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Finitely Generated Annihilating-Ideal Graph of Commutative Rings

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Abstract

Let R be a commutative ring and $\mathbb{A}(R)$ be the set of all ideals with non-zero annihilators. Assume that $\mathbb{A}^*(R) = \mathbb{A}(R)\setminus\{(0)\}$ and $\mathbb{F}(R)$ denote the set of all finitely generated ideals of R. In this paper, we introduce and investigate the *finitely generated annihilating-ideal graph* of R, denoted by $\mathbb{AG}_F(R)$. It is the (undirected) graph with vertices $\mathbb{A}_F(R) = \mathbb{A}^*(R) \cap \mathbb{F}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. First, we study some basic properties of $\mathbb{AG}_F(R)$. For instance, it is shown that if R is not a domain, then $\mathbb{AG}_F(R)$ has ascending chain condition on vertices if and only if R is Noetherian. We characterize all rings for which $\mathbb{AG}_F(R)$ is a finite, complete, star or bipartite graph. Next, we study diameter and girth of $\mathbb{AG}_F(R)$. It is proved that $\mathrm{diam}(\mathbb{AG}_F(R)) \leqslant \mathrm{diam}(\mathbb{AG}(R))$ and $\mathrm{gr}(\mathbb{AG}_F(R)) = \mathrm{gr}(\mathbb{AG}(R))$.

Keywords : Commutative ring; Annihilating-ideal; Finitely generated ideal; Graph.

1 Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past years. There are many papers on assigning a graph to a ring (see for example [6, 7, 9, 10, 13]). Let R be a commutative ring. We call an ideal I of R is an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = (0)$ and use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. By the annihilating-ideal graph $\mathbb{AG}(R)$ of R we mean the graph with vertices $\mathbb{A}^*(R) = \mathbb{A}(R)\setminus\{(0)\}$ such that there is an (undirect) edge between vertices I and J if and only if $I \neq J$ and $IJ = (0)$. Thus $\mathbb{AG}(R)$ is an empty graph if and only if R is an integral domain. The concept of the annihilating-ideal graph of a commutative ring was first introduced by Behboodi and Rakeei in [11, 12]. Recently, this notation of the annihilating-ideal graph has been extensively studied by various authors (see for instance, [1, 2, 3, 4, 5, 14, 15] and many others). In [15], Taheri, Behboodi and Theranian, introduce and investigate the spectrum graph of the annihilating-ideal graph of a commutative ring, denoted by $\mathbb{AG}_s(R)$, that is, the graph whose vertices are all non-zero prime ideals of R with non-zero annihilator, denoted by $\mathbb{A}_s(R)$ and two distinct vertices P_1, P_2 are adjacent if and only if $P_1P_2 = (0)$. This is an induced subgraph of the annihilating-ideal graph of R.

In this paper, we introduce and study the *finitely generated annihilating-ideal graph* of a commutative ring R, denoted by $\mathbb{AG}_F(R)$, that is, the graph whose vertices are all non-zero finitely generated ideals of R with non-zero annihilator and two distinct vertices I, J are adjacent if and only if $IJ = (0)$. This is an induced subgraph of

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the annihilating-ideal graph of R. It is clear that, if R is a Noetherian ring, then $\mathbb{AG}_F(R) = \mathbb{AG}(R)$.

Throughout this paper, all rings are commutative with identity and all modules are unital. For a ring R we denote by (R), (R), $Z(R)$, $\mathbb{I}(R)$ and $\mathbb{F}(R)$ the set of all prime ideals, the set of all minimal prime ideals, the set of all zero divisors, the set of all non-zero proper ideals and the set of all finitely generated ideals of R, respectively. Let X be either an element or subset of R. The *annihilator* of X is the ideal $\mathrm{Ann}(X) = \{a \in R \mid aX = 0\}$.

Let G be any graph. We denote the vertex set of G by $\mathrm{V}(G)$. Sometimes, two graphs G and H have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs G and H are *isomorphic* and we write $G \cong H$. The graph G is called *connected* if there is a path between every two distinct vertices. For distinct vertices P, Q of G, let $\mathrm{d}(P, Q)$ be the length of the shortest path from P to Q and, if the is no such path, we define $\mathrm{d}(P, Q) = \infty$. The *diameter* of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(P, Q) : P \text{ and } Q \text{ are distinct vertices of } G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is defined as the length of the shortest cycle in G and $\mathrm{gr}(G) = \infty$ if G contains no cycles. A *complete graph* is a graph in which any two distinct vertices are adjacent. A complete graph with n vertices denoted by K_n. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets A and B such that every edge connects a vertex in A to one in B. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. In this case, if $|A| = n$ and $|B| = m$, we denote the graph by $K_{n,m}$. If $|A| = 1$ or $|B| = 1$, then the graph is said to be a *star graph*. For a graph G the degree of a vertex I, is the number of vertices adjacent to I. If the degree of all vertices of G is equal, we say G is a regular graph. For every positive integer n, we denote by P_n a path of order n.

