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Resource allocation based on DEA for distance improvement to MPSS points considering environmental factors

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Abstract
This paper proposes a new resource allocation model which is based on data envelopment analysis (DEA) and concerns systems with several homogeneous units operating under supervision of a central unit. The previous studies in DEA literature deal with reallocating/allocating organizational resource to improve performance or maximize the total amount of outputs produced by individual units. In those researches, it is assumed that all data are discretionary. Resource allocation problem has a multiple criteria nature; thus to solve it, many intervening factors should be regarded. This paper not only develops resource allocation plan for systems with both discretionary and non discretionary data in their inputs, but also considers environmental factors as well. In addition, the overall distance from the decision making units (DMUs) to their most productive scale size (MPSS) points is taken into account and is minimized in this method. To find the best allocation plan, this paper applies multiple objective programming (MOLP). Numerical examples are employed to illustrate the application of this approach on real data.
Keywords: Resource allocation, DEA, MOLP, MPSS point, Undesirable outputs, Discretionary inputs.

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1. Introduction

Resource allocation is an important issue in corporation management. In real life, the resources are always limited, so how to allocate them plays a pivotal role in determining a corporation's growth. As a result, resource allocation has been an interesting topic for both corporations and researchers.

Data envelopment analysis (DEA) is a methodology for measuring the relative efficiencies of a set of decision making units (DMUs) that use multiple inputs to produce multiple outputs. It was first introduced by Charnes et al.[1].

DEA is a valid method from both theoretical and empirical sides and it is applicable to management process, performance estimation and behavior analysis. DEA models bring dramatic chances when used as a method for analyzing or solving resource allocation problems. DEA assumes homogeneity among DMUs in terms of the nature of the operations they perform, the measures of their efficiency and the conditions under which they operate. When DMUs are not homogeneous, the efficiency scores may reflect the underlying differences in environments rather than inefficiencies. One strategy to tackle this issue is to separate DMUs into homogenous groups [2].

In recent years, DEA has been wildly used by managerial researchers to study how to better allocate resource. The research about resource allocation using DEA may be classified into two categories. One category assumes the efficiency of DMUs to be constant [3, 4, 5, 6] while the other assumes the efficiency of DMUs to be changeable [4, 7, 8, 9]. In this paper, the latter category where the efficiency of DMUs are changeable is studied. Golany and Tamir [10] have proposed a DEA- based output oriented resource allocation model which includes constraints that imposes upper bounds on the total input consumption of the target points. Care must be taken since constraining the model too much could result in infeasibility. Since this flexible model aims at trading off efficiency, effectiveness and equality in resource consumption, it is rather complex. Athanassopoulos [11] has presented a DEA-based goal programming model (GODEA) for centralized planning. Global targets for the total consumption of each input and the total production of each output are approximated by solving a series of independent DEA models for each input and each output. These goals are usually not simultaneously attainable but deviations from them can be minimized, resulting in an appropriate resource allocation. Although existing DMUs are jointly projected, the individual units are not necessarily projected onto the efficient frontier. Athanassopoulos[12] has also proposed another goal programming model yet this time not based on the envelopment DEA formulation but on the multiplier form. The global targets for each input and output are computed as in GODEA and the deviation from them constitute the highest priority element in the objective function. The second priority is to maximize operational efficiency onto the part of the computed input and output quantities. In the third place, the objective function also tries to minimize the inequality of resource consumption among units. Beasley [7] presents a non-linear resource allocation model to jointly compute inputs and outputs for each DMU for the next period with the objective of maximizing the average efficiency. The approach is based on a non-linear ratio form formulation, (which can lead to alternative optima) and requires explicit upper limits on the total amount of every input and of every output. Nevertheless, there is no guarantee that the projected points lie on the efficient frontier.

Lozano and Villa [13] have proposed two

new models for centralized resource allocation. The first model seeks radial reductions of the total consumption of every input while the second model seeks separate reduction levels for the total amount of each input according to a preference structure. The two key features of the proposed models are their simplicity and the fact that both of them project all DMUs onto the efficient frontier. The radial model jointly projects each of the existing DMUs onto the pareto efficiency frontier. Despite a high probability that for every input the total consumption is necessarily lower than the sum of the input levels of the independently input-oriented projected units, this cannot be proved.

Korhonen and Syrjanen [14] have developed an interactive approach based on DEA and MOLP, which is capable of dealing with several assumptions imposed by DM.

Bi et al. [15] propose a methodology for resource allocation and target setting based on DEA for parallel production systems. It deals with any organization with several production units which have parallel production lines.

Crisp input and output data are fundamentally indispensable in conventional DEA. However, the observed values of the input and output data in real word problems are sometimes imprecise or vague. Many Researchers have proposed various method for dealing with the imprecise and ambiguous data in DEA [16].

Although in most available resource allocation models the key point is to maximize the total amount of outputs produced by individual units, they are not concerned about the improvement in the efficiency score of individual units or even the efficiency of the whole system. Nasrabadi et al. [17] present a model to investigate the resource allocation problem based on efficiency improvement. In their model, the parameters used are not necessarily unique in the case of alternative optimal solution. However, each optimal solution can be applied in this model to achieve performance improvement. This is a shortcoming, of their model, since finding all alternative optimal solutions and solving the model for each one seems unreasonable.

The two factors which often have some relation with each other and play important roles in resource allocation models are economic and environmental factors. Economic factors usually refer to the desirable outputs generated in the production process, such as profit. Environmental factors usually refer to the undesirable outputs such as smoke pollution and waste. Jie and et al. [18] have proposed some new DEA models which consider both economic and environmental factors in the allocation of a given resource. Research on undesirable inputs and outputs has also been actively pursued by means of DEA. Environmental factors are very important in resource allocation, few papers have provided methods in this regard. For the first time, the current paper not only considers environmental and economic factors in resource allocation in the light of DEA, but also it deals with discretionary and non discretionary data, simultaneously.

In the previous literature it was assumed that all inputs and outputs can be varied at the discretion of management or other users. These may be called "discretionary variables" . "Non discretionary variables", which are not subject to management control, may also need to be considered. This paper aims to involve this variable as well.

The assumptions that concern the units' ability to change their input-output mix and efficiency are clearly some of the key factors affecting the results of resource allocation. Although many valuable ideas have been proposed concerning these assumptions, the DMUs ability to change their input-output mix and efficiency has not been discussed thoroughly in the literature. In addition, the multiple criteria nature of the resource allocation problem has drawn only limited attention. Resource allocation, which is a decision problem in which the decision maker (DM) allocates future available resources to a number of similar units, has a multiple criteria nature. This paper aims to decrease the distance from units to their MPSS points through the application of DEA and MOLP.

This study assumes that several homogeneous units operating under the supervision of a central unit such that these units consume some desirable and undesirable inputs to produce very desirable and undesirable outputs. Moreover, a few of the inputs can be of non discretionary type. Thus resource allocation here includes both desirable and undesirable inputs and outputs. The literature in this area may be classified into two categories: direct approaches and indirect ones. Direct approaches are based on the work of Fare et al. [19], which replaced the strong disposability of outputs by the assumption that outputs are weakly disposable, and have then been largely extended. Indirect approaches may be further classified into two groups. The first group deals with the undesirable outputs as inputs for processing [20, 21]; while the second ones transform data for undesirable outputs and then use the traditional efficiency model for their evaluation [22]. The above mentioned indirect approaches are used to handle the undesirable inputs and outputs.

The proposed model intends to address the arisen problems when resources are allocated to various DMUs such that the distance between DMUs and their MPSSs is minimized. On the other hand, the current method improves efficiency and return to scale of DMUs. In the new method not only the economic and environmental factors are considered but also it is assumed that some data are non discretionary.

The rest of this paper is organized as follows:

Some theoretical aspects of MOLP, DEA and MPSS points are discussed in section 2. In section 3, the proposed approach for resource allocation is described. Section 4 shows the application of this method in agriculture and petroleum industry. Next in section 5, the proofs of the theorems from section 3 are stated. Finally, conclusions are presented in section 6.

2. Preliminary Considerations

2.1. Envelopment DEA technology

A system with n DMUs, each consuming m inputs and producing p outputs is considered. Assume a production process in this system where desirable and undesirable inputs are consumed and desirable and undesirable outputs are jointly produced. In addition, some of the inputs probability are discretionary.

Let $x_k = [(x_{ik})_{i\in DDI}, (x_{jk})_{j\in DUI}, (x_{lk})_{l\in NDDI}, (x_{mk})_{m\in NDUI}]$

for $k = 1, \dots, n$, denotes the vector of inputs of $DMU_k; (k = 1, \dots, n)$ which DDI, DUI, NDDI, NDUI are the index sets which include indexes of discretionary desirable inputs, discretionary undesirable inputs, non discretionary desirable inputs and non discretionary undesirable inputs, respectively. Also, denote the vector of outputs of DMU_k by $y_k = [(y_{tk})_{t\in DO}, (y_{sk})_{s\in UO}]$ where DO and UO are the index sets of desirable outputs and undesirable outputs, respectively. The production technology can be described as:

$$P = \{(x^{DDI}, x^{DUI}, x^{NDDI}, x^{NDUI}, y^{DO}, y^{UO})|$$

$$(x^{DDI}, x^{DUI}, x^{NDDI}, x^{NDUI}) \text{ can produce } (y^{DO}, y^{UO})\}$$

In order to reasonably model a production technology that consumes both desirable and undesirable inputs to produce desirable and undesirable outputs, the assumptions proposed by Liu et al are adopted [20]. Under these assumptions, the extended strong disposability can formally be stated as:
If $(x^{DDI}, x^{DUI}, x^{NDDI}, x^{NDUI}, y^{DO}, y^{UO}) \in P$, then for every $(w^{DDI}, w^{DUI}, w^{NDDI}, w^{NDUI}, z^{DO}, z^{UO})$ such that

$$w^{DDI} \geq x^{DDI}, \quad w^{DUI} \leq x^{DUI}, \quad w^{NDDI} \geq x^{NDDI},$$
$$w^{NDUI} \leq x^{NDUI} \text{ and } z^{DO} \leq y^{DO} \quad z^{UO} \geq y^{UO},$$

we will have:
$(w^{DDI}, w^{DUI}, w^{NDDI}, w^{NDUI}, z^{DO}, z^{UO}) \in P$.
Hence, the corresponding production possibility set (PPS) is as follows:

$$P = \left\{ (x^{DDI}, x^{DUI}, x^{NDDI}, x^{NDUI}, y^{DO}, y^{UO}) \left| \begin{array}{l} x^{DDI} \geq \sum_{j=1}^{n} \lambda_j x_j^{DDI} \\ x^{DUI} \leq \sum_{j=1}^{n} \lambda_j x_j^{DUI} \\ y^{DO} \leq \sum_{j=1}^{n} \lambda_j y_j^{DO} \\ y^{UO} \geq \sum_{j=1}^{n} \lambda_j y_j^{UO} \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \ldots, n. \end{array} \right. \right\}$$

