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Efficiency of decision making units in network DEA using interval data

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Abstract
In this paper, the difference between multiplicative and envelopment models of network DEA is examined, in which network DEA multiplicative model is able to calculate efficiency and the envelopment model can calculate the projection on the frontier. Here, a model is presented that can calculate both frontier projection and efficiency in network DEA. Since in real world, many data are interval data, we present a model in this article that calculates the efficiency of the units being evaluated by such these interval data. Since data are as intervals, the resulting efficiencies are calculated as intervals. We present two models for calculating the lower and upper bounds for any DMU and prove that these models give upper and lower bounds of efficiency.

Keywords*:* data envelopment analysis, efficiency, interval data.

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1. Introduction

As it was mentioned, DEA (Data Envelopment Analysis) is a technique to evaluate efficiency of congruent decision making units with several inputs and outputs. However in classical models, DEA of the unit being examined is considered as a black box and inter-organizational relations are ignored. However, these relations affect the performance evaluation of the units being examined. Therefore, many scientists of DEA gradually paid attention to the evaluation of multi-stage units and many researches are conducted in this context, including Seiford and Zhu [6], Chen and Zhu [3], Kao and Hwang [4], Chen et al. [2], Tone and Tsutsui [7] and Cook et al. [1]. All of these techniques are based on performance evaluation of multi-stage units. Of course, they have various weaknesses and strengths.

Finally, Kao and Hwang [5] persented a radial model with the following problems:

- It does not guarantee relative efficiency.
- In variable efficiency to scale conditions, it becomes a nonlinear model and it is not possible to linearize it with normal methods.
- The projection presented by these methods is not necessarily efficient.
- It can not be generalized to multi-phase structures.

According to these problems, we presented a new model that can solve the above problems in addition to calculating the efficiency in real world, many data may not be accurate and may be available as intervals [4]. So, we present models that can calculate the efficiency of the decision making units in a network with such these data, so that the i-th input and output are as intervals, and intermediate products are as sharp values.

In section 2, 3, interval theorems and definitions are presented, in section 4, a practical example and finally conclusions are presented.

2. Network DEA Model

DEA is described as a technique for evaluating efficiency of congruent decision making units with several inputs and outputs. Decision making units can be in different forms such as hospitals, universities, banks, etc. These units can act as a two-phase process in some cases. The first phase uses some inputs and produces some outputs. These outputs form the second phase inputs. Outputs of the first stage are also called "mediators". Using mediators, the second stage produces the final outputs of the system. According to the models presented by Kao and Hwang [4], the hybrid mode is presented in a way that the following multiplicative model is presented by calculating the efficiency in first and second phases in the network and combining them.

$$
\begin{aligned}
&Max.(\textstyle\sum_{d=1}^{D} \tilde{w}_d z_{do} - \tilde{w}_o)+\sum_{r=1}^{s} \hat{u}_r y_{ro} \\
&s.t.\ \textstyle\sum_{i=1}^{m} \tilde{v}_i x_{io} = 1, \\
&\textstyle\sum_{d=1}^{D} \hat{w}_d z_{do} + \hat{w}_o = 1 \\
&E_O \textstyle\sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s} u_r y_{ro} = 0, \\
&\textstyle\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \\
&(j = 1, \dots, n), \\
&\textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \\
&(j = 1, \dots, n), \\
&\textstyle\sum_{d=1}^{D} \tilde{w}_d z_{do} - \tilde{w}_o - \sum_{i=1}^{m} \tilde{v}_i x_{io} \leq 0 \\
&\textstyle\sum_{r=1}^{s} \hat{u}_r y_{rj} - \sum_{d=1}^{D} \hat{w}_d z_{dj} - \hat{w}_o \leq 0, \forall j
\end{aligned}
$$

And its envelope dual is as follows:

$$
\begin{aligned}
&Dual\ min.\theta_1 + \theta_2 \\
&s.t.\ \textstyle\sum_{j=1}^{n} \alpha_j x_{ij} \leq \theta_1 x_{io}, \\
&\textstyle\sum_{j=1}^{n} \beta_j z_{dj} \leq \theta_2 z_{do}, \\
&\textstyle\sum_{j=1}^{n} \alpha_j = 1, \quad \sum_{j=1}^{n} \beta_j = \theta_2 \\
&\textstyle\sum_{j=1}^{n} \alpha_j z_{dj} \geq z_{do}, \quad (d = 1, \dots, D), \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq \varphi E_o x_{io}, (i = 1, \dots, m), \\
&\textstyle\sum_{j=1}^{n} \beta_j y_{rj} \geq y_{ro}\ (d = 1, \dots, D), \\
&\textstyle\sum_{j=1}^{n} \mu_j y_{rj} \geq \varphi y_{ro}, (r = 1, \dots, s), \\
&\textstyle\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{dj} \geq 0, (d = 1, \dots, D)
\end{aligned}
$$