Let R be a ring. In this paper, we denote the vertex set of $\mathbb{AG}_F(R)$ by $\mathbb{A}_F(R)$. In fact, $V(\mathbb{AG}_F(R)) = \mathbb{A}_F(R) = \mathbb{A}^*(R) \cap \mathbb{F}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Unlike the spectrum graph, for every ring R, $\mathbb{AG}_F(R)$ is a connected graph. In section 2, first we give some basic properties of the finitely generated annihilating-ideal graph. For instance, it is shown that if R is non-domain, $\mathbb{AG}_F(R)$ has ACC on vertices if and only if R is a Noetherian ring. Also we show that there is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$ if and only if there exists a vertex of $\mathbb{AG}_F(R)$ which is adjacent to every other vertex of $\mathbb{AG}_F(R)$ (see Proposition 2.5). Moreover we show that $\mathbb{AG}(R)$ is a complete (star) graph if and only if $\mathbb{AG}_F(R)$ is a complete (star) graph (see Proposition 2.4 and 3.4). Also we show that finitely generated annihilating-ideal graph can not be a cycle (see Proposition 2.6). In section 3, diameter and girth of the $\mathbb{AG}_F(R)$ are studied. It is shown that for every ring R, $\mathrm{diam}(\mathbb{AG}_F(R)) \leqslant \mathrm{diam}(\mathbb{AG}(R))$ (see Corollary 3.1) and $\mathrm{gr}(\mathbb{AG}_F(R)) = \mathrm{gr}(\mathbb{AG}(R))$ (see Proposition 3.5). Also it is shown that if $\mathrm{gr}(\mathbb{AG}_F(R)) = 4$, then $\mathbb{AG}(R)$ is a complete bipartite graph if and only if $\mathbb{AG}_F(R)$ is a complete bipartite graph (see Theorem 3.1). Consequently, if R is a reduced ring such that $\mathbb{AG}_F(R)$ is a complete bipartite graph with $\mathrm{gr}(\mathbb{AG}_F(R)) = 4$, then $|\mathrm{Min}(R)| = 2$ (see Corollary 3.5).

2 Some basic properties of finitely generated annihilating-ideal graph

By [11, Example 1.9], there exists a local zero-dimensional ring R such that $\mathbb{A}^*(R) \neq \emptyset$, but $\mathbb{AG}_s(R)$ is an empty graph. Also, [15, Example 2.5] gives a non-connected spectrum graph of a local ring. The following proposition shows that for each non-domain ring R, finitely generated graph, $\mathbb{AG}_F(R)$, is a non-empty connected graph, in general.

Proposition 2.1 *For every non-domain ring R, $\mathbb{AG}_F(R)$ is a non-empty connected graph.*

Proof. Since R is a non-domain, there exists $0 \neq a \in Z(R)$. Then $Ra \in \mathbb{A}_F(R)$ and hence $\mathbb{AG}_F(R) \neq \emptyset$. Assume that I and J are two distinct vertex of $\mathbb{AG}_F(R)$. If $IJ = (0)$, then there is nothing to prove. Suppose that $IJ \neq (0)$. Since $I, J \in \mathbb{A}^*(R)$ and $\mathbb{AG}(R)$ is a connected graph (see [11, Theorem 2.1]), there exist $I_1, J_1 \in \mathbb{A}^*(R)$ such that $I_1 I = J_1 J = (0)$. First assume that $I_1 = J_1$. Let $a \in I_1 = J_1$, then $I — Ra — J$ is a path in $\mathbb{AG}_F(R)$. Now assume that $I_1 \neq J_1$. Without loss of generality suppose that $a \in I_1 \setminus J_1$ and $b \in J_1$, so $Ra \neq Rb$. If $(Ra)(Rb) = (0)$, then

$I — Ra — Rb — J$ is a path in $\mathbb{AG}_F(R)$. If $(Ra)(Rb) \neq (0)$, then $I — Rab — J$ is a path in $\mathbb{AG}_F(R)$. □

Let R be a ring. We say that finitely generated annihilating-ideal graph has ACC on vertices if R has ACC on $\mathbb{A}_F(R)$. The following result is a generalization of [11, Theorem 1.1].

Theorem 2.1 *Let R be a non-domain ring. Then $\mathbb{AG}_F(R)$ has ACC on vertices if and only if R is a Noetherian ring.*

Proof. ($\Leftarrow$) It is trivial.

($\Rightarrow$) Assume that $\mathbb{AG}_F(R)$ has ACC on vertices. By contrary, suppose that R is not Noetherian ring. Therefore by [11, Theorem 1.1], $\mathbb{AG}(R)$ has not ACC on vertices. So, there exists ideals $I_i \in \mathbb{A}^*(R)$ for $i \in \mathbb{N}$, such that $I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq \dots$ form an ascending chain such that is an infinite chain. Now assume that $a_1 \in I_1$, then $Ra_1 \subseteq I_1$. Since $I_1 \subsetneqq I_2$, there exists $a_2 \in I_2$ such that $a_2 \notin I_1$ and hence $Ra_1 \subsetneqq Ra_1 + Ra_2$. By continuing this process, we have a chain as a following:

$$Ra_1 \subsetneqq Ra_1 + Ra_2 \subsetneqq Ra_1 + Ra_2 + Ra_3 \subsetneqq \dots$$

which is an infinite chain of elements of $\mathbb{A}_F(R)$, a contradiction. □

Corollary 2.1 *Assume that R is a non-domain ring. Then $\mathbb{AG}_F(R)$ is a finite graph if and only if $\mathbb{AG}(R)$ is finite.*

Proof. ($\Leftarrow$) It is trivial.

($\Rightarrow$) Assume that $\mathbb{AG}_F(R)$ is a finite graph, so $\mathbb{AG}_F(R)$ has ACC on vertices. By Theorem 2.1, R is a Noetherian ring and hence $\mathbb{AG}(R) = \mathbb{AG}_F(R)$, therefore $\mathbb{AG}(R)$ is a finite graph. □

Next, we characterize non-domain rings R for which the finitely generated annihilating-ideal graph is a finite graph.

