0,1]$ for Example 5.1.

then equation (4.13) can be written as $\mathbb{L}(u) = f(x)$. It is clear that $\mathbb{L} : {}^oW^m[0,1] \longrightarrow W^1[0,1]$ is a bounded linear operator and $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$. We shall give the representation of analytical solution of Eq. (1.1) in the space $W^1[0,1]$.
Now, we choose a countable dense subset $\{x_i\}_{i=1}^{\infty}$ in $[0,1]$, and define

$$\varphi_i(x) = R_y(x_i),$$

$$\psi_i(x) = [\mathbb{L}_y R_y(x)](x_i) = \mathbb{L}^* R_y(x_i), i = 1, 2, \cdots, \tag{4.14}$$

where the subscript y in the operator $\mathbb{L}$ indicates that the operator $\mathbb{L}$ applies to the function of y.

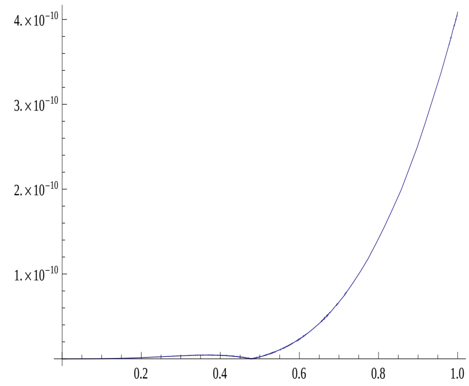

Figure 4: The absolute errors in spaces $W^8[0,1]$ for Example 5.1.

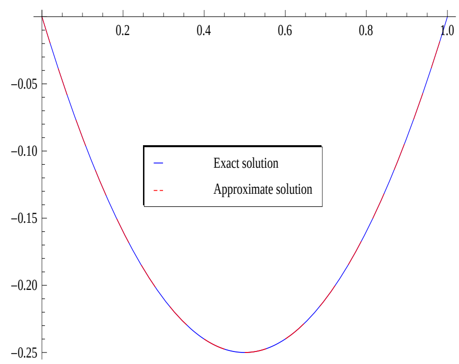

Figure 5: The comparisons between numerical and exact solution for $m = 6$, $\gamma = 1$ and $\alpha = 0.95$ for Example 5.2.

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of ${}^oW^m[0,1]$ is constructed from $\{\psi_i(x)\}_{i=1}^{\infty}$ by using the Gram-Schmidt algorithm, and then the approximate solution will be obtained by calculating a truncated series based on these functions, such that

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0,\ i = 1, 2, \cdots), \tag{4.15}$$

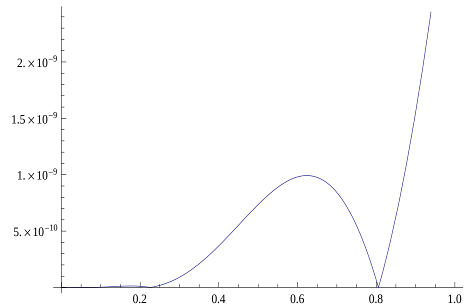

Figure 6: The absolute errors in space $W^6[0,1]$ and $\alpha = 0.5$ for Example 5.2.

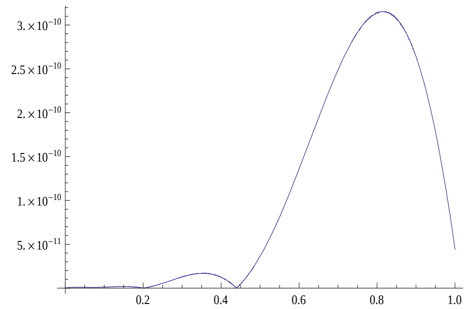

Figure 7: The absolute errors in space $W^6[0,1]$ and $\alpha = 0.75$ for Example 5.2.