The above discussion shows that although θ_1 and θ_2 are independent, efficiency of the first and second phases is calculated and finally, the total efficiency and frontier projection are calculated based on the above model.

4. Efficiency of interval models in two-stage network

According to the composite model, we have the following interval model:

$$
\begin{aligned}
& \min \theta_1 + \theta_2 \\
& s.t \sum_{j=1}^{n} \alpha_j \tilde{x}_{ij} \leq \theta_1 \tilde{x}_{io}, \ (i = 1, \dots, m), \\
& \sum_{j=1}^{n} \beta_j z_{dj} \leq \theta_2 z_{do}, \ (d = 1, \dots, D), \\
& \sum_{j=1}^{n} \alpha_j = 1, \quad \sum_{j=1}^{n} \beta_j = \theta_2 \qquad (1) \\
& \sum_{j=1}^{n} \alpha_j z_{dj} \geq z_{do}, \quad (d = 1, \dots, D), \\
& \sum_{j=1}^{n} \beta_j \tilde{y}_{rj} \geq \tilde{y}_{ro} \quad (r = 1, \dots, s), \\
& \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \varphi E_o \tilde{x}_{io}, \ (i = 1, \dots, m), \\
& \sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \geq \varphi \tilde{y}_{ro}, \quad (r = 1, \dots, s), \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{dj} \geq 0, \ (d = 1, \dots, D),
\end{aligned}
$$

Since the input and outputs are as intervals, the objective function value of the above model is obtained as an interval. Its value can be obtained using the upper and lower bounds. Since the input is as:

$x_i^l \leq \tilde{x}_i \leq x_i^u$

and the output is as:

$y_i^l \leq \tilde{y}_i \leq y_i^u$

to find the lower bound, DMU_0 must be in its worst conditions and the remaining DMUs must be in their best conditions. In other words, DMU_0 must be as (x_0^u, z_0, y_0^l) and other DMUs must be as $(x_j^l, z_j, y_j^u), j \neq 0$. Then, we have:

$$
\begin{aligned}
& z_l = \min \theta_1 + \theta_2 \\
& s.t \sum_{\substack{j=1 \\ j \neq 0}}^{n} \alpha_j x_{ij}^l + \alpha_o x_{io}^u \leq \theta_1 x_{io}^u, \\
& (i = 1, \dots, m), \\
& \sum_{j=1}^{n} \beta_j z_{dj} \leq \theta_2 z_{do} \quad (d = 1, \dots, D), \\
& \sum_{j=1}^{n} \alpha_j = 1 \qquad (2) \\
& \sum_{j=1}^{n} \beta_j = \theta_2 \\
& \sum_{j=1}^{n} \alpha_j z_{dj} \geq z_{do}, \quad (d = 1, \dots, D) \\
& \sum_{\substack{j=1 \\ j \neq 0}}^{n} \beta_j y_{rj}^u + \beta_o y_{ro}^l \geq y_{ro}^l, \\
& (r = 1, \dots, s), \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^l + \lambda_o x_{io}^u \leq \varphi E_O x_{io}^u, \\
& (i = 1, \dots, m), \\
& \sum_{j=1}^{n} \mu_j y_{rj}^u + \mu_o y_{ro}^l \geq, \ \varphi y_{ro}^l, \\
& (r = 1, \dots, s), \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{dj} \geq 0, \quad (d = 1, \dots, D), \\
& \alpha_j, \beta_j, \lambda_j, \mu_j \geq 0, \quad j = 1, \dots, n., \\
& \theta_1 \leq 1, \theta_2 \leq 1
\end{aligned}
$$