Proposition 2.2 *Let R be a non-domain ring. Then the following statements are equivalent.*
(1) *$\mathbb{AG}_F(R)$ is a finite graph.*
(2) *R has only finitely many ideal.*
(3) *R has only finitely many finitely generated ideals.*
(4) *Every vertex of $\mathbb{AG}_F(R)$ has finite degree.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear with Corollary 2.1 and [11, Theorem 1.4].

(4) $\Rightarrow$ (1) Assume that every vertex of $\mathbb{AG}_F(R)$ has finite degree. By contrary suppose that $\mathbb{AG}_F(R)$ is an infinite graph. Corollary 2.1 implise that $\mathbb{AG}(R)$ is an infinite graph and by [11, Theorem 1.4], there exists ideal $I \in \mathbb{A}^*(R)$ such that vertex I has infinite degree. Suppose $a \in I$ and $I_0 = Ra$, so I_0 is a vertex of $\mathbb{AG}_F(R)$ with infinite degree, a contradiction. Therefore $\mathbb{AG}_F(R)$ is a finite graph. □

Proposition 2.3 *Let R be a ring. Then the following statements are equivalent:*
(1) *There is a vertex of $\mathbb{AG}_F(R)$ which is adjacent to every other vertex of $\mathbb{AG}_F(R)$.*
(2) *There is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$.*
(3) *Either $R = F \oplus D$, where F is a field and D is an integral domain, or $Z(R) = \mathrm{Ann}(x)$ for some $0 \neq x \in R$.*

Proof. (1) $\Rightarrow$ (2) Suppose that I is a vertex of $\mathbb{AG}_F(R)$ which is adjacent to every other vertex of $\mathbb{AG}_F(R)$. We claim that for every $I \neq J \in \mathbb{A}^*(R)$, $IJ = (0)$. By contrary, suppose that there exists $I \neq J \in \mathbb{A}^*(R)$ such that $IJ \neq (0)$, so there exists $0 \neq a \in I$ and $0 \neq b \in J$ such that $ab \neq 0$. Let $I_1 = Ra \subseteq I$ and $I_2 = Rb \subseteq J$. First assume that $I_1 \neq I_2$. Since $II_2 = (0)$, $I_2I_1 = (0)$ and hence $ab = 0$, a contradiction (with supposing which $I \neq I_2$, for case $I = I_2$, we let $I_2 = Rb+Rc$, where $c \in J \setminus I_2$). Now assume that $I_1 = I_2$. Since $J \notin \mathbb{F}(R)$, there exists $0 \neq c \in J$ such that $c \notin Rb = I_2$, thus $I_2 = Rb \subsetneqq Rb + Rc \subsetneqq J$. Since $I(Rb + Rc) = (0)$, we can conclude that $ab = 0$, a contradiction.

(2) $\Rightarrow$ (3) It is [11, Theorem 2.2].

(3) $\Rightarrow$ (1) If $R = F \oplus D$, where F is a field and D is an integral domain, then $F \oplus (0)$ is adjacent to every other vertex of $\mathbb{AG}_F(R)$. If $Z(R) = \mathrm{Ann}(x)$ for some $0 \neq x \in R$, then Rx is adjacent to every other vertex of $\mathbb{AG}_F(R)$. □

Now we characterize all rings for which finitely generated annihilating-ideal graph is a complete graph.

Proposition 2.4 *Let R be a ring. Then the following statements are equivalent:*
(1) *$\mathbb{AG}_F(R)$ is a complete graph.*

(2) *$\mathbb{AG}(R)$ is a complete graph.*
(3) *Either $R \cong F_1 \oplus F_2$, where F_1, F_2 are fields, or $Z(R)$ is an ideal of R, $(Z(R))^3 = (0)$ and for each ideal $I \subset Z(R)$, $IZ(R) = (0)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathbb{AG}_F(R)$ is a complete graph. We claim that for every $I, J \in \mathbb{A}^*(R)$, $IJ = (0)$. By contrary, suppose that there exist two distinct ideals $I, J \in \mathbb{A}^*(R)$ such that $IJ \neq (0)$, where at least one of them is not finitely generated, therefore there exist $a \in I$ and $b \in J$ such that $ab \neq 0$. Let $I_0 = Ra$ and $J_0 = Rb$, so $I_0, J_0 \in \mathbb{A}_F(R)$ and $I_0 J_0 \neq (0)$. If $I_0 \neq J_0$, then we have a contradiction. Suppose that $I_0 = J_0$ and $J \notin \mathbb{F}(R)$. Since $J_0 \subsetneq J$, there is $c \in J$ such that $c \notin J_0$ and hence $I_0 \neq Rb + Rc$ and $I_0(Rb + Rc) \neq (0)$. Since $I_0, Rb + Rc \in \mathbb{A}_F(R)$, we have a contradiction. Therefore $\mathbb{AG}(R)$ is a complete graph.