where β_{ik} are orthogonal coefficients. In order to obtain β_{ik}, take

$$\psi_i(x) = \sum_{k=1}^{i} B_{ik}\bar{\psi}_k(x). \tag{4.16}$$

$$\langle \psi_i(x), \psi_i(x) \rangle = \sum_{k=1}^{i-1} B_{ik}^2 + B_{ii}^2, \tag{4.17}$$

$$B_{ii}^2 = \sqrt{\langle \psi_i(x), \psi_i(x) \rangle - \sum_{k=1}^{i-1} B_{ik}^2}. \tag{4.18}$$

$$\beta_{ii} = \frac{1}{\sqrt{\langle \psi_i(x), \psi_i(x) \rangle}}. \tag{4.19}$$

$$\beta_{ij} = \beta_{ii}\left(-\sum_{k=j}^{i-1} B_{ik}\beta_{kj}\right). \tag{4.20}$$

Theorem 4.1 *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system in ${}^oW^m[0,1]$.*

Theorem 4.2 *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and*

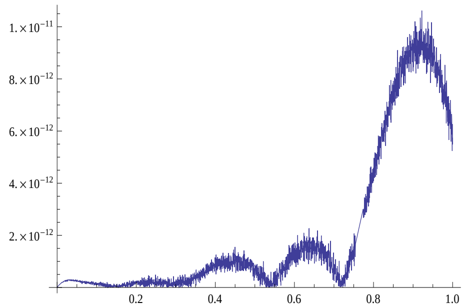

Figure 8: The absolute errors in space $W^6[0,1]$ and $\alpha = 0.95$ for Example 5.2.

the solution of Eq. (1.1) is unique, then the solution of Eq. (1.1) is

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x). \tag{4.21}$$

Proof. $u(x)$ can be expanded to Fourier series in terms of the normal orthogonal basis $\bar{\psi}_i(x)$ in ${}^oW^m[0,1]$,

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \psi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \mathbb{L}^*\varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle \mathbb{L}u(x), \varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle f(x), \varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x). \end{aligned}$$

The proof is complete. □

By truncating the series of the left-hand side of (4.21), we obtain the approximate solution of Eq. (1.1)

$$u_N(x) = \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}f(x_k)\bar{\psi}_i(x). \tag{4.22}$$

$u_N(x)$ in (4.22) is the N-term intercept of $u(x)$ in (4.21), so $u_N(x) \longrightarrow u(x)$ in $W^m[0,1]$ as $N \longrightarrow \infty$.

Lemma 4.1 *If $u(x) \in {}^oW^m[0,1]$, then there exists a constant c such that $|u(x)| \leq c\|u\|_m$.*

Proof.

$|u(x)| = |\langle u(y), R_x(y)|\rangle \leq \|u(y)\|\|R_x(y)\|_m$,

there exists a constant c such that

$|u(x)| \leq c\|u\|_m$.

The proof of the lemma is complete.□

Theorem 4.3 *Assume that $\|u_N(x)\|$ is bounded and Eq. (1.1) has a unique solution. If $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[0,1]$, then N-term approximate solution $u_N(x)$ converges to the exact solution $u(x)$ of Eq. (1.1) and the exact solution is expressed as*

$$u(x) = \sum_{i=1}^{\infty}B_i\bar{\psi}_i(x), \tag{4.23}$$

where $B_i = \sum_{k=1}^{i}\beta_{ik}f(x_k)$.

Proof. First, we will prove the convergence of $\|u_N(x)\|$ from (4.22). We infer that

$$u_N(x) = u_{N-1}(x) + B_N\bar{\psi}_N(x). \tag{4.24}$$

From the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\|u_N(x)\|^2_{{}^oW^m} = \|u_{N-1}(x)\|^2_{{}^oW^m} + \|B_N\|^2. \tag{4.25}$$

From (4.25), it holds that $\|u_N\|_{{}^oW_2^m} \geq \|u_{N-1}\|_{{}^oW_2^m}$. Due to the condition that $\|u_N\|_{{}^oW_2^m}$ is bounded, $\|u_N\|_{{}^oW_2^m}$ is convergent as soon as $N \longrightarrow \infty$. Then, there exists a constant c such that

$$\sum_{i=1}^{\infty}B_i^2 = c. \tag{4.26}$$

If $M > N$, in view of $(u_M - u_{M-1}) \perp (u_{M-1} - u_{M-2}) \perp \ldots \perp (u_{N+1} - u_N)$, it follows that

$$\begin{aligned} &\|(u_M - u_N)\|^2_{{}^oW^m} \\ &= \|u_M - u_{M-1} + u_{M-1} - u_{M-2} \\ &+ \cdots + u_{N+1} - u_N\|^2_{{}^oW^m} \\ &= \|u_M - u_{M-1}\|^2_{{}^oW^m} + \|u_{M-1} - u_{M-2}\|^2_{{}^oW^m} \\ &+ \cdots + \|u_{N+1} - u_N\|^2_{{}^oW^m} \\ &= \sum_{i=N+1}^{M}(B_i)^2 \longrightarrow 0, (N \longrightarrow \infty). \end{aligned} \tag{4.27}$$