To find the upper bound, DMU_0 must be in its best conditions and the others must be in their worst conditions. In other words, DMU_0 must be as (x_0^l, z_0, y_0^u) and other DMUs must be as $(x_j^u, z_j, y_j^l), j \neq 0$. Then, we have:

$$
\begin{aligned}
& z_u = \min \theta_1 + \theta_2 \\
& s.t \sum_{\substack{j=1 \\ j \neq 0}}^{n} \alpha_j x_{ij}^u + \alpha_o x_{io}^l \leq \theta_1 x_{io}^l \\
& (i = 1, \dots, m), \\
& \sum_{j=1}^{n} \beta_j z_{dj} \leq \theta_2 z_{do}, \quad (d = 1, \dots, D), \\
& \sum_{j=1}^{n} \alpha_j = 1 \qquad (3) \\
& \sum_{j=1}^{n} \beta_j = \theta_2 \\
& \sum_{j=1}^{n} \alpha_j z_{dj} \geq z_{do}, \quad (d = 1, \dots, D), \\
& \sum_{\substack{j=1 \\ j \neq 0}}^{n} \beta_j y_{rj}^l + \beta_o y_{ro}^u \geq y_{ro}^u, \\
& (r = 1, \dots, s), \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^u + \lambda_o x_{io}^l \leq \varphi E_O x_{io}^l, \\
& (i = 1, \dots, m), \\
& \sum_{j=1}^{n} \mu_j y_{rj}^l + \mu_o y_{ro}^u \geq, \ \varphi y_{ro}^u, \\
& (r = 1, \dots, s), \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{dj} \geq 0, \quad (d = 1, \dots, D), \\
& \alpha_j, \beta_j, \lambda_j, \mu_j \geq 0, \quad j = 1, \dots, n., \\
& \theta_1 \leq 1, \theta_2 \leq 1
\end{aligned}
$$

Theorem 1: if $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\varphi}, \tilde{\lambda}, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta})$, $(\theta_1^l, \theta_2^l, \varphi^l, \lambda^l, \mu^l, \alpha^l, \beta^l)$ and $(\theta_1^u, \theta_2^u, \varphi^u, \lambda^u, \mu^u, \alpha^u, \beta^u)$ are optimal solutions of the models (1), (2) and (3), respectively, then:

1) $\theta_1^l + \theta_2^l \leq \tilde{\theta}_1 + \tilde{\theta}_2 \leq \theta_1^u + \theta_2^u$

2) $\theta_1^l \leq \tilde{\theta}_1 \leq \theta_1^u$

3) $\theta_2^l \leq \tilde{\theta}_2 \leq \theta_2^u$

Proof: to prove that $z^l \leq z$, we show that the optimal solution of the model (1) is a possible solution of the model (2).

We assume that $(\theta^*_1, \theta^*_2, \alpha^*, \beta^*, \lambda^*, \varphi^*, \mu^*)$ is the optimal solution of the model (1), then we have

$$\sum_{j=1}^{n} \alpha_j^* = 1$$
$$\sum_{j=1}^{n} \beta_j^* = \theta_2$$
$$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \alpha_j^* x_{ij}^l + \alpha_o^* x_{io}^u \leq \sum_{\substack{j=1 \\ j\neq 0}}^{n} \alpha_j^* \tilde{x}_{ij} + \alpha_o^* x_{io}^u - \alpha_o^* \tilde{x}_{io} + \alpha_o^* \tilde{x}_{io}$$
$$= \sum_{j=1}^{n} \alpha_j^* \tilde{x}_{ij} + \alpha_o^* x_{io}^u - \alpha_o^* x_{io}^u \leq \theta_1^* \tilde{x}_{io} + \alpha_o^* x_{io}^u - \alpha_o^* \tilde{x}_{io} + \theta_1^* x_{io}^u - \theta_1^* x_{io}^u$$
$$\rightarrow \sum_{\substack{j=1 \\ j\neq 0}}^{n} \alpha_j^* x_{ij}^l + \alpha_o^* x_{io}^u \leq \theta_1^* x_{io}^u + \theta_1^* \tilde{x}_{io} + \alpha_o^* x_{io}^u - \alpha_o^* \tilde{x}_{io} - \theta_1^* x_{io}^u$$