2.2. Most Productive Scale Size (MPSS) points

Definition 1.(Banker's definition [23]): (x_o, y_o) is MPSS if and only if for every $(\alpha x_o, \beta y_o) \in P$ we have $\alpha \geq \beta$.
Jahanshahloo and Khodabakhshi [24] used the input-output orientation model for determining MPSS points corresponding to DMUs. For this aim, they solved the following model:

$$\begin{array}{ll} \text{Max} & \varphi_o - \theta_o \\ \text{s.t} & \varphi_o y_o \leq \sum_{j=1}^{n} \lambda_j y_j \\ & \theta_o x_o \geq \sum_{j=1}^{n} \lambda_j x_j \\ & \sum_{j=1}^{n} \lambda_j = 1 \qquad (1-\text{I}) \\ & \lambda_j \geq 0; j = 1, \ldots, n. \end{array}$$

Theorem 1. DMU_o is MPSS if and only if the two following conditions are satisfied:
(a) The optimal amount of objective function is zero.
(b) The amounts of slacks in alternative optimal solutions are zero.
Proof: Refer to Reference[24].
For determining MPSS points using model $(1-\text{I})$, the non- Archimedean form is applied. The non- Archimedean model is defined as follows:

$$\begin{array}{ll} \text{Max} & \varphi_o - \theta_o + \varepsilon(1.S^- + 1.S^+) \\ \text{s.t} & X\lambda + S^- - \theta_o x_o = 0 \qquad (1-\text{II}) \\ & -Y\lambda + S^+ + \varphi_o y_o = 0 \\ & 1.\lambda = 1 \\ & \lambda \geq 0, S^- \geq 0, S^+ \geq 0 \end{array}$$

This model is solved by a pre-emptive approach such that at first the max $(\varphi_o - \theta_o)$ is obtained without any attention to slacks and then in the second stage the slacks are maximized by fixing φ_o^*, θ_o^* amounts instead of φ_o, θ_o. Therefore, in this approach it is not necessary to devote any amount for ε.
Remark 1. For DMU_o with (x_o, y_o) input and output combinations, figurative DMU with $(\theta^* x_o - S^{-*}, \varphi^* y_o + S^{+*})$ input and output combinations is MPSS.
Also Khodabakhshi [25] estimates the most productive scale size in stochastic data envelopment analysis (DEA).To estimate the most productive scale size

with stochastic data, he develops the input-output orientation model that was introduced by Jahanshahloo and Khodabakhshi [24] in classic DEA in stochastic data envelopment analysis. The deterministic equivalent of stochastic model is obtained which is generally non-linear, and it can be converted to a quadratic problem.

In the following, we extend this model for desirable and undesirable inputs and outputs.

First, the following model is solved:

$$
\begin{aligned}
\text{Max} \quad & \varphi_{1o} - \theta_{1o} - \varphi_{2o} + \theta_{2o} && \text{(2–I)} \\
\text{s.t} \quad & \varphi_{1o} y_{ro} \le \sum_{j=1}^{n} \lambda_j y_{rj} && ; r \in DO \\
& \theta_{1o} x_{io} \ge \sum_{j=1}^{n} \lambda_j x_{ij} && ; i \in DDI \\
& \varphi_{2o} y_{ro} \ge \sum_{j=1}^{n} \lambda_j y_{rj} && ; r \in UO \\
& \theta_{2o} x_{io} \le \sum_{j=1}^{n} \lambda_j x_{ij} && ; i \in DUI \\
& x_{io} \ge \sum_{j=1}^{n} \lambda_j x_{ij} && ; i \in NDDI \\
& x_{io} \le \sum_{j=1}^{n} \lambda_j x_{ij} && ; i \in NDUI \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 && j = 1, \dots, n.
\end{aligned}
$$

Then the following model is solved:

$$
\begin{aligned}
\text{Max} \quad & \Sigma_{i \in DDI}\, s_i + \Sigma_{i \in DUI}\, \overline{s_i} + \Sigma_{r \in DO}\, t_r + \Sigma_{r \in UO}\, \overline{t_r} + \Sigma_{i \in NDDI}\, \widehat{s_i} + \Sigma_{i \in NDUI}\, \widetilde{s_i} \\
\text{s.t} \quad & \varphi_{1o}{}^{*} y_{ro} = \sum_{j=1}^{n} \lambda_j y_{rj} - t_r; \ r \in DO \\
& \theta_{1o}{}^{*} x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i; i \in DDI \\
& \varphi_{2o}{}^{*} y_{ro} = \sum_{j=1}^{n} \lambda_j y_{rj} + \overline{t_r}; \ r \in UO && \text{(2-II)} \\
& \theta_{2o}{}^{*} x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} - \overline{s_i}; i \in DUI \\
& x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + \widehat{s_i}; i \in NDDI \\
& x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} - \widetilde{s_i}; i \in NDUI \\
& \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \ge 0; j = 1, \dots, n.
\end{aligned}
$$

Similarly, it can be proven for DMU_o with $(x_o^{DDI}, x_o^{DUI}, x_o^{NDDI}, x_o^{NDUI}, y_o^{DO}, y_o^{UO})$ combination, figurative DMU with the following form:

$$(\theta_1^* x_o^{DDI} - S^*, \theta_2^* x_o^{DUI} + \overline{S^*}, x_o^{NDDI} - \widehat{S^*}, x_o^{NDUI} + \widetilde{S^*}, \varphi_1^* y_o^{DO} + t^*, \varphi_2^* y_o^{UO} - \overline{t^*})$$

is MPSS.

Here, for determining the distance between DMU_o and its MPSS, the following formulation is applied:

$$
\begin{aligned}
\rho_o = {} & \Sigma_{i \in DDI}\, (x_{io}(1 - \theta_{1o}^*) + s_i^*) + \\
& \Sigma_{i \in DUI}\, (x_{io}(\theta_{2o}^* - 1) + \overline{s_i^*}) + \\
& \Sigma_{r \in DO} (y_{ro}(\varphi_{1o}^* - 1) + t_r^*) + && (3) \\
& \Sigma_{r \in UO} (y_{ro}(1 - \varphi_{2o}^*) + \overline{t_r^*}) + \\
& \Sigma_{i \in NDDI}\, \widehat{s_i^*} + \Sigma_{i \in NDUI}\, \widetilde{s_i^*}.
\end{aligned}
$$

2.3 Multi-objective programming

Here, some fundamentals of MOP problems and the weighted sum method for solving them are reviewed, which will be used throughout the remainder of this paper.

The MOP problem can be presented as follows:

$$
\begin{aligned}
\max \quad & f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\
\text{s.t.} \quad & \\
& x \in \chi && (4)
\end{aligned}
$$

where χ is a feasible set of an optimization problem (4) and $f_k: \chi \to \mathbb{R}$ for $k = 1, \dots, p$ are criteria or objective functions. The fundamental importance of efficiency (Pareto optimality) is based on the observation that any x which is not efficient cannot represent a most preferred alternative for a DM, because there exists at least one other feasible solution x' such that $f_k(x') \ge f_k(x)$ for all $k = 1, \dots, p$, where strict inequality holds at least once, i.e., x' should clearly be preferred to x.

The multi-objective linear programming (MOLP) problems are specified by linear functions which are to be maximized subject to a set of linear constraints. The standard form of MOLP can be written as follows:

$$\begin{aligned} \max \quad & f(x) = Cx \\ \text{s.t} \quad & \\ & x \in \chi = \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\} \end{aligned} \tag{5}$$

where C is an $p \times n$ objective function matrix, A is $m \times n$ constraint matrix, b is an m- vector of right hand side, and x is an n-vector of decision variables.

Definition 2. (Ehrgott [26]) Let x^* be a feasible solution to the problem (5), if there is no feasible solution x of (5) such that $(Cx^*)_k \leq (Cx)_k$ for all $k = 1,2,\ldots,p$ and $(Cx^*)_k < (Cx)_k$ for at least one i, then we say x^* is a strongly efficient solution of (5).

There are many methods for solving MOLP problems. One of the non scalarizing methods is weighted sum method. In this method, an MOP problem (5) can be solved (i.e. its efficient solutions be found) by solving a single objective problem of this type:

$$\max \quad \{\lambda^T Cx \mid x \in \chi\} \tag{6}$$

where $\lambda_k \in \Lambda$ and $\Lambda = \{\lambda \in \mathbb{R}^k \mid \lambda_i \geq 0, \Sigma_{i=1}^k \lambda_i = 1\}$.

Theorem 2: $\bar{x} \in \chi$ is efficient if and only if there exists a $\bar{\lambda} \in \Lambda$ such that $\bar{x}$ is a maximized solution of $\max\{\lambda^T Cx \mid x \in \chi\}$.

Proof: Refer to Reference [27].

3. Development of resource allocation model

In this section, a new model for the resource allocation is proposed. Here resource allocation means a decision problem in which the decision maker wishes to allocate extra resources as new inputs and additional market demands as new outputs for a set of homogeneous units operating in a system to achieve more small distances to MPSS points of these units.

Assume that $(x_k^{DDI} + \Delta x_k^{DDI}, x_k^{DUI} + \Delta x_k^{DUI}, x_k^{NDDI}, x_k^{NDUI}, y_k^{DO} + \Delta y_k^{DO}, y_k^{UO} + \Delta y_k^{UO})$ represents the activity vector of DMU_k after the planning period, in which Δx_k^{DDI} and Δx_k^{DUI} denote the vector of discretionary desirable input changes and discretionary undesirable input changes of DMU_k, for $k = 1,\ldots,n$ respectively. Also, Δy_k^{DO} and Δy_k^{UO} denote the vector of desirable output changes and undesirable output changes of DMU_k, for $k = 1,\ldots,n$.