Considering the completeness of ${}^oW^m[0,1]$. We have

$$u_N(x) \overset{\|\cdot\|_{{}^oW^m}}{\longrightarrow} u(x), \quad N \longrightarrow \infty.$$

It will be proved that $u(x)$ is the solution of Eq. (1.1).
From (4.23), it follows

$$\begin{aligned}(\mathbb{L}u)(x_j) &= \sum_{i=1}^{\infty} B_i \langle \mathbb{L}\bar{\psi}_i(x), \varphi_j(x)\rangle \\ &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \mathbb{L}^* \varphi_j(x))\rangle \\ &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \psi_j(x)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{N} \beta_{Nj} (\mathbb{L}\varphi)(x_j) &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \sum_{j=1}^{N} \beta_{Nj} \psi_j(x)\rangle_{{}^oW^m} \\ &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \bar{\psi}_N(x)\rangle_{{}^oW^m} \\ &= B_N.\end{aligned}$$

If $n = 1$, then $(\mathbb{L}u)(x_1) = f(x_1)$.

If $n = 2$ then $\beta_{21}(\mathbb{L}u)(x_1) + \beta_{22}(\mathbb{L}u)(x_2) = \beta_{21} f(x_1) + \beta_{22} f(x_2)$.

It is clear that $(\mathbb{L}u)(x_2) = f(x_2)$.
Moreover, it is easy to see by induction that

$$(\mathbb{L}u)(x_j) = f(x_j), j = 1, 2, \cdots. \tag{4.28}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, for any $x \in [0, 1]$. We have

$$(\mathbb{L}u)(x) = f(x). \tag{4.29}$$

That is, $u(x)$ is the solution of Equation (1.1) and

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x). \tag{4.30}$$

The proof is complete. □

5 Numerical experiments

To test the accuracy of the present method, some numerical examples are presented in this section. All examples are done by taking parameter $N = 20$ where N is the number of terms of the Fourier series of the unknown function $u(x)$. Parameter n is the number of terms of the Taylor series and we choose $m > n$ to solve these examples. The examples are computed using Mathematica 8.0. Results obtained by the method are compared with the exact solution of each example and found to be in a good agreement.

Example 5.1 *As the first example, we consider the following initial value problem in the case of the inhomogeneous Lane-Emden equation [14]:*

$$\begin{cases} D^{\alpha_1} u(x) + \frac{c}{x^{\alpha_1-\alpha_2}} D^{\alpha_2} u(x) + \frac{1}{x^{2-\alpha_1}} u(x) = f(x), \\ u(0) = 0, \qquad u'(0) = 0. \end{cases}$$

The exact solution of this problem is $u(x) = x^3 - x^2$. In this problem $\alpha_1 = \frac{3}{2}, \alpha_2 = \frac{1}{2}$,

$$\begin{aligned} f(x) &= 6x^{3-\alpha_1}\left(\frac{\Gamma(4-\alpha_2)+c\Gamma(4-\alpha_1)}{\Gamma(4-\alpha_2)\Gamma(4-\alpha_1)} + \frac{x^2}{6}\right) \\ &-2\left(\frac{\Gamma(3-\alpha_2)+c\Gamma(3-\alpha_1)}{\Gamma(3-\alpha_2)\Gamma(3-\alpha_1)} + \frac{x^2}{2}\right) x^{2-\alpha_1}. \end{aligned}$$

We find the approximate solution by the proposed method for $n = 2$. The absolute errors obtained by reproducing Kernel in space $W^6[0, 1], W^7[0, 1], W^8[0, 1]$ are shown in Table 1. This is an indication of stability on the reproducing Kernel. However, by increasing m, the behavior improves.
The comparisons between the numerical solutions and the exact solution for $m = 8$ and $\gamma = 2$ are given in Figure 1. We can clearly see that the numerical solutions and exact solution are coincided completely. Figures 2 - 4 reveals the absolute errors in spaces $W^6[0, 1], W^7[0, 1], W^8[0, 1]$, respectively.
Comparing between the numerical solutions presented in [14] with these numerical solutions, we find that the reproducing Kernel method can reach a higher degree of accuracy when solving the same equation.