(2) $\Rightarrow$ (1) It is clear.
(2) $\Leftrightarrow$ (3) It is [11, Theorem 2.7]. □

Proposition 2.5 *Let R be a non-domain ring. Then*

(1) *$\mathbb{AG}_F(R) \cong K_1$ if and only if R has only one non-zero proper ideal.*

(2) *Assume that $\mathbb{AG}_F(R) \cong K_2$, then $R \cong F_1 \oplus F_2$, where F_1, F_2 are two fields or $(R, \mathcal{M})$ is a local ring with $\mathbb{A}_F(R) = \{\mathcal{M}^2, \mathcal{M}\}$.*

(3) *Assume that $\mathbb{AG}_F(R) \cong K_n$, where $n \geqslant 3$. Then $n \geqslant 4$ and $Z(R)$ is an ideal of R with $(Z(R))^3 = (0)$.*

(4) *Let $\mathbb{AG}_F(R) \cong P_n$, where $n \geqslant 3$. Then $n \in \{3, 4\}$.*

Proof. (1) ($\Rightarrow$) Suppose that $\mathbb{AG}_F(R) \cong K_1$, by Theorem 2.1, R is a Noetherian ring and hence $\mathbb{AG}(R) \cong K_1$, which implise $|\mathbb{I}(R)| = 1$.

($\Leftarrow$) Suppose that $\mathbb{I}(R) = \{I\}$, since R is an Artinian ring, $\mathbb{A}^*(R) = \{I\}$. Since $\mathbb{AG}_F(R)$ is a non-empty graph, $\mathbb{A}_F(R) = \{I\}$ and so $\mathbb{AG}_F(R) \cong K_1$.

(2) ($\Rightarrow$) Assume that $\mathbb{AG}_F(R) \cong K_2$. By Theorem 2.1, R is a Noetherian ring and hence $\mathbb{AG}(R) \cong K_2$. By [11, Corollary 2.9], proof is complete.

($\Leftarrow$) It is easy.

(3) Assume that $\mathbb{AG}_F(R) \cong K_n$, where $n \geqslant 3$. If $n = 3$, then it is clear that $\mathbb{AG}(R) \cong K_3$, a contradiction (see [3, Corollary 9]). Therefore $n \geqslant 4$. By Proposition 2.4, $Z(R)$ is an ideal of R with $(Z(R))^3 = (0)$.

(4) Assume that $\mathbb{AG}_F(R) \cong P_n$, where $n \geqslant 3$. Since each vertex of $\mathbb{AG}_F(R)$ has finite degree, by Proposition 2.4, R is a Noetherian ring and hence $\mathbb{AG}(R) \cong P_n$, where $n \geqslant 3$. Since $\operatorname{diam}(\mathbb{AG}(R)) \leqslant 3$ (see [11, Theorem 2.1]), $n \in \{3, 4\}$. □

Now we are in position to characterize rings for which finitely generated annihilating-ideal graph is a path.

Corollary 2.2 *Let R be a ring such that $\mathbb{AG}_F(R) \cong P_n$, where $n \leq 4$, then R is one of the following three types of rings.*

(1) *$R \cong F_1 \oplus F_2$, where F_1, F_2 are two fields.*

(2) *$R \cong F \oplus S$, where F is a field and S is a ring with exactly one non trivial ideal.*

(3) *R is a local ring.*

Proof. It is clear with Proposition 2.5 and [3, Theorem 11]. □

Lemma 2.1 *Let R be a ring. If $\mathbb{AG}_F(R)$ is a regular graph of finite degree, then $\mathbb{AG}_F(R)$ is a complete graph.*

Proof. Assume that $\mathbb{AG}_F(R)$ is a regular graph of finite degree. By Proposition 2.4, R is a Noethrian ring and hence $\mathbb{AG}(R)$ is a regular graph of finite degree, by [3, Theorem 8], $\mathbb{AG}(R)$ is a complete graph and so Proposition 2.4, implise that $\mathbb{AG}_F(R)$ is a complete graph. □

Proposition 2.6 *Let R be a non-domain ring. Then $\mathbb{AG}_F(R)$ can not be a cycle.*

Proof. Assume that $\mathbb{AG}_F(R) \cong C_n$, where $n \geqslant 3$. By Lemma 2.1, $\mathbb{AG}_F(R) \cong K_3$, a contradiction (see Proposition 2.5 (3)). □

For every Noetherian ring R, it is clear that

$\mathbb{AG}_F(R) = \mathbb{AG}(R)$, the following example shows a non-Noetherian ring R for which, $\mathbb{AG}_F(R)$ is a proper subgraph of the annihilating-ideal graph.

Example 2.1 Let F be a field. We consider the ring

$$R = F[[X, Y, Z_1, Z_2, \dots]] / < XY, XZ_i, Z_i$$

$Y, Z_i^2 | \ i = 1, 2, \cdots >$. Let $I =< XY, XZ_i, Z_iY, Z_i^2 | \ i = 1, 2, \cdots >$. Then R is a non-Noetherian ring. Set $P_1 = Rx + \sum Rz_i$, $P_2 = Ry + \sum Rz_i$ where $i = 1, 2, \dots$, $x = X + I, y = Y + I$ and $z_i = Z_i + I$. It is clear that $P_1P_2 = (0)$, so $P_1, P_2 \in \mathbb{A}^*(R)$, but $P_1, P_2 \notin \mathbb{A}_F(R)$, so $\mathbb{AG}_F(R)$ is a proper subgraph of $\mathbb{AG}(R)$.

Let R be a Noetherian ring. Then it is clear that $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_F(R)$. Now by note to Cohen's theorem a natural question is posed: If for every $P \in \mathbb{A}_s(R)$, P is finitely generated (i.e, $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_F(R)$), is R a Noetherian ring? The following example shows that the answer of this question is negative.