Definition 3. The system containing $DMU_1,\ldots,DMU_n$ has a smaller distance to MPSS points, after the planning period if $(\frac{\rho_k}{\rho_{k\prime}}) > 1$ for $k: 1,\ldots n$ where ρ_k and $\rho_{k\prime}$ are the distance measure between DMU_k and its MPSS, before and after the planning period, respectively.

It should be maximized $\{\frac{\rho_1}{\rho_1},\ldots,\frac{\rho_k}{\rho_{k\prime}},\ldots,\frac{\rho_n}{\rho_{n\prime}}\}$, while maintaining the feasibility of all units and imposing the DM's constraints, to achieve a smaller distance to MPSS points.

$$\begin{aligned} \text{Max} \quad & \{\frac{\rho_1}{\rho_1},\ldots,\frac{\rho_k}{\rho_{k\prime}},\ldots,\frac{\rho_n}{\rho_{n\prime}}\} \\ \text{s.t} \quad & (x_k^{DDI} + \Delta x_k^{DDI}, x_k^{DUI} + \Delta x_k^{DUI}, \\ & x_k^{NDDI}, x_k^{NDUI}, y_k^{DO} + \Delta y_k^{DO}, y_k^{UO} + \Delta y_k^{UO}) \in P \\ & ; k = 1,\ldots,n \\ & (\Delta x_1^{DDI}, \Delta x_2^{DDI},\ldots,\Delta x_n^{DDI}) \in \Delta^{DDI} \\ & (\Delta x_1^{DUI}, \Delta x_2^{DUI},\ldots,\Delta x_n^{DUI}) \in \Delta^{DUI} \\ & (\Delta y_1^{DO}, \Delta y_2^{DO},\ldots,\Delta y_n^{DO}) \in \Delta^{Do} \\ & (\Delta y_1^{UO}, \Delta x_2^{UO},\ldots,\Delta x_n^{UO}) \in \Delta^{UO}. \end{aligned} \tag{7}$$

Model (7) has n objectives and is a multiple objective problem. To solve this model, the weighted sum method is used. Therefore the following model is obtained:

$$\begin{aligned} &\text{Max} \quad \sum_{k=1}^{n} w_k\left(\frac{\rho_k}{\rho_{k'}}\right) \\ &\text{s.t} \quad \text{Constraints of Model (7),} \end{aligned} \tag{8}$$

where w_ks are positive weights representing the DM's preferences such that $\sum_{k=1}^{n} w_k = 1$.

Here, it is important to know, if a decrease in the distance measure occurs from all units to their MPSS points, the system will surely have a smaller distance, whereas, in general the converse does not hold true. For considering this fact in the new model, it was assumed that the DM is not willing to have deterioration in distance measure from any units to its MPSS, which implies $\rho_{k'} \le \rho_k$ for $k = 1, \ldots, n$. Then the resource allocation model is introduced as follows:

$$\begin{aligned} &\text{Max} \quad \sum_{k=1}^{n} w_k\left(\frac{\rho_k}{\rho_{k'}}\right) \\ &\text{s.t} \quad \text{Constraints of Model (7),} \\ &\qquad \rho_{k'} \le \rho_k; \quad k = 1, \ldots, n \end{aligned} \tag{9}$$

where sets $\Delta^{DDI}, \Delta^{DUI}, \Delta^{DO}, \Delta^{UO}$, refer to the restrictions imposed on the vectors of discretionary desirable and undesirable input changes and desirable and undesirable output changes, respectively.

Note that we can choose these sets according to the DM's preferences. Here we consider them as follows:

$\Delta^{DDI} = \{(\Delta x_{i1}^{DDI}, \Delta x_{i2}^{DDI}, \ldots, \Delta x_{in}^{DDI})|$
$\alpha_{ik} \le \Delta x_{ik}^{DDI} \le \beta_{ik}$; $k = 1, \ldots, n$
$\Sigma_{k=1}^{n} \Delta x_{ik}^{DDI} = c_i$; $i \in DDI\}$

$\Delta^{DUI} = \{(\Delta x_{i1}^{DUI}, \Delta x_{i2}^{DUI}, \ldots, \Delta x_{in}^{DUI})|$
$\alpha_{ik}' \le \Delta x_{ik}^{DUI} \le \beta_{ik}'$; $k = 1, \ldots, n$

$, \Sigma_{k=1}^{n} \Delta x_{ik}^{DUI} = \acute{c}_i$; $i \in DUI\}$
$\Delta^{DO} = \{(\Delta y_{r1}^{DO}, \Delta y_{r2}^{DO}, \ldots, \Delta y_{rn}^{DO})|$
$\gamma_{rk} \le \Delta y_{rk}^{DO} \le \delta_{rk}$; $k = 1, \ldots, n$
$, \Sigma_{k=1}^{n} \Delta y_{rk}^{DO} = d_r$; $r \in Do\}$

$\Delta^{UO} = \{(\Delta y_{r1}^{UO}, \Delta y_{r2}^{UO}, \ldots, \Delta y_{rn}^{UO})|$
$\gamma_{rk}' \le \Delta y_{rk}^{UO} \le \delta_{rk}'$; $k = 1, \ldots, n$
$\Sigma_{k=1}^{n} \Delta y_{rk}^{UO} = \acute{d}_r$; $r \in Uo\}$.

Using resource allocation model (9), first models (2-I) and (2-II) should be solved to determine MPSS points, for $o = 1, \ldots, n$. Then, by applying the obtained optimal solutions to these models in equation (3) , ρ_k ; $k = 1, \ldots, n$ are found. On the other hand, for finding $\acute{\rho}_k; = 1, \ldots, n$ the following models should be solved:

$$\begin{aligned} &\text{Max} \quad \eta_{1o} - \eta_{2o} - \zeta_{1o} + \zeta_{2o} \\ &\text{s.t} \quad \eta_{1o}(y_{ro} + \Delta y_{ro}) \le \sum_{j=1}^{n} \lambda_j y_{rj} \quad ; r \in DO \\ &\qquad \zeta_{1o}(x_{io} + \Delta x_{io}) \ge \sum_{j=1}^{n} \lambda_j x_{ij} \quad ; i \in DDI \\ &\qquad \eta_{2o}(y_{ro} + \Delta y_{ro}) \ge \sum_{j=1}^{n} \lambda_j y_{rj} \quad ; r \in UO \\ &\qquad \zeta_{2o}(x_{io} + \Delta x_{io}) \le \sum_{j=1}^{n} \lambda_j x_{ij} \quad ; i \in DUI \\ &\qquad x_{io} \ge \sum_{j=1}^{n} \lambda_j x_{ij} \quad ; i \in NDDI \\ &\qquad x_{io} \le \sum_{j=1}^{n} \lambda_j x_{ij} \quad ; i \in NDUI \\ &\qquad \sum_{j=1}^{n} \lambda_j = 1 \\ &\qquad \lambda_j \ge 0, \ j = 1, \ldots, n \\ &\qquad \Delta(x_{ik}) \in \Delta^{DDI} \\ &\qquad i \in DDI, \ k: 1, \ldots, n \\ &\qquad \Delta(x_{ik}) \in \Delta^{DDI} \\ &\qquad i \in DUI, \ k: 1, \ldots, n \\ &\qquad \Delta(y_{rk}) \in \Delta^{DO} \\ &\qquad r \in DO, \ k: 1, \ldots, n \\ &\qquad \Delta(y_{rk}) \in \Delta^{UO} \\ &\qquad i \in UO, \ k: 1, \ldots, n \end{aligned} \tag{10–I}$$

And

$$\begin{aligned} &\text{Max} \quad \Sigma_{i \in DDI} \ l_i + \Sigma_{i \in DUI} \ \bar{l}_i + \\ &\qquad \Sigma_{r \in DO} \ h_r + \Sigma_{r \in UO} \ \bar{h}_r + \\ &\qquad \Sigma_{i \in NDDI} \ \hat{l}_i + \Sigma_{i \in NDUI} \ \tilde{l}_i \\ &\text{s.t} \\ &\eta_{1o}{}^{*}(y_{ro} + \Delta y_{ro}) = \sum_{j=1}^{n} \lambda_j y_{rj} - h_r ; \ r \in DO \\ &\zeta_{1o}{}^{*}(x_{io} + \Delta x_{io}) = \sum_{j=1}^{n} \lambda_j x_{ij} + l_i ; i \in DDI \end{aligned}$$

$$\eta_{2o}{}^{*}(y_{ro}+\Delta y_{ro})=\sum_{j=1}^{n}\lambda_j y_{rj}+\overline{h_r};\ r\in UO$$

$$\zeta_{2o}{}^{*}(x_{io}+\Delta x_{io})=\sum_{j=1}^{n}\lambda_j x_{ij}-\bar{l}_i;\ i\in DUI \quad (10-II)$$

$$x_{io}=\sum_{j=1}^{n}\lambda_j x_{ij}+\hat{l}_i;\quad i\in NDDI$$
$$x_{io}=\sum_{j=1}^{n}\lambda_j x_{ij}-\tilde{l}_i;\ i\in NDUI$$
$$\sum_{j=1}^{n}\lambda_j=1$$
$$\lambda_j\geq 0;\ j=1,\dots,n$$
$$\Delta(x_{ik})\in\Delta^{DDI};\ i\in DDI,\quad k:1,\dots,n$$
$$\Delta(x_{ik})\in\Delta^{DDI};\ i\in DUI,\quad k:1,\dots,n$$
$$\Delta(y_{rk})\in\Delta^{DO};\ r\in DO,\quad k:1,\dots,n$$
$$\Delta(y_{rk})\in\Delta^{UO};\ i\in UO,\quad k:1,\dots,n.$$

These models are the parametric linear program and their optimal solutions are a function of $\Delta x_o^{DDI}=(\Delta x_{io})_{i\in DDI}$, $\Delta x_o^{DUI}=(\Delta x_{io})_{i\in DUI}$, $\Delta y_o^{DO}=(\Delta y_{ro})_{r\in DO}$, and $\Delta y_o^{UO}=(\Delta y_{ro})_{r\in UO}$. Then ρ_o, can be obtained as follows:

$$\acute{\rho_o}=\Sigma_{i\in DDI}\ [\ (x_{io}+\Delta x_{io})(1-\zeta_{1o}^{*})+l_i^{*}\ ]\ +\ \Sigma_{i\in DUI}\ [\ (x_{io}+\Delta x_{io})(\zeta_{2o}^{*}-1)+\overline{l_i^{*}}\ ]\ +\Sigma_{r\in DO}\ [\ (y_{ro}+\Delta y_{ro})(\eta_{1o}^{*}-1)+h_r^{*}]+\Sigma_{r\in UO}\ [(y_{ro}+\Delta y_{ro})(1-\eta_{2o}^{*})+\overline{h_r^{*}}\ ]+\Sigma_{i\in NDDI}\ \widehat{l_i^{*}}\ +\ \Sigma_{i\in NDUI}\ \widetilde{l_i^{*}}. \quad (11)$$

Theorem 3. Model (2-I) and model(10-I) are the feasible and bounded problems.
Proof: Refer to Section 5.