Example 5.2 *Consider the following semi differential equation [16, 21]:*

$$\begin{cases} D^{\alpha} u(x) + u(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x, \\ u(0) = 0, \qquad 0 < \alpha \leq 1. \end{cases}$$

The exact solution is $u(x) = x^2 - x$.
We find the approximate solution by the proposed method for $n = 2$ and $\gamma = 1$. The absolute errors obtained by reproducing Kernel in space $W^6[0, 1]$ and $\alpha = 0.5, \alpha = 0.75, \alpha = 0.95$ are shown in Table 2. Figure 6 reveals the absolute errors in space

$W^6[0,1]$ and $\alpha = .5, \alpha = .75, \alpha = .95$, respectively. Note that as α approaches 1, the numerical solution converges to the exact solution. i.e. in the limit, the solution of the fractional differential equations approaches to that of the integer-order differential equations. The comparisons between the numerical solutions and the exact solution for $m = 6$, $\gamma = 1$ and $\alpha = 0.95$ are given in Figures 6 - 8, respectively. We can clearly see that the numerical solutions and exact solution are coincide completely.

6 Concluding

In this study, we developed an efficient and computationally attractive method to solve the fractional differential equations. Using the definition of Caputo-type fractional derivative, Taylor series and the properties of proposed method; we transform the initial problem into an equivalent equation. The error analysis of reproducing kernel method is introduced. The comparisons between the numerical solutions and the exact solution show the present method is accurate.

References

[1] A. Kilbas, H.M. Srivastava, J. J. Trujillo, Theory and application of fractional differntial equations, *Elsevier* 2006.

[2] J. H. He, Nonlinear oscillation with fractional derivative and its applications, International Conference on Vibrating Engineering, Dalian, China, 98 (1998) 288-291.

[3] J. H. He, Some applications of nonlinear fractional differential equations and their approximations, *Bulletin of Science Technology* 15 (1999) 86-90.

[4] R. L. Bagley, P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *Journal of Rheology* 27 (1983) 201-210.

[5] R. L. Bagley, P. J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, *American Institute of Aeronautics and Astronautics* 23 (1985) 918-925.

[6] F. Mainardi, Fractals and fractional calculus in continuum mechanics, *Springer Verlag* 378 (1997) 291-348.

[7] B. Mandelbrot, Some noises with 1/f spectrum, a bridge between direct current and white noise, *Institute of Electrical and Electronics Engineers Transactions on Information Theory* 13 (1967) 289-298.

[8] Y. A. Rossikhin, M. V. Shitikova, Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, *Applied Mechanics Reviews* 50 (1997) 15-67.

[9] R. T. Baillie, Long memory processes and fractional integration in econometrics, *Journal of Econometrics* 73 (1996) 5-59.

[10] R. L. Magin, Fractional calculus in bioengineering, *Redding: Begell House*, 2006.

[11] R. L. Magin, Fractional calculus in bioengineering-part 2, *Critical Reviews in Biomedical Engineering* 32 (2004) 105-193.

[12] R. Metzler, J. Klafter, Recent developments in the description of anomalous transport by fractional dynamics, *Journal of physics A* 37 (2004) 161-208.

[13] T. S. Chow, Fractional dynamics of interfaces between soft-nanoparticles and rough substrates, *Physics Letters A* 342 (2005) 148-155.

[14] M. S. Mechee, N. Senu, Numerical study of fractional differential equations of Lane-Emden type by method of collocation, *Applied Mathematics* 3 (2012) 851-856.

[15] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Computers and Mathematics with Applications* 59 (2010) 1326-1336.

[16] S. Kazem, Exact solution of some linear fractional differential equations by Laplace transform, *International Journal of Nonlinear Science* 16 (2013) 3-11.