Example 2.2 Let $R = \{\{a_n\}_{n\in\mathbb{N}} \mid a_n \in \mathbb{Z}_2$ such that $\{a_n\}$ is eventually constant$\}$. Then with pointwise addition and multiplication, R is a Boolean ring. Let $P_i = \{\{a_n\} \in R \mid a_i = 0\}$ and $P_\infty = \{\{a_n\} \in R \mid$ there exists $m \in \mathbb{N}$ such that $a_n = 0$ for $n \geq m\}$. Then P_∞ is not finitely generated, so R is a non-Noetherian ring. One can easily see $P_i \in (R)$, $\mathbb{A}_s(R) = \{P_i \mid i \geq 1\}$, $\mathbb{AG}_s(R) \cong N_\infty$. Since each P_i is a principal ideal, $P_i \in \mathbb{A}_F(R)$ and hence $\mathbb{A}_s(R) \subseteq \mathbb{A}_F(R)$, so $\mathbb{AG}_s(R)$ is a subgraph of $\mathbb{AG}_F(R)$ but R is not Noetherian.

Let R be a ring. In [11, Theorem 2.10], it is shown that $\mathbb{AG}_s(R) = \mathbb{AG}(R)$ if and only if either $R = F_1 \oplus F_2$ for a pair of fields F_1 and F_2 or R has only one non-zero proper ideal. Therefore if $\mathbb{AG}_s(R) = \mathbb{AG}(R)$, then $\mathbb{AG}_F(R) = \mathbb{AG}(R)$. The following example shows that the converse is not hold.

Example 2.3 Let F be a field. We consider the ring

$$R = F[[X, Y, Z]] / < XY, XZ, ZY, Z^2 >.$$

Then R is a local ring with the maximal ideal $\mathcal{M} = Rx + Ry + Rz$ (where $x = X+ < XY, XZ, ZY, Z^2 >$, $y = Y+ < XY, XZ, ZY, Z^2 >$ and $z = Z+ < XY, XZ, ZY, Z^2 >$). Set $P_1 = Rx + Rz$ and $P_2 = Ry + Rz$, and since $P_1P_2 = (0)$, we conclude that $\mathrm{Min}(R) = \{P_1, P_2\}$. One can easily see that $\mathrm{Spec}(R) = \{\mathcal{M}, P_1, P_2\}$ and $P_1\mathcal{M} \neq (0)$, $P_2\mathcal{M} \neq (0)$. Also, since $z\mathcal{M} = (0)$, $\mathcal{M}$ is also an annihilating-ideal. Thus $\mathbb{AG}_s(R) \cong K_2 \cup N_1$, so $\mathbb{AG}_s(R) \neq \mathbb{AG}(R)$, but $\mathbb{AG}_F(R) = \mathbb{AG}(R)$ (since R is a Noetherian ring).

3 Diameter and girth of finitely generated annihilating-ideal graph

In this section, we express some properties of diameter and girth of finitely generated annihilating-ideal graph. Let H be a subgraph of G. In general there is no any relation between $\mathrm{diam}(H)$ and $\mathrm{diam}(G)$. We note that for each non-domain ring R, the annihilating-ideal graph $\mathbb{AG}(R)$ is connected and $0 \leq \mathrm{diam}(\mathbb{AG}(R)) \leq 3$ (see [11, Theorem 2.1]). The following proposition more or less summarizes the over all situation for the diameter of the finitely generated annihilating-ideal graph of a ring. We begin with the following lemma.

Lemma 3.1 *For every non-domain ring R, $0 \leqslant \mathrm{diam}(\mathbb{AG}_F(R)) \leqslant 3$.*

Proof. It is clear by Proposition 2.1. □

Proposition 3.1 *Let R be a non-domain ring. Then*
(1) $\mathrm{diam}(\mathbb{AG}(R)) = 0$ *if and only if* $\mathrm{diam}(\mathbb{AG}_F(R)) = 0$.
(2) $\mathrm{diam}(\mathbb{AG}(R)) = 1$ *if and only if* $\mathrm{diam}(\mathbb{AG}_F(R)) = 1$.
(3) *If* $\mathrm{diam}(\mathbb{AG}(R)) = 2$, *then* $\mathrm{diam}(\mathbb{AG}_F(R)) = 2$.
(4) *If* $\mathrm{diam}(\mathbb{AG}(R)) = 3$, *then* $\mathrm{diam}(\mathbb{AG}_F(R)) = 2$ *or 3.*

Proof. (1) By Proposition 2.5, it is clear that $\mathbb{AG}(R) \cong K_1$ if and only if $\mathbb{AG}_F(R) \cong K_1$, so there is nothing to prove.

(2) It is clear by Proposition 2.4.