Conclusion 1. The dual problems of models (2-I) and (10-I) have the finite optimal solutions.
In the following, we introduce the Theorem 3, which states that the mathematical relationship between MPSS points corresponding to DMU_o, before and after resource allocation.

Theorem 4.
If $((\lambda_j^{*})_{j=1,\dots,n};\varphi_{1o}^{*};\theta_{1o}^{*};\ \varphi_{2o}^{*};\ \theta_{2o}^{*})$ is an optimal solution to the model (2-I), then for every $j=1,\dots,n$, there exists μ_{jo} such that $((\widehat{\lambda_j}^{*}=\mu_{jo}\lambda_j^{*})_{j=1,\dots,n};\eta_{1o}^{*};\zeta_{1o}^{*};\ \eta_{2o}^{*};\ \zeta_{2o}^{*})$ is an optimal solution to the model(10-I); which is introduced as follows :

$$\widehat{\lambda_j}^{*}=\mu_{jo}\lambda_j^{*}\quad;j=1,\dots,n$$
$$\eta_{1o}^{*}=\min_{r\in DO}\left\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}y_{rj}}{y_{ro}+\Delta y_{ro}}\right\};$$
$$\zeta_{1o}^{*}=\max_{i\in DDI}\left\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}x_{ij}}{x_{io}+\Delta x_{io}}\right\}$$
$$\eta_{2o}^{*}=\max_{r\in UO}\left\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}y_{rj}}{y_{ro}+\Delta y_{ro}}\right\};$$
$$\zeta_{2o}^{*}=\min_{i\in DUI}\left\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}x_{ij}}{x_{io}+\Delta x_{io}}\right\}.$$

Proof: Refer to Section 5.

Conclusion 2. Based on Theorem 1, if $((\lambda_j^{*})_{j=1,\dots,n};\varphi_{1o}^{*};\theta_{1o}^{*};\varphi_{2o}^{*};\theta_{2o}^{*};(t_r^{*})_{r\in DO};(s_i^{*})_{i\in DDI};(\overline{t_r}^{*})_{r\in UO};(\overline{s_i}^{*})_{i\in DUI};(\widehat{s_i}^{*})_{i\in NDDI};(\widetilde{s_i}^{*})_{i\in NDUI})$ is an optimal solution to a non-Archimedean form of the model(2-I) and model (2-II), then: $((\widehat{\lambda_j}^{*})_{j=1,\dots,n};\eta_{1o}^{*};\zeta_{1o}^{*};\eta_{2o}^{*};\zeta_{2o}^{*};(h_r^{*})_{r\in DO};(l_i^{*})_{i\in DDI};(\overline{h_r}^{*})_{r\in UO};(\overline{l_i}^{*})_{i\in DUI};(\widehat{l_i}^{*})_{i\in NDDI};(\widetilde{l_i}^{*})_{i\in NDUI})$ is an optimal solution to a non-Archimedean form of the model (10-I) and model(10-II), such that:

$$t_r^{*}=\sum_{j=1}^{n}\ \lambda_j^{*}y_{rj}-\varphi_{1o}^{*}\ y_{ro};\ \ r\in DO$$
and $\overline{t_r}^{*}=\varphi_{2o}^{*}\,y_{ro}-\sum_{j=1}^{n}\ \lambda_j^{*}y_{rj};\ r\in UO$

$$s_i{}^{*}=\theta_{1o}^{*}\ x_{io}-\sum_{j=1}^{n}\ \lambda_j^{*}x_{ij};\ i\in DDI$$
and $\overline{s_i}^{*}=\sum_{j=1}^{n}\lambda_j^{*}x_{ij}-\theta_{2o}^{*}\ x_{io};\ \ i\in DUI$
$\widehat{s_i}^{*}=x_{io}-\sum_{j=1}^{n}\ \lambda_j^{*}x_{ij};\quad i\in NDDI$and

$$h_r^{*}=\sum_{j=1}^{n}\widehat{\lambda_j^{*}}y_{rj}-\min_{r\in DO}\left\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}y_{rj}}{y_{ro}+\Delta y_{ro}}\right\}(y_{ro}+\Delta y_{ro});\ \ r\in DO$$
$$\overline{h_r}^{*}=\max_{r\in UO}\left\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}y_{rj}}{y_{ro}+\Delta y_{ro}}\right\}(y_{ro}+\Delta y_{ro})-\sum_{j=1}^{n}\ \widehat{\lambda_j^{*}}y_{rj};\ \ r\in UO$$

$\underline{l_i}^* = \max_{i\in DDI}\{ \frac{\sum_{j=1}^{n} \widehat{\lambda_j}^* x_{ij}}{x_{io}+\Delta x_{io}}\}\}\ (x_{io} + \Delta x_{io}) - \sum_{j=1}^{n} \widehat{\lambda_j^*} x_{ij};\ i \in DDI$

$\overline{l_i}^* = \sum_{j=1}^{n} \widehat{\lambda_j^*} x_{ij} - \min_{i\in DUI}\{ \frac{\sum_{j=1}^{n} \widehat{\lambda_j}^* x_{ij}}{x_{io}+\Delta x_{io}}\}\}\ (x_{io} + \Delta x_{io});\ i \in DUI$

$\widehat{l_i}^* = x_{io} - \sum_{j=1}^{n} \widehat{\lambda_j^*} x_{ij};\ i \in NDDI$

and $\widetilde{l_i}^* = \sum_{j=1}^{n} \widehat{\lambda_j^*} x_{ij} - x_{io};\ i \in NDUI.$

Based on Theorem 3 and Conclusion 2, , there s no need to solve the parametric models(10-I) and (10-II) and by using the optimal solution to the models (2-I) and (2-II), the resource allocation model (9) is as follows:

$$\text{Max} \sum_{k=1}^{n} w_k \left(\frac{\begin{bmatrix} \sum_{i\in DDI} [x_{ik}(1-\theta_{1o}^*)+s_i^*\ ] + \\ \sum_{i\in DUI} [x_{ik}(\theta_{2k}^*-1)+\overline{s_i^*}\ ] \end{bmatrix}}{\begin{bmatrix} \sum_{i\in DDI} [\ (x_{ik}+\Delta x_{ik})(1-\zeta_{1k}^*)+l_i^*\ ] + \\ \sum_{i\in DUI} [\ (x_{ik}+\Delta x_{ik})(\zeta_{2k}^*-1)+\overline{l_i^*}\ ] \end{bmatrix}} + \frac{\begin{bmatrix} \sum_{r\in DO} [y_{rk}(\varphi_{1k}^*-1)+t_r^*\ ] + \\ \sum_{r\in UO} [y_{rk}(1-\varphi_{2k}^*)+\overline{t_r^*}\ ] \end{bmatrix}}{\begin{bmatrix} \sum_{r\in DO} [\ (y_{rk}+\Delta y_{rk})(\eta_{1k}^*-1)+h_r^*] + \\ \sum_{r\in UO} [\ (y_{rk}+\Delta y_{rk})(1-\eta_{2k}^*)+\overline{h_r^*}\ ] \end{bmatrix}} + \frac{\sum_{i\in NDDI} \widehat{s_i^*} + \sum_{i\in NDUI} \widetilde{s_i^*}}{\sum_{i\in NDDI} \widehat{l_i^*} + \sum_{i\in NDUI} \widetilde{l_i^*}} \right)$$

S.t. $(y_{rk} + \Delta y_{rk}) \le \sum_{j=1}^{n} \lambda_j^k y_{rj}$
$;r \in DO, k = 1,\dots,n$
$(x_{ik} + \Delta x_{ik}) \ge \sum_{j=1}^{n} \lambda_j^k x_{ij}$
$;i \in DDI, k = 1,\dots,n$
$(y_{rk} + \Delta y_{rk}) \ge \sum_{j=1}^{n} \lambda_j^k y_{rj}$
$;r \in UO, k = 1,\dots,n$
$(x_{ik} + \Delta x_{ik}) \le \sum_{j=1}^{n} \lambda_j^k x_{ij}$
$;i \in DUI, k = 1,\dots,n$
$x_{ik} \ge \sum_{j=1}^{n} \lambda_j^k x_{ij}$
$;i \in NDDI, k = 1,\dots,n$
$x_{ik} \le \sum_{j=1}^{n} \lambda_j^k x_{ij}$
$;i \in NDUI, k = 1,\dots,n$
$\sum_{j=1}^{n} \lambda_j^k = 1\ ;k = 1,\dots,n$
$\sum_{i\in DDI} [x_{ik}(1-\theta_{1o}^*)+s_i^*\ ] + \sum_{i\in DUI} [x_{ik}(\theta_{2k}^*-1)+\overline{s_i^*}\ ] + \sum_{r\in DO} [y_{rk}(\varphi_{1k}^*-1)+t_r^* + \sum_{r\in UO} [y_{rk}(1-\varphi_{2k}^*)+\overline{t_r^*} + \sum_{i\in NDDI} \widehat{s_i^*} + \sum_{i\in NDUI} \widetilde{s_i^*}$
$\ge$
$\sum_{i\in DDI} [(x_{ik}+\Delta x_{ik})(1-\zeta_{1k}^*)+l_i^*\ ] + \sum_{i\in DUI} [\ (x_{ik}+\Delta x_{ik})(\zeta_{2k}^*-1)+\overline{l_i^*}\ ] + \sum_{r\in DO} [\ (y_{rk}+\Delta y_{rk})(\eta_{1k}^*-1)+h_r^*] + \sum_{r\in UO} [\ (y_{rk}+\Delta y_{rk})(1-\eta_{2k}^*)+\overline{h_r^*}\ ] + +\sum_{i\in NDDI} \widehat{l_i^*} + \sum_{i\in NDUI} \widetilde{l_i^*};\ k = 1,\dots,n$
$(\Delta x_1^{DDI}, \Delta x_2^{DDI},\dots,\Delta x_n^{DDI}) \in \Delta^{DDI}$
$(\Delta x_1^{DUI}, \Delta x_2^{DUI},\dots,\Delta x_n^{DUI}) \in \Delta^{DUI}$
$(\Delta y_1^{DO}, \Delta y_2^{DO},\dots,\Delta y_n^{DO}) \in \Delta^{Do}$
$(\Delta y_1^{UO}, \Delta x_2^{UO},\dots,\Delta x_n^{UO}) \in \Delta^{UO}.$