[17] Q. Wang, Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method, *Applied Mathematics and Computation* 182 (2006) 1048-1055.

[18] M. Inc, The approximate and exact solutions of the space and time-fractional Burgers equations with initial conditions by variational iteration method, *Journal of Mathematical Analysis and Applications* 345 (2008) 476-484.

[19] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, *International Journal of Nonlinear Sciences and Numerical Simulation* 7 (2006) 27-34.

[20] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, *Physics Letters A* 365 (2007) 345-350.

[21] M. Hamarsheh, E.A. Rawashdeh, A numerical method for solution of semidiffential equations, *Matematicki Vesnik* 62 (2010) 117-126.

[22] P. Kumar, O.P. Agrawal, An approximate method for numerical solution of fractional differential equations, *Signal Processing* 86 (2006) 2602-2610.

[23] N.H. Sweilam, M.M. Khader, R.F. Al-Bar, Numerical studies for a multi-order fractional differential equation, *Physics Letters A* 371 (2007) 26-33.

[24] V. Daftardar-Gejji, H. Jafari, Solving a multi-order fractional differential equation, *Applied Mathematics and Computation* 189 (2007) 541-548.

[25] H. Du, J. Shen, Reproducing kernel method of solving singular integral equation with cosecant kernel, *Journal of Mathematical Analysis and Applications* 348 (2008) 308-314.

[26] H. Du, M. Cui, Approximate solution of the Fredholm integral equation of the first kind in a reproducing kernel Hilbert space, *Applied Mathematics Letters* 21 (2008) 617-623.

[27] H. Du, G. Zhao, C. Zhao, Reproducing kernel method for solving singular Fredhom integro-differential equations with weakly singularity, *Applied Mathematics* 255 (2014) 122-132.

[28] E. Moradi, A. Yusefi, A. Abdollahzadeh, E. Tila, New implementation of reproducing kernel Hilbert space method for solving a class of third-order differential equations, *Journal of mathematics and computer science* 12 (2014) 253-262.

[29] O. Abu Arqub, M. Al-Smadi, S. Momani, Application of reproducing kernel method for solving nonlinear Fredholm Volterra integro-differential equations, *Abstract and Applied Analysis* 2012 (2012) 1-16.

[30] S. Bushnaq, S. Momani, Y. Zhou, A reproducing kernel Hilbert space method for solving integro-differential equations of fractional order, *Journal of Optimization Theory and Applications* (2013) 1-10.

[31] F.Z. Geng, S.P. Qian, S. Li, A numerical method for singularly perturbed turning point problems with an interior layer, *Journal of Computational and Applied Mathematics* 255 (2014) 97-105.

[32] A. Alvandi, T. Lotfi, M. Paripour, Reproducing kernel method for solving Wiener-Hopf equations of the second kind, *Journal of Hyperstructures* 5 (2016) 56-68.

[33] A. Alvandi, M. Paripour,The combined reproducing kernel method and Taylor series to solve nonlinear Abels integral equations with weakly singular kernel, *Cogent Mathematics* 3 (2016) 125-130.

[34] A. Alvandi, M. Paripour, The combined reproducing kernel method and Taylor series for solving nonlinear Volterra-Fredholm integro-differential equations, *International Journal of Mathematical Modelling & Computations* 6 (2016) 301-312.

[35] A. Alvandi, M. Paripour, Reproducing kernel method with Taylor expansion for linear Volterra integro-differential equations, *Communications in Numerical Analysis* 1 (2017) 1-10.

[36] X.Y. Li, B.Y. Wu, Error estimation for the reproducing kernel method to solve linear boundary value problems, *Journal of Computational and Applied Mathematics* 243 (2013) 10-15

[37] M. Cui, Y. Lin, Nonlinear Numerical Analysis in Reproducing Kernel Space, *Nova Science Pub. Inc., Hauppauge,* 2009.

Azizallah Alvandi is PhD of applied mathematics, Islamic Azad University, Hamedan Branch, Iran. Interested in numerical solution of nonlinear ordinary differential equations and integral equations.

Mahmoud Paripour is Assistant Professor in Applied Mathematics, Hamedan University of Technology, Hamedan, Iran. Interested in numerical solution of functional equations, numerical linear algebra, fuzzy mathematics and finance mathematics.