(3) Assume that $\operatorname{diam}(\mathbb{AG}(R)) = 2$. By before section, $\operatorname{diam}(\mathbb{AG}_F(R)) \neq 0, 1$ and hence $2 \leqslant \operatorname{diam}(\mathbb{AG}_F(R)) \leqslant 3$. Let $I, J \in \mathbb{A}_F(R)$ such that $IJ \neq (0)$. Since $I, J \in \mathbb{A}^*(R)$ and $\operatorname{diam}(\mathbb{AG}(R)) = 2$, there is $K \in \mathbb{A}^*(R)$ such that $I - K - J$ is a path in $\mathbb{AG}(R)$. Now let $0 \neq a \in K$, so $I — Ra — J$ is a path in $\mathbb{AG}_F(R)$. Therefore $\operatorname{diam}(\mathbb{AG}(R)) = 2 = \operatorname{diam}(\mathbb{AG}_F(R))$.
(4) Suppose that $\operatorname{diam}(\mathbb{AG}(R)) = 3$, so $\operatorname{diam}(\mathbb{AG}_F(R)) \neq 0, 1$. By Lemma 3.1, $\operatorname{diam}(\mathbb{AG}_F(R)) = 2$ or 3. □

Corollary 3.1 *For every non-domain ring R, $\operatorname{diam}(\mathbb{AG}_F(R)) \leqslant \operatorname{diam}(\mathbb{AG}(R))$.*

Proof. Immediate from Proposition 3.1. □

Corollary 3.2 *Let R be a ring. If $\operatorname{diam}(\mathbb{AG}_F(R)) = 2$ or 3, then $(Z(R))^2 \neq (0)$. The converse is also true if $\mathbb{AG}(R) \ncong K_2$.*

Proof. Suppose that $\operatorname{diam}(\mathbb{AG}_F(R)) = 2$ or 3, then there exist $I, J \in \mathbb{A}_F(R)$ such that $IJ \neq (0)$, so for some $a, b \in Z(R)$, $ab \neq (0)$, thus $(Z(R))^2 \neq (0)$. Now assume that $\mathbb{AG}(R) \ncong K_2$, by [11, Theorem 2.7] and Proposition 3.1, $\operatorname{diam}(\mathbb{AG}_F(R)) \neq 0, 1$ and so by Lemma 3.1 $\operatorname{diam}(\mathbb{AG}_F(R)) = 2$ or 3. □

Let H be a subgraph of G. Then it is clear that $\operatorname{gr}(G) \leq \operatorname{gr}(H)$. In the following lemma we show that the converse is also hold for finitely generated annihilating-ideal graph.

Proposition 3.2 *Let R be a ring. Then $\operatorname{gr}(\mathbb{AG}_F(R)) = \operatorname{gr}(\mathbb{AG}(R))$.*

Proof. It is sufficient to prove that $\operatorname{gr}(\mathbb{AG}_F(R)) \leq \operatorname{gr}(\mathbb{AG}(R))$. We know that $\operatorname{gr}(\mathbb{AG}(R)) = \infty, 3$ or 4 (see [11, Theorem 2.1]). If $\operatorname{gr}(\mathbb{AG}(R)) = \infty$, then it is trivial that $\operatorname{gr}(\mathbb{AG}_F(R)) = \infty$. Assume that $\operatorname{gr}(\mathbb{AG}(R)) = 3$ and $I_1 — I_2 — I_3 — I_1$ is a cycle in $\mathbb{AG}(R)$. We claim that $\mathbb{AG}_F(R)$ contains a triangle. We consider the following cases:

case 1: If I_1, I_2 and I_3 are finitely generated, then $\operatorname{gr}(\mathbb{AG}_F(R)) = 3$.

case 2: Suppose that I_1 is not finitely generated and I_2, I_3 are finitely generated. Let $a_1 \in I_1$ and $J_1 = Ra_1$. If $J_1 \neq I_2, I_3$, then $J_1 — I_2 — I_3 — J_1$ is a triangle in $\mathbb{AG}_F(R)$ and hence $\operatorname{gr}(\mathbb{AG}_F(R)) = 3$. Let $J_1 = I_2$, since $I_2 = J_1 \subsetneqq I_1$, there exists $a_2 \in I_1$ such that $a_2 \notin I_2$, so $I_2 = J_1 \subsetneqq Ra_1 + Ra_2 = J_2$. Now if $J_2 \neq I_3$, then $J_2 — I_2 — I_3 — J_2$ is a cycle in $\mathbb{AG}_F(R)$. If $J_2 = I_3$, then there exists $a_3 \in I_1$ such that $I_3 = J_2 \subsetneqq Ra_1 + Ra_2 + Ra_3 = J_3$, in this case $J_3 — I_1 — I_2 — J_3$ is a cycle in $\mathbb{AG}_F(R)$. Therefore in every cases we have a triangle in $\mathbb{AG}_F(R)$ and hence $\operatorname{gr}(\mathbb{AG}_F(R)) = 3$.

case 3: Assume that I_1 and I_2 are not finitely generated and I_3 is a finitely generated ideal of R. Let $a_1 \in I_1$ and $J_1 = Ra_1$. If $J_1 \neq I_3$, then $J_1 — I_3 — I_2 — J_1$ is a triangle in $\mathbb{AG}(R)$, where $J_1, I_3 \in \mathbb{A}_F(R)$ and $I_2 \notin \mathbb{A}_F(R)$. By same argument in case 2, the proof is complete. Now assume that $Ra_1 = J_1 = I_3$. Since $I_1 \notin \mathbb{F}(R)$, there is $a_2 \in I_1$ such that $a_2 \notin J_1 = Ra_1$, so $I_3 = J_1 \subsetneqq Ra_1 + Ra_2 = J_2$. Therefore $J_2 — I_2 — I_3 — J_2$ is a cycle in $\mathbb{AG}_F(R)$ such that $J_2, I_3 \in \mathbb{A}_F(R)$ and $I_2 \notin \mathbb{A}_F(R)$. By same argument in case 2, we have $\operatorname{gr}(\mathbb{AG}_F(R)) = 3$.