Remark 2. Note that the objective function of the above model is fractional but based on equation (3)

$$\sum_{r\in DO} [\ y_{rk}(\varphi_{1k}^*-1)+t_r^*\ \ ] + \sum_{r\in UO} [y_{rk}(1-\varphi_{2k}^*) + \overline{t_r^*}\ ]\ \sum_{i\in NDDI} \widehat{s_i^*} + \sum_{i\in NDUI} \widetilde{s_i^*} = \rho_k^* \tag{13}$$

therefore the numerator of this function is a constant value; thus, by considering:

$$\sum_{i\in DDI} [(x_{ik}+\Delta x_{ik})(1-\zeta_{1k}^*)+l_i^*\ ] + \sum_{i\in DUI} [\ (x_{ik}+\Delta x_{ik})(\zeta_{2k}^*-1)+\overline{l_i^*}\ ] + \sum_{r\in DO} [(y_{rk}+\Delta y_{rk})(\eta_{1k}^*-1) + h_r^*] + \sum_{r\in UO} [\ (y_{rk}+\Delta y_{rk})(1-\eta_{2k}^*)+\overline{h_r^*}\ ] + \sum_{i\in NDDI} \widehat{l_i^*} + \sum_{i\in NDUI} \widetilde{l_i^*} = \rho'_k, \tag{14}$$

we can rewrite this model as follows:

Max $\sum_{k=1}^{n} w_k\ \rho'_k$
S.t. $(y_{rk} + \Delta y_{rk}) \le \sum_{j=1}^{n} \lambda_j^k y_{rj}$
$;r \in DO, k = 1,\dots,n$
$(x_{ik} + \Delta x_{ik}) \ge \sum_{j=1}^{n} \lambda_j^k x_{ij}$
$;i \in DDI, k = 1,\dots,n$
$(y_{rk} + \Delta y_{rk}) \ge \sum_{j=1}^{n} \lambda_j^k y_{rj}$
$;r \in UO, k = 1,\dots,n$
$(y_{rk} + \Delta y_{rk}) \ge \sum_{j=1}^{n} \lambda_j^k y_{rj}$
$;r \in UO, k = 1,\dots,n$
$x_{ik} \ge \sum_{j=1}^{n} \lambda_j^k x_{ij}$
$;i \in NDDI, k = 1,\dots,n$ (15)
$x_{ik} \le \sum_{j=1}^{n} \lambda_j^k x_{ij}$

$; i \in NDUI, k = 1, \dots, n$
$\sum_{j=1}^{n} \lambda_j^k = 1;$
$k = 1, \dots, n$
$\rho'_k \le \acute{\rho}_k^*$
$\lambda_j^k \ge 0; \quad k = 1, \dots, n$
$(\Delta x_1^{DDI}, \Delta x_2^{DDI}, \dots, \Delta x_n^{DDI}) \in \Delta^{DDI}$
$(\Delta x_1^{DUI}, \Delta x_2^{DUI}, \dots, \Delta x_n^{DUI}) \in \Delta^{DUI}$
$(\Delta y_1^{DO}, \Delta y_2^{DO}, \dots, \Delta y_n^{DO}) \in \Delta^{Do}$
$(\Delta y_1^{UO}, \Delta x_2^{UO}, \dots, \Delta x_n^{UO}) \in \Delta^{UO}.$

Remark 3. For using of formulations (13), (14), we can use Theorem 4 and replace the amounts of $\zeta_{1k}^*, \zeta_{2k}^*, \eta_{1k}^*, \eta_{2k}^*$ for $k = 1, \dots, n$ in this model.
However, by considering the sets Δ^{DDI}, Δ^{DUI}, Δ^{DO} and Δ^{UO} for every $i \in DDI,\ r \in DO,$ we have:

$$-\max_{i\in DDI}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\alpha_{io}}\right\} \le -\max_{i\in DDI}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\Delta x_{io}}\right\} \le -\max_{i\in DDI}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\beta_{io}}\right\}$$

$$\min_{r\in DO}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\delta_{ro}}\right\} \le \min_{r\in DO}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\Delta y_{ro}}\right\} \le \min_{r\in DO}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\gamma_{ro}}\right\}$$

and for every $i \in DUI,\ i \in UO,$ we have:

$$\min_{i\in DUI}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\beta'_{io}}\right\} \le \min_{i\in DUI}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\Delta x_{io}}\right\} \le \min_{i\in DUI}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\alpha'_{io}}\right\}$$

$$-\max_{r\in UO}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\gamma'_{ro}}\right\} \le -\max_{r\in UO}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\Delta y_{ro}}\right\} \le -\max_{r\in UO}\left\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\delta'_{ro}}\right\}.$$

Then, for achieving the minimum value's objective function of model (12), we replace $(-\zeta_{1o}^*)$ with $(-\max_{i\in DDI}\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\alpha_{io}}\})$ and (η_{1o}^*) with $(\min_{r\in DO}\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\delta_{ro}}\})$ and (ζ_{2o}^*) with $(\min_{i\in DUI}\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*x_{ij}}{x_{io}+\beta'_{io}}\})$ and $(-\eta_{2o}^*)$ with $(-\max_{r\in UO}\{\frac{\sum_{j=1}^{n}\mu_j\lambda_j^*y_{rj}}{y_{ro}+\gamma'_{ro}}\})$.

4. Application Examples:

In this section the proposed method in Section 3 is used for allocating resources in two real world situations. In one of these examples, there are data of desirable and undesirable type and in the other example in addition to these types, some of the data are of discretionary and non discretionary type.

Example 1:

Here, we illustrate the resource allocation method discussed in this paper through the analysis data from petroleum companies. The data set consists of 20 gas companies located in 18 regions in Iran. The data for this analysis are derived from operations during 2005. There are six variables from the data set as inputs and outputs in this example. Inputs include capital (x_1), number of staff (x_2), and operational costs (excluding staff costs) (x_3) and outputs include number of subscribers (y_1), length of gas network (y_2) and the sold-out gas income (y_3). Table 1 contains a listing of the original data.

Table1: Dat a of Inputs and outputs for Iranian gascompanies

DMU	x_1	x_2	x_3	y_1	y_2	y_3
1	124313	129	198598	30242	565	61836
2	67545	117	131649	14139	153	46233
3	47208	165	228730	13505	211	42094
4	43494	106	165470	8508	114	44195
5	48308	141	180866	7478	248	45841
6	55959	146	194470	10818	230	136513
7	40605	145	179650	6422	127	70380
8	61402	87	94226	18260	182	36592
9	87950	104	91461	22900	170	47650
10	33707	114	88640	3326	85	13410
11	100304	254	292995	14780	318	79833
12	94286	105	98302	19105	273	32553
13	67322	224	287042	15332	241	172316
14	102045	104	155514	18082	441	30004
15	177430	401	528325	77564	801	201529
16	221338	1094	1186905	44136	803	840446
17	267806	1079	1323325	27690	251	832616
18	160912	444	648685	45882	816	251770
19	177214	801	909539	72676	654	34158
20	146325	686	545115	19839	177	341585

There is central company (DM) which supervises the above gas companies. This company has a notable additional capital and staff. The amount of additional capital is 15000 and the number of additional staff is 300. The central company wants to allocate them to those companies so that the number of subscribers reaches 500684 and the amounts of sold-out gas income reaches 2936415. The proposed method is applied in order to determine the contribution of each of company.

Here one must note that some data are desirable and others are undesirable. For example capital and operational costs are discretionary desirable input and non discretionary desirable input,respectively. Also, the number of staff is undesirable Input because one of the DM's aims is to help solve the unemployment issue and the number of subscribers and sold-out gas income are desirable outputs but the length of gas network is undesirable output.

First the model $(2 - \text{I})$ is solved and the following results are achieved: (Table2)

Table2: The results of model $(2-\mathrm{I})$

DMU	φ_{1k}	θ_{1k}	φ_{2k}	θ_{2k}
1	0.24151	0.48936	0.18967	1.95175
2	0.34528	0.65612	0.61221	1.4349
3	1.1226	1.2147	0.86304	1.4014
4	1.9473	1.8704	1.2782	1.9153
5	2.0524	1.6632	0.56955	1.5853
6	0.65558	1.0689	0.46230	1.6891
7	1.1361	1.4290	0.83687	1.5695
8	0.19321	0.57140	0.50334	1.1301
9	0.14970	0.39117	0.50334	1.1301
10	1	1	1	1
11	1.8896	1.1112	0.71563	1.2670
12	1.1962	0.94214	0.62310	1.0741
13	0.82394	1.1767	0.6085	1.5369
14	1.2426	0.94265	0.38773	1.79014
15	0.24795	0.80135	0.21675	1.6583
16	0.83067	1.0842	0.41239	0.91003
17	1	1	1	1
18	0.75967	0.95507	0.38304	1.6187
19	0.27298	0.84967	0.27064	0.85643
20	1	1	1.475	1