case 4: Assume that I_1, I_2 and I_3 are not finitely generated. Let $a \in I_1$, $J = Ra$, by using of same argument in case 2 for triangle $J — I_2 — I_3 — J$ where $J \in \mathbb{A}_F(R)$ and $I_2, I_3 \notin \mathbb{A}_F(R)$, we have $\operatorname{gr}(\mathbb{AG}_F(R)) = 3$.
For case $gr(\mathbb{AG}(R)) = 4$, we have a similar argument and conclude that $\operatorname{gr}(\mathbb{AG}_F(R)) \leq 4$. Therefore in every cases, $\operatorname{gr}(\mathbb{AG}_F(R)) = \operatorname{gr}(\mathbb{AG}(R))$. □

Corollary 3.3 *For every non-domain ring R, if $\mathbb{AG}_F(R)$ contains a cycle, then $\operatorname{gr}(\mathbb{AG}_F(R)) \leqslant 4$.*

Proof. It is clear with Proposition 3.2 and [11, Theorem 2.1]. □

In [2], the authors studied rings for which annihilating-ideal graph is bipartite and star graph, the following two proposition shows that finitely generated annihilating-ideal graph is bipartite (star) if and only if annihilating-ideal graph is bipartite (star). We need the following two lemmas.

Lemma 3.2 *([8, Theorem 3.4]) Let G be a graph. Then G is a bipartite graph if and only if contains no odd cycles.*

Lemma 3.3 *([2, Corollary 25]) Let R be a ring. Then $\mathbb{AG}(R)$ is a bipartite graph if and only if $\mathbb{AG}(R)$ is triangle-free.*

Proposition 3.3 *Let R be a ring. Then $\mathbb{AG}_F(R)$ is a bipartite graph if and only if $\mathbb{AG}(R)$ is a bipartite graph.*

Proof. ($\Rightarrow$) Assume that $\mathbb{AG}_F(R)$ is a bipartite graph. By contrary suppose that $\mathbb{AG}(R)$ is not bipartite. By Lemma 3.3, $\text{gr}(\mathbb{AG}(R)) = 3$. Thus by Proposition 3.2, $\text{gr}(\mathbb{AG}_F(R)) = 3$, which implise that $\mathbb{AG}_F(R)$ contains an odd cycles, so $\mathbb{AG}_F(R)$ is not bipartite (see Lemma 3.2), a contradiction.
($\Leftarrow$) It is trivial. □

Proposition 3.4 *Let R be a ring. Then $\mathbb{AG}(R)$ is a star graph if and only if $\mathbb{AG}_F(R)$ is a star graph.*

Proof. ($\Rightarrow$) Assume that $\mathbb{AG}(R)$ is a star graph. Since $\mathbb{AG}_F(R)$ is an induced subgraph of $\mathbb{AG}(R)$, $\mathbb{AG}_F(R)$ is also a star graph.
($\Leftarrow$) Suppose that $\mathbb{AG}_F(R)$ is a star graph and I is a vertex of $\mathbb{AG}_F(R)$ which is adjacent to every other vertex in $\mathbb{AG}_F(R)$. Let $J \in \mathbb{A}^*(R) \setminus \{I\}$. We claim that J is only adjacent to I. By same argument in Proposition 2.5, $IJ = (0)$. Now assume that there is $K \in \mathbb{A}^*(R) \setminus \{I\}$ such that $KJ = (0)$. Therefore $I - J - K - I$ is a triangle in $\mathbb{AG}(R)$ and so $\text{gr}(\mathbb{AG}(R)) = 3$. By Proposition 3.5, $\text{gr}(\mathbb{AG}_F(R)) = 3$, a contradiction (since $\mathbb{AG}_F(R)$ is a star graph). □

Corollary 3.4 *Let R be a reduced ring. Then the following statements are equivalent.*
(1) *There is a vertex of $\mathbb{AG}_F(R)$ which is adjacent to every other vertex.*
(2) *There is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex.*
(3) *$R \cong F \oplus D$, where F is a field and D is an integral domain.*
(4) *$\mathbb{AG}(R)$ is a star graph.*
(5) *$\mathbb{AG}_F(R)$ is a star graph.*

Proof. Immediate from Proposition 2.5, Proposition 3.4 and [11, Corollary 2.3]. □

Theorem 3.1 *Let R be a ring such that $\text{gr}(\mathbb{AG}_F(R)) = 4$. Then $\mathbb{AG}(R)$ is a complete bipartite graph if and only if $\mathbb{AG}_F(R)$ is a complete bipartite graph.*

Proof. ($\Rightarrow$) It is trivial (since $\mathbb{AG}_F(R)$ is an induced subgraph of $\mathbb{AG}(R)$).