Now, by applying Theorem 4 we obtain μ^*_{jo} and then the $\widehat{\lambda_j}^*$ are achieved as follows:

$$\widehat{\lambda_j}^* = \begin{cases} 0.99 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 1;$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.97 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 2$$

$$\widehat{\lambda_j}^* = \begin{cases} 1.004 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 3;$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.63 & ; \quad j = 9 \\ 0.37 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10,9 \end{cases} \quad \text{For } o = 4$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.562 & ; \quad j = 9 \\ 0.454 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10,9 \end{cases} \quad \text{For } o = 5;$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.99 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 6$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.043 & ; \quad j = 9 \\ 0.973 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10,9 \end{cases} \quad \text{For } o = 7;$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.95 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 8$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.98 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 9;$$

$$\widehat{\lambda_j}^* = \begin{cases} 1 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 10$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.99 & ; \quad j = 9 \\ 0 & ; \quad j \neq 9 \end{cases} \quad \text{For } o = 11;$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.99 & ; \quad j = 9 \\ 0 & ; \quad j \neq 9 \end{cases} \quad \text{For } o = 12$$

$$\widehat{\lambda_j}^* = \begin{cases} 0.99 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 13;$$

$$\widehat{\lambda_j}^* = \begin{cases} 1 & ; \quad j = 9 \\ 0 & ; \quad j \neq 9 \end{cases} \quad \text{For } o = 14$$

$$\widehat{\lambda_j}^* = \begin{cases} 1 & ; \quad j = 10 \\ 0 & ; \quad j \neq 10 \end{cases} \quad \text{For } o = 15;$$

$$\widehat{\lambda_j}^* = \begin{cases} 1.02 & ; \quad j = 19 \\ 0.511 & ; \quad j = 20 \\ 0 & ; \quad j \neq 19,20 \end{cases} \text{Foro} = 16$$

$$\widehat{\lambda_j}^* = \begin{cases} 1 & ; \quad j = 17 \\ 0 & ; \quad j \neq 17 \end{cases} \quad \text{For } o = 17;$$

$$\widehat{\lambda_j}^* = \begin{cases} 1 & ; \quad j = 20 \\ 0 & ; \quad j \neq 20 \end{cases} \quad \text{For } o = 18$$

$$\widehat{\lambda_j}^* = \begin{cases} 1.18 & ; \quad j = 20 \\ 0 & ; \quad j \neq 20 \end{cases} \quad \text{For } o = 19;$$

$$\widehat{\lambda_j}^* = \begin{cases} 1 & ; \quad j = 20 \\ 0 & ; \quad j \neq 20 \end{cases} \quad \text{For } o = 20$$

and we have: (Table3)

and according to Conclusion 2 and Remark 3, we obtain h_1^*, $\overline{h}_2^*$, h_3^* and l_1^*, l_2^*,$\hat{l}_3^*$. Then we replace these parameters in the model (15), the following results are achieved: (Table4)

Table3: Efficien cyscores for DMUs afterre source allocation DMU

DMU	η_{1k}	η_{2k}	ζ_{1k}	ζ_{2k}
1	0.54	0.16	0.27	0.25
2	0.61	0.75	0.49	0.25
3	0.064	0.51	0.72	0.23
4	0.165	1.94	1.57	0.25
5	0.154	0.65	1.34	0.24
6	0.044	0.45	0.6	0.24
7	0.014	1.07	0.91	0.24
8	0.062	0.58	0.52	0.26
9	0.060	0.65	0.38	0.26
10	0.074	2	1.005	0.26
11	0.192	0.62	0.88	0.18
12	0.235	0.73	0.93	0.24
13	0.039	0.43	0.5	0.21
14	0.24	0.43	0.86	0.24
15	0.031	0.11	0.19	0.14
16	0.52	1	1.16	0.82
17	0.47	1.2	1	0.77
18	0.26	0.23	0.91	0.89
19	0.23	0.34	0.98	0.83
20	0.39	1.32	1	0.8

Table4: Data of additional inputs and additional outputs for DMUs DMU

DMU	Δx_{1k}	Δx_{2k}	Δy_{1k}	Δy_{2k}	Δy_{3k}
1	402.8	54.36722	0	0	26726.38
2	887.15	25.16787	2985.19	7.303861	1552.059
3	887.15	5.797695	144.9329	0	2391.259
4	887.15	60.58406	0	15.55526	0
5	887.15	2.394052	1717.488	0	18906.59
6	887.15	0	0	0	0
7	887.15	1.011556	0	109654.4	0
8	887.15	0.6775440	152.3884	0	423.7117
9	887.15	0	0	0	0
10	0	0	0	0	0
11	887.15	0	0	84248.1	0
12	887.15	0	0	486754.3	0
13	0	0	0	0	0
14	887.15	0	0	1243889	0
15	887.15	150	5000	0	50000
16	887.15	0	0	819897.6	0
17	402.8	0	0	0	0
18	887.15	0	0	1047185	0
19	887.15	0	0	6461624	0
20	887.15	0	0	0	0

According to the optimal resource allocation plan, company 17 and company 10 will receive less resources than the other companies because these units are more efficient than the other DMUs.

Example 2: A manufacturer of agricultural products manages three farms. These farms consume resources such as land, seeds, animal manure, carbon monoxide (co), which is a toxic gases and etc to produce several types of fruits and vegetables. The information of these farms, such as the quantities of consumed resources and produced products, is introduced in Table 5.

Table5: Consumed inputs and produced outputs of the farms $farm_1$

	$farm_1$	$farm_2$	$farm_3$
land(m^2)	100	150	200
Seeds (kg)	12	17	20
rainfall(mm)	30	20	18
Car bonmonoxide(m^3)	1	1.5	2
animalmanure(kg)	5	8	12
desirablefruits(kg)	20	28	32
undesirablefruits(kg)	3	4	5
Desirablevegetables(kg)	15	20	28
undesirablefruits(kg)	2	3	2

This manufacturer has a significant manure and seeds excess. The DM wants to allocate them to the farms so that the production level reaches a special level. This paper's method is applied to determine the allocation resources to each farm.

For this aim, we consider the farms as DMUs; the land, seeds, animal manure and carbon monoxide (co) are considered as inputs and the produced products are assumed as outputs. Note that some data are desirable and others are undesirable. Also two classes for inputs can be considered: discretionary and non discretionary inputs. The decomposition of data is as follows: (Table6)

The manufacturer expresses the following conditions for resource allocation:

$\Delta^{DDI} = \{(\Delta x_{i1}, \Delta x_{i2}, \Delta x_{i3});$
$-0.4x_{ik} \leq \Delta x_{ik} \leq 0.5x_{ik};$
$k = 1,2,3; i = 1,2;$
$\sum_{k=1}^{3} \Delta x_{1k} = 120 \quad \sum_{k=1}^{3} \Delta x_{2k} = 10\}$
$\Delta^{DUI} = \{(\Delta x_{51}, \Delta x_{52}, \Delta x_{53});$
$-0.2x_{5k} \leq \Delta x_{5k} \leq 0.3x_{5k}; k = 1,2,3;$
$\sum_{k=1}^{3} \Delta x_{5k} = 2\}$
$\Delta^{DO} = \{(\Delta y_{r1}, \Delta y_{r2}, \Delta y_{r3});$
$-0.2y_{rk} \leq \Delta y_{rk} \leq 0.3y_{rk};$
$k = 1,2,3; \; r = 1,3;$
$\sum_{k=1}^{3} \Delta y_{1k} = 8 \quad \sum_{k=1}^{3} \Delta y_{3k} = 5\}$
$\Delta^{UO} = \{(\Delta y_{r1}, \Delta y_{r2}, \Delta y_{r3});$
$-0.1y_{rk} \leq \Delta y_{rk} \leq 0.3\mathrm{y}_{rk};$
$k = 1,2,3; \; r = 2,4;$
$\sum_{k=1}^{3} \Delta y_{2k} = 2 \quad \sum_{k=1}^{3} \Delta y_{4k} = 2\}.$

After applying the algorithm which is proposed in this paper, the following results are achieved. (Table7)

Table6: The decomposed data

DiscretionaryDesirable Inputs(DDI)	land and seeds
Nondiscretionary Desirable Input(NDI)	rainfall
Discretionary Undesirable Input(DUI)	**animal** manure
NondiscretionaryUndesirable Input(NUI)	**co**
UndesirableOutput(DO) **Undesirable fruits and vegetables**	

Table7: The result of the propose

	farm_1	farm_2	farm_3
Δ^{DDI}land(m^2)	120	0	0
Δ^{DDI}Seeds (kg)	9	1	0
Δ^{DUI}animalmanure(kg)	0	2	0
Δ^{DO} desirablefruits(kg)	2	6	0
Δ^{UO}undesirablefruits(kg)	2	0	0
Δ^{DO}desirablevegetables(kg)	0.5	4.5	0
Δ^{UO}undesirablevegetables(kg)	1.5	0.5	0

5. Appendix

In this section Theorem 3 and Theorem 4 which were introduced in Section 3 are proved.

Proof of Theorem 3: Suppose that:

$$\overline{\lambda_j} = \begin{cases} 0 & ; j \neq o \\ 1 & ; j = o \end{cases}$$

$$\overline{\eta_{1o}} = \overline{\zeta_{2o}} = 0$$

and

$$\overline{\zeta_{1o}} = \max_{i \in DDI} \left\{ \frac{x_{io}}{x_{io} + \Delta x_{io}} \right\} ,$$

$$\overline{\eta_{2o}} = \max_{r \in UO} \{ \frac{y_{ro}}{y_{ro} + \Delta y_{ro}} \}.$$

This is a feasible solution to the model(10-I). On the other hand, the constraints of model(10-I) imply that there is an upper bound for the amount of objective function in every feasible solutions of model (10-I), as follows:

$$\eta_{1o} - \zeta_{1o} - \eta_{2o} + \zeta_{2o} \leq \min_{r \in DO} \left\{ \frac{\sum_{j=1}^{n} \lambda_j y_{rj}}{y_{ro} + \Delta y_{ro}} \right\} - \max_{i \in DDI} \{ \frac{\sum_{j=1}^{n} \lambda_j x_{ij}}{x_{io} + \Delta x_{io}} \} - \max_{r \in UO} \left\{ \frac{\sum_{j=1}^{n} \lambda_j y_{rj}}{y_{ro} + \Delta y_{ro}} \right\} + \min_{i \in DUI} \{ \frac{\sum_{j=1}^{n} \lambda_j x_{ij}}{x_{io} + \Delta x_{io}} \}.$$

Therefore model (10-I) is a bounded problem. Similarly, it can be shown that model (2-I) is also feasible and bounded.

Proof of Theorem 4. Suppose that set $J = \{1, \ldots, n\}$ is decomposed as follows:

$J = J_1 \cup J_2$

such that:

$$J_1 = \{ j \in J | \max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} \leq \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \} \}.$$

Note that if $NDUI = \emptyset$ then $J_1 = J$ and $J_2 = J$.

and

$$J_2 = \{ j \in J | \max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} > \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \} \}.$$

Set $J_3 = \{ j \in J_1 | max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \} > 1 \}$. $J_3 \neq \emptyset$, Because $o \in J_3$, for $j = 1, \ldots, n$, μ_j can be defined as follows:

$$\mu_{jo} = \begin{cases} \dfrac{\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}}{\sum_{j \in J_3} (\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}) \lambda_j^*}; & j \in J_3 \\ 0; & j \in J - J_3 \end{cases}$$

Here, we will show that $((\widehat{\lambda_j}^* = \mu_{jo} \lambda_j^*)_{j=1,\ldots,n}; \eta_{1o}^*; \zeta_{1o}^*; \ \eta_{2o}^*; \ \zeta_{2o}^*)$ is a feasible solution to model(10-I). For this aim, we are going to investigate the constraints of model(10-I).

We note that for $j \in J_3$,

$$\mu_{jo} = \frac{\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}}{\sum_{j \in J_3} (\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}) \lambda_j^*} = \frac{\dfrac{\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}}{\sum_{j \in J_3} \lambda_j^*}}{\dfrac{\sum_{j \in J_3} (\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}) \lambda_j^*}{\sum_{j \in J_3} \lambda_j^*}}.$$

Based on the definition of J_3 and that $((\lambda_j^*)_{j=1,\ldots,n}; \varphi_{1o}^*; \theta_{1o}^*; \ \varphi_{2o}^*; \ \theta_{2o}^*)$ being an optimal solution to the model (2-I), we know that:

$$\frac{\sum_{j \in J_3} (\max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \} + \min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}) \lambda_j^*}{\sum_{j \in J_3} \lambda_j^*} > 1$$

and this implies that for every $j \in J_3$,

$$\frac{max_{i \in NDUI} \{ \frac{x_{io}}{x_{ij}} \}}{\sum_{j \in J_3} \lambda_j^*} \leq \mu_{jo} \leq \frac{min_{i \in NDDI} \{ \frac{x_{io}}{x_{ij}} \}}{\sum_{j \in J_3} \lambda_j^*}.$$

For every $i \in NDDI$, we have:

$$\sum_{j=1}^{n} \widehat{\lambda_j}^* x_{ij} = \sum_{j=1}^{n} \mu_{jo}\lambda_j^* x_{ij} = \sum_{j\in J_3} \mu_{jo}\lambda_j^* x_{ij} \le \sum_{j\in J_3} \frac{\min_{i\in NDDI}\{\frac{x_{io}}{x_{ij}}\}}{\sum_{j\in J_3}\lambda_j^*}\lambda_j^* x_{ij} \le \frac{1}{\sum_{j\in J_3}\lambda_j^*}\sum_{j\in J_3}\frac{x_{io}}{x_{ij}}\lambda_j^* x_{ij} \le \frac{x_{io}}{\sum_{j\in J_3}\lambda_j^*}\sum_{j\in J_3}\lambda_j^* = x_{io}.$$

Similarly, for every $i \in NDUI$, we have:

$$\sum_{j=1}^{n} \widehat{\lambda_j}^* x_{ij} = \sum_{j=1}^{n} \mu_{jo}\lambda_j^* x_{ij} = \sum_{j\in J_3} \mu_{jo}\lambda_j^* x_{ij} \ge \sum_{j\in J_3} \frac{\max_{i\in NDUI}\{\frac{x_{io}}{x_{ij}}\}}{\sum_{j\in J_3}\lambda_j^*}\lambda_j^* x_{ij} \ge \frac{1}{\sum_{j\in J_3}\lambda_j^*}\sum_{j\in J_3}\frac{x_{io}}{x_{ij}}\lambda_j^* x_{ij} \ge \frac{x_{io}}{\sum_{j\in J_3}\lambda_j^*}\sum_{j\in J_3}\lambda_j^* = x_{io}.$$

For every $r \in DO$:

$$\eta_{1o}^*(y_{ro}+\Delta y_{ro}) = \min_{r\in DO}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* y_{rj}}{y_{ro}+\Delta y_{ro}}\}(y_{ro}+\Delta y_{ro}) \le \frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* y_{rj}}{y_{ro}+\Delta y_{ro}}(y_{ro}+\Delta y_{ro}) = \sum_{j=1}^{n}\widehat{\lambda_j}^* y_{rj}$$

and for every $i \in DDI$:

$$\zeta_{1o}^*(x_{io}+\Delta x_{io}) = \max_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* x_{ij}}{x_{io}+\Delta x_{io}}\}(x_{io}+\Delta x_{io}) \ge \frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* x_{ij}}{x_{io}+\Delta x_{io}}(x_{io}+\Delta x_{io}) = \sum_{j=1}^{n}\widehat{\lambda_j}^* x_{ij}.$$

For $r \in UO$:

$$\eta_{2o}^*(y_{ro}+\Delta y_{ro}) = \max_{r\in UO}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* y_{rj}}{y_{ro}+\Delta y_{ro}}\}(y_{ro}+\Delta y_{ro}) \ge \frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* y_{rj}}{y_{ro}+\Delta y_{ro}}(y_{ro}+\Delta y_{ro}) = \sum_{j=1}^{n}\widehat{\lambda_j}^* y_{rj}$$

and for every $i \in DUI$:

$$\zeta_{2o}^*(x_{io}+\Delta x_{io}) = \min_{i\in DUI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* x_{ij}}{x_{io}+\Delta x_{io}}\}(x_{io}+\Delta x_{io}) \le \frac{\sum_{j=1}^{n}\widehat{\lambda_j}^* x_{ij}}{x_{io}+\Delta x_{io}}\}(x_{io}+\Delta x_{io}) \le \sum_{j=1}^{n}\widehat{\lambda_j}^* x_{ij}$$

and it is clear that:

$$\mu_{jo} \ge 0; j = \{1,2,\dots,n\} \quad and \quad \sum_{j=1}^{n}\widehat{\lambda_j^*} = \sum_{j\in J_3}\widehat{\lambda_j^*} = 1.$$

Then $\widehat{\lambda_j}^* = \mu_{jo}\lambda_j^*)_{j=1,\dots,n}; \eta_{1o}^*; \zeta_{1o}^*;\ \eta_{2o}^*;\ \zeta_{2o}^*)$ is a feasible solution to the model (10-I). Now, suppose that $((\widehat{\lambda_j}^{**}{}_{j=1,\dots,n}; \eta_{1o}^{**}; \zeta_{1o}^{**};\ \eta_{2o}^{**};\ \zeta_{2o}^{**})$ is an optimal solution to the model(10-I), then we have:

$\eta_{1o}^{**} \ge \eta_{1o}^*, \quad \zeta_{1o}^{**} \le \zeta_{1o}^*,$
$\eta_{2o}^{**} \le \eta_{2o}^*, \quad ; \zeta_{2o}^{**} \ge \zeta_{2o}^*.$

In the following, we will show that:

$\eta_{1o}^{**} \le \eta_{1o}^*, \quad \zeta_{1o}^{**} \ge \zeta_{1o}^*,$
$\eta_{2o}^{**} \ge \eta_{2o}^*, \quad ; \ \zeta_{2o}^{**} \le \zeta_{2o}^*$

which implies that:

$\eta_{1o}^{**} = \eta_{1o}^*, \zeta_{1o}^{**} = \zeta_{1o}^*, \eta_{2o}^{**} = \eta_{2o}^*, \zeta_{2o}^{**} = \zeta_{2o}^*.$

or in other words $((\widehat{\lambda_j}^* = \mu_{jo}\lambda_j^*)_{j=1,\dots,n}; \eta_{1o}^*;\ \zeta_{1o}^*;\ \eta_{2o}^*;\ \zeta_{2o}^*)$ is an optimal solution to the model(2-I). For this aim, we first prove that

$$2 - \max_{r\in DO}\left\{\frac{y_{ro}}{\sum_{j=1}^{n}\lambda_j^* y_{rj}}\right\}\varphi_{1o}^* = 1 \quad and$$

$$2 - \min_{r\in UO}\{\frac{y_{ro}}{\sum_{j=1}^{n}\lambda_j^* y_{rj}}\}\varphi_{2o}^* = 1$$

$$2 - \min_{i\in DDI}\left\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\right\}\theta_{1o}^* = 1 \ \ and$$

$$2 - \max_{i\in DUI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\}\theta_{2o}^* = 1$$

because $((\lambda_j^*)_{j=1,\dots,n}; \varphi_{1o}^*; \theta_{1o}^*;\ \varphi_{2o}^*;\ \theta_{2o}^*)$ is an optimal solution to the model(2-I), then we have:

$$\varphi_{1o}^* y_{ro} \le \sum_{j=1}^{n}\lambda_j^* y_{rj};\ \ r \in DO$$

$$\theta_{1o}^{*}x_{io} \geq \sum_{j=1}^{n} \lambda_j^{*}x_{ij} \;;\; i \in DDI$$

$$\varphi_{2o}^{*}y_{ro} \geq \sum_{j=1}^{n} \lambda_j^{*}y_{rj} \;;\; r \in UO$$

$$\theta_{2o}^{*}x_{io} \leq \sum_{j=1}^{n} \lambda_j^{*}x_{ij} \quad i \in DUI$$