($\Leftarrow$) Assume that $\mathbb{AG}_F(R)$ is a complete bipartite graph with two section $\mathbf{X}$, $\mathbf{Y}$. We claim that $\mathbb{AG}(R)$ is a complete bipartite graph. If $\mathbb{AG}_F(R) = \mathbb{AG}(R)$, then there is nothing to prove. Assume $I \in \mathbb{A}^*(R) \setminus \mathbb{A}_F(R)$. We claim that, either for each $J \in \mathbf{X}$, $IJ = (0)$ or for each $K \in \mathbf{Y}$, $IK = (0)$. Since $\mathbb{AG}(R)$ is a connected graph with $\text{diam}(\mathbb{AG}(R)) \leq 3$ and $\text{gr}(\mathbb{AG}(R)) = 4$ (see [11, Theorem 2.1] and Lemma 3.2), we have only one of the following cases:

case 1: For some $J \in \mathbf{X}$, $IJ = (0)$. In this case we claim that for each $J \in \mathbf{X}$, $IJ = (0)$. By contrary, suppose that for $J_1 \in \mathbf{X}$, $IJ_1 \neq (0)$. So there is $0 \neq x \in I$ such that $(Rx)J_1 \neq (0)$. Since $Rx \in \mathbb{A}_F(R)$, $Rx \in \mathbf{X}$ and hence $(Rx)J \neq (0)$, a contradiction (since $Rx \subseteq I$ and $JI = (0)$).

case 2: For some $K \in \mathbf{Y}$, $IK = (0)$, By similar argument in case 1, for each $K \in \mathbf{Y}$, $IK = (0)$.

case 3: There exists $K \in \mathbb{A}^*(R)$ such that $IK = (0)$, where either for each $J \in \mathbf{X}$, $KJ = (0)$, or for each $L \in \mathbf{Y}$, $KL = (0)$ and for each $J \in \mathbf{X}$, $L \in \mathbf{Y}$, $IJ \neq (0)$, $IL \neq (0)$. Without loss of generality suppose that for every $J \in \mathbf{X}$, $KJ = (0)$. We claim that for each $L \in \mathbf{Y}$, $IL = (0)$, by contrary suppose that for $L_0 \in \mathbf{Y}$, $IL_0 \neq (0)$, so for some $0 \neq x \in I$, $(Rx)L_0 \neq (0)$, since $(Rx) \in \mathbb{A}_F(R)$, $Rx \in \mathbf{Y}$ and $K - J - Rx - K$ form a triangle in $\mathbb{AG}(R)$, a contradiction. Therefore for each $L \in \mathbf{Y}$, $IL = (0)$, a contradiction. So this case implise a contradiction in general.

Therefore for every $I \in \mathbb{A}^*(R) \setminus \mathbb{A}_F(R)$, either $IJ = (0)$ for each $J \in \mathbf{X}$, or $IK = (0)$ for each $K \in \mathbf{Y}$. Let $\overline{\mathbf{X}} = \mathbf{X} \cup \{I \in \mathbb{A}^*(R) : \text{for each } J \in \mathbf{Y}, IJ = (0)\}$ and $\overline{\mathbf{Y}} = \mathbf{Y} \cup \{J \in \mathbb{A}^*(R) : \text{for each } I \in \mathbf{X}, IJ = (0)\}$. Suppose $I \in \overline{\mathbf{X}} \setminus \mathbf{X}$ and $J \in \overline{\mathbf{Y}} \setminus \mathbf{Y}$. By contrary suppose that $IJ \neq (0)$, so there exists $x \in I$ such that $(Rx)J \neq (0)$. Since $Rx \in \mathbb{A}_F(R)$, $Rx \in \mathbf{Y}$ and $(Rx)I = (0)$, but for each $L \in \mathbf{Y}$, $IL = (0)$. Since $Rx \subseteq I$, for each $L \in \mathbf{Y}$, $(Rx)L = (0)$, a contradiction. Therefore $\mathbb{AG}(R)$ is a complete bipartite graph with two section $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$. □

Corollary 3.5 *Let R be a reduced ring such that* $\mathrm{gr}(\mathbb{AG}_F(R)) = 4$ *and* $\mathbb{AG}_F(R)$ *is a complete bipartite graph. Then* $|\mathrm{Min}(R)| = 2$.

Proof. It is clear with [12, Corollary 2.5] and Theorem 3.1. □

Let R be a ring. Then the spectrum graph is tree in every cases, i.e, $\mathrm{gr}(\mathbb{AG}_s(R)) = \infty$ (see [11, Corollary 2.4]). The following proposition shows that, if $I, J \in \mathbb{A}_F(R)$ and $IJ = (0)$, where I and J are not principal ideal, then $\mathbb{AG}_F(R)$ is not a tree.

Proposition 3.5 *Let R be a ring and $G \cong K_2$ is a subgraph of $\mathbb{AG}_F(R)$, with $V(G) = \{I, J\}$, where I and J are not principal ideal. Then* $\mathrm{gr}(\mathbb{AG}_F(R)) \neq \infty$.

Proof. Assume that $V(G) = \{I, J\} \subseteq \mathbb{A}_F(R)$ such that $IJ = (0)$ and I, J are not principal ideal. Thus there exist $0 \neq x \in I$ and $0 \neq y \in J$ such that $Rx \subsetneqq I$ and $Ry \subsetneqq J$. If $Rx = Ry$, then $I — J — Rx — I$ is a triangle in $\mathbb{AG}_F(R)$. If $Rx \neq Ry$, $I — Rx — Ry — J$ is a cycle in $\mathbb{AG}(R)$, so in every cases, $\mathrm{gr}(\mathbb{AG}_F(R)) \in \{3, 4\}$. □

We conclude this paper with the following corollary.

Corollary 3.6 *Let R be a reduced ring and $G \cong K_2$ is a subgraph of $\mathbb{AG}_F(R)$, with $V(G) = \{I, J\}$, where I and J are not principal ideals, then $R \ncong F \oplus D$, where F is a field and D is an integral domain.*

Proof. Immediate from Proposition 3.5, Proposition 3.2 and [11, Corollary 3.11]. □

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