But by using the complementary theorem, there exists $k_1 \in DO$, $k_2 \in DDI$, $k_3 \in UO$ and $k_4 \in DUI$ such that:

$$\varphi_{1o}^{*}y_{k_1 o} = \sum_{j=1}^{n} \lambda_j^{*}y_{k_1 j}$$

$$\theta_{1o}^{*}x_{k_2 o} = \sum_{j=1}^{n} \lambda_j^{*}x_{k_2 j}$$

$$\varphi_{2o}^{*}y_{k_3 o} = \sum_{j=1}^{n} \lambda_j^{*}y_{k_3 j}$$

$$\theta_{2o}^{*}x_{k_4 o} = \sum_{j=1}^{n} \lambda_j^{*}x_{k_4 j}$$

because if, for example, there is no binding constraint of $r \in DO$ in the optimal solution, then their corresponding dual variables$(\gamma_r; r \in DO)$ will be zero and the constraint corresponding φ_{1o} in the dual problem to the model(2-I) will be as follows:

$$\sum_{r \in DO} (\gamma_r y_{ro}) = 1$$

or in other words $0 = 1$ and this is a contradiction. Similarly, we can show that there exist $k_2 \in DDI$, $k_3 \in UO$ and $k_4 \in DUI$. However, based on the above explanations , it is obvious that:

$$\frac{y_{k_1 o}}{\sum_{j=1}^{n} \lambda_j^{*}y_{k_1 j}}\varphi_{1o}^{*} = \max_{r \in DO}\frac{y_{ro}}{\sum_{j=1}^{n} \lambda_j^{*}y_{rj}}\varphi_{1o}^{*} \;,$$

$$\frac{x_{k_2 o}}{\sum_{j=1}^{n} \lambda_j^{*}x_{k_2 j}}\theta_{1o}^{*} = \min_{i \in DDI}\frac{x_{io}}{\sum_{j=1}^{n} \lambda_j^{*}x_{ij}}\theta_{1o}^{*}$$

$$\frac{y_{k_3 o}}{\sum_{j=1}^{n} \lambda_j^{*}y_{k_3 j}}\varphi_{2o}^{*} = \min_{r \in UO}\frac{y_{ro}}{\sum_{j=1}^{n} \lambda_j^{*}y_{rj}}\varphi_{2o}^{*} \;,$$

$$\frac{x_{k_4 o}}{\sum_{j=1}^{n} \lambda_j^{*}x_{k_4 j}}\theta_{2o}^{*} = \max_{i \in DUI}\frac{x_{io}}{\sum_{j=1}^{n} \lambda_j^{*}x_{ij}}\theta_{2o}^{*}$$

thus:

$$2 - \max_{r \in DO}\left\{\frac{y_{ro}}{\sum_{j=1}^{n} \lambda_j^{*}y_{rj}}\right\}\varphi_{1o}^{*} = 1 \quad and$$

$$2 - \min_{r \in UO}\{\frac{y_{ro}}{\sum_{j=1}^{n} \lambda_j^{*}y_{rj}}\}\varphi_{2o}^{*} = 1$$

$$2 - \min_{i \in DDI}\left\{\frac{x_{io}}{\sum_{j=1}^{n} \lambda_j^{*}x_{ij}}\right\}\theta_{1o}^{*} = 1 \;\; and$$

$$2 - \max_{i \in DUI}\{\frac{x_{io}}{\sum_{j=1}^{n} \lambda_j^{*}x_{ij}}\}\theta_{2o}^{*} = 1$$

then, we can rewrite:

$$\eta_{1o}^{*} = \frac{\min_{r \in DO}\{\frac{\sum_{j=1}^{n}\mu_{jo}\lambda_j^{*}y_{rj}}{y_{ro}+\Delta y_{ro}}\}}{2 - \max_{r \in DO}\{\frac{y_{ro}}{\sum_{j=1}^{n}\lambda_j^{*}y_{rj}}\}\varphi_{1o}^{*}}$$

$$\eta_{2o}^{*} = \frac{\max_{r \in UO}\{\frac{\sum_{j=1}^{n}\mu_{jo}\lambda_j^{*}y_{rj}}{y_{ro}+\Delta y_{ro}}\}}{2 - \min_{r \in UO}\{\frac{y_{ro}}{\sum_{j=1}^{n}\lambda_j^{*}y_{rj}}\}\varphi_{2o}^{*}}$$

$$\zeta_{2o}^{*} = \frac{\min_{i \in DUI}\{\frac{\sum_{j=1}^{n}\mu_{jo}\lambda_j^{*}x_{ij}}{x_{io}+\Delta x_{io}}\}}{2 - \max_{i \in DUI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^{*}x_{ij}}\}\theta_{2o}^{*}}$$

$$\zeta_{1o}^{*} = \frac{\max_{i \in DDI}\{\frac{\sum_{j=1}^{n}\mu_{jo}\lambda_j^{*}x_{ij}}{x_{io}+\Delta x_{io}}\}}{2 - \min_{i \in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^{*}x_{ij}}\}\theta_{1o}^{*}}.$$

Here, we will show that $((\lambda_j^{*})_{j=1,\dots,n}, \widetilde{\varphi_{1o}}, \widetilde{\theta_{1o}}, \widetilde{\varphi_{2o}}, \widetilde{\theta_{2o}})$ such that:

$$\widetilde{\varphi_{1o}} = \frac{2 - \frac{\min_{r \in DO}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}y_{rj}}{y_{ro}+\Delta y_{ro}}\}}{\eta_{1o}^{**}}}{\max_{r \in DO}\{\frac{y_{ro}}{\sum_{j=1}^{n}\lambda_j^{*}y_{rj}}\}}$$

$$\widetilde{\theta_{1o}} = \frac{2 - \frac{\max_{i \in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{1o}^{**}}}{\min_{i \in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^{*}x_{ij}}\}}$$

$$\widetilde{\varphi_{2o}} = \frac{2 - \frac{\max_{r \in UO}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}y_{rj}}{y_{ro}+\Delta y_{ro}}\}}{\eta_{2o}^{**}}}{\min_{r \in UO}\{\frac{y_{ro}}{\sum_{j=1}^{n}\lambda_j^{*}y_{rj}}\}}$$

$$\widetilde{\theta_{2o}} = \frac{2 - \frac{\min_{i \in DUI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{2o}^{**}}}{\max_{i \in DUI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^{*}x_{ij}}\}}$$

is a feasible solution to the model (2-I). Note that $\frac{max_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{1o}^{**}} \leq 1$ because for every $i \in DDI, \zeta_{1o}^{**}(x_{io}+\Delta x_{io}) \geq \sum_{j=1}^{n} \widehat{\lambda_j}^{**} x_{ij}$.

$$\widetilde{\theta_{1o}}x_{io} = \frac{2-\frac{\max\limits_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{1o}^{**}}}{\min\limits_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\}}x_{io} \geq \frac{1}{\min\limits_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\}}x_{io}$$

$$\geq \frac{1}{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}}x_{io} = \frac{\sum_{j=1}^{n}\lambda_j^* x_{ij}}{x_{io}}x_{io} = \sum_{j=1}^{n}\lambda_j^* x_{ij}$$

and, similarly:

$$\widetilde{\theta_{2o}}x_{io} \leq \sum_{j=1}^{n}\lambda_j^* x_{ij};\ \ i \in DUI$$

$$\widetilde{\varphi_{1o}}y_{ro} \leq \sum_{j=1}^{n}\lambda_j^* y_{rj}\ ;\ \ r \in DO$$

$$\widetilde{\varphi_{2o}}y_{ro} \geq \sum_{j=1}^{n}\lambda_j^* y_{rj}\ ;r \in UO.$$

Clearly, this solution is also satisfied by the other constraints of model (2-I); hence, it is a feasible solution to the model (2-I) thus:

$$\theta_{1o}^* \leq \widetilde{\theta_{1o}} = \frac{2-\frac{\max\limits_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{1o}^{**}}}{\min\limits_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\}}$$

and consequently:

$$\theta_{1o}^*\min_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\} \leq 2-\frac{\max\limits_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{1o}^{**}}$$

$$\frac{\max\limits_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{\zeta_{1o}^{**}} \leq 2-\theta_{1o}^*\min_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\}$$

$$\frac{\max\limits_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{2-\theta_{1o}^*\min\limits_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\lambda_j^* x_{ij}}\}} \leq \zeta_{1o}^{**}.$$

On the other hand, because $\widehat{\lambda_j}^{**}_{j=1,\dots,n}$ are optimal solutions and $\widehat{\lambda_j}^{*}_{j=1,\dots,n}$ are feasible solutions of model(8-I) and the quantity of slack variables in the optimal solution must be smaller than those in the feasible solution, then:

$$\max_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\} \geq \max_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}x_{ij}}{x_{io}+\Delta x_{io}}\}$$

and

$$2-\theta_{1o}^*\min_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}x_{ij}}\} = 1$$

then:

$$\frac{\max\limits_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}{x_{io}+\Delta x_{io}}\}}{2-\theta_{1o}^*\min\limits_{i\in DDI}\{\frac{x_{io}}{\sum_{j=1}^{n}\widehat{\lambda_j}^{**}x_{ij}}\}} \geq \max_{i\in DDI}\{\frac{\sum_{j=1}^{n}\widehat{\lambda_j}^{*}x_{ij}}{x_{io}+\Delta x_{io}}\}$$

i.e:

$$\zeta_{1o}^* \leq \zeta_{1o}^{**}$$

thus:

$$\zeta_{1o}^* = \zeta_{1o}^{**}$$

and therefore the proof is completed.■

6. Conclusion

In this paper, resource allocation problem for systems with a supervising center is investigated. It is assumed that the units under supervision are homogeneous because otherwise, the efficiency scores may reflect the underlying differences in environments rather than inefficiencies. In these systems (i.e. organizations) homogeneous units consume both desirable and undesirable inputs to produce outputs some of which might be undesirable. The proposed method not only considers environmental factors such as smoke pollution and waste but also considers economic ones, both of which are of fundamental importance to resource allocation.

As it was discussed, the previous methods in the literature were not designed to solve problems with non discretionary data; while the proposed method , as it was shown, can do so. Another difference is that the new method puts emphasis on efficiency and return to scale simultaneously as opposed to the previous methods in which resources are allocated between units to improve efficiency or maximize the total amount of outputs.

To consider both return to scale and efficiency simultaneously, our model uses the notion of MPSS points and allocates available resources between units in a way that not only the overall system distance is minimized, but also the distance from any unit to it's MPSS points is minimized as well. In addition to illustrate the applicability of the proposed model, two real-life examples were solved.

The majority of resource allocation situations are dealt with in the proposed method. What remains can be of interest to other scientists and thus subjects for further researches. Devising a method for resource allocation in an organization which deals with fuzzy type data is proposed as an inspiring further research question. Moreover, the way data is handled in this paper might be used as a guideline for further researches in solving transportation and/or network problems.

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