We should just show that:

$$\theta_1^* \tilde{x}_{io} + \alpha_o^* x_{io}^u - \alpha_o^* \tilde{x}_{io} - \theta_1^* x_{io}^u \leq 0 \leftrightarrow (\theta_1^* - \alpha_o^*)\, \tilde{x}_{io} + (\alpha_o^* - \theta_1^*)\, x_{io}^u \leq 0$$
$$\leftrightarrow \quad (\theta_1^* - \alpha_o^*)(\tilde{x}_{io} - x_{io}^u) \leq 0 \leftrightarrow \theta_1^* - \alpha_o^* \geq 0$$
$$\rightarrow \theta_1^* \geq \alpha_o^* \rightarrow \sum_{\substack{j=1 \\ j\neq 0}}^{n} \alpha_j^* \tilde{x}_{ij} + \alpha_0^* x_{io} \leq \theta_1 x_{io}$$
$$\rightarrow \sum_{\substack{j=1 \\ j\neq 0}}^{n} \alpha_j^* \tilde{x}_{ij} \leq (\theta_1^* - \alpha_o^*) x_{io}$$

Since α_j^* and $\tilde{x}_{ij}$ are nonnegative, $\sum_{j=1,\neq 0}^{n} \alpha_j^* \tilde{x}_{ij}$ is positive. On the other hand, x_{io} is positive. So, we conclude that:

$$\rightarrow \theta_1^* - \alpha_o^* \geq 0$$

And the remaining conditions also hold true. So, the optimal value of the objective function of the model (2) is less than that of the model (1). Also, the objective function of the model (1) is less than that of the model (3).

Finally, for any unit of the efficiency intervals of the first, second and total phases, we have:

$$[\,\theta_1^l, \theta_1^u\,], [\,\theta_2^l, \theta_2^u\,], [\,\theta_1^l + \theta_2^l, \theta_1^u + \theta_2^u\,]$$

The DMU in the lower and upper bounds is known as efficient DMU. We consider the following sets:

$$E^{++} = \{\, DMUj \,|\, \theta_{1j}^l = \theta_{1j}^u = \theta_{2j}^l = \theta_{2j}^u = 1 \,\}$$

The set E^{++} includes DMUs that are efficient in both lower and upper bounds and it is evident that the DMUs are efficient in both first and second phases.

$$E^{+} = \{\, DMUj \,|\, \theta_{1j}^u = \theta_{2j}^u = 1, \theta_{1j}^l . \theta_{2j}^l < 1 \,\}$$

The set E^{+} includes DMUs that are inefficient in both lower and upper bounds. In other words, DMU is efficient in both phases but in lower bound, it is inefficient at least in one of the phases.

$$E^{-} = \{\, DMUj \,|\, \theta_{1j}^u . \theta_{2j}^u < 1 \,\}$$

The set E^{-} includes DMUs that are inefficient in both lower and upper bounds. In other words, they are inefficient at least in one of the phases in both lower and upper bounds.

$$E_1^{++} = \{\, DMUj \,|\, \theta_{1j}^l = \theta_{1j}^u = 1 \,\}$$

The set E_1^{++} includes DMUs that are efficient in both upper and lower bounds in the first phase. It is evident that the DMUs are efficient in the first phase.

$$E_1^{+} = \{\, DMUj \,|\, \theta_{1j}^u = 1, \theta_{1j}^l < 1 \,\}$$

The set E_1^{+} includes DMUs that are efficient in the upper bound and are inefficient in the lower bound in the first phase. Hence, they are inefficient in the first phase.

$$E_1^{-} = \{\, DMUj \,|\, \theta_{1j}^u < 1 \,\}$$

The set E_1^- includes DMUs that are inefficient in both upper and lower bounds in the first phase. Hence, they are inefficient in the first phase.

$$E_2^{++} = \{DMUj \mid \theta_{2j}^l = \theta_{2j}^u = 1\}$$

The set E_2^{++} includes DMUs that are efficient in both upper and lower bounds in the second phase. Hence, they are efficient in the second phase.

$$E_2^{+} = \{DMUj \mid \theta_{2j}^u = 1, \theta_{2j}^l < 1\}$$

The set E_2^+ includes DMUs that are efficient in the upper bound and are inefficient in the lower bound in the second phase. Hence, they are inefficient in the second phase.

$$E_2^{-} = \{DMUj \mid \theta_{2j}^u < 1\}$$

The set E_2^- includes DMUs that are inefficient in the upper bound in the second phase. Hence, they are inefficient in the second phase.

5. Practical Example

A it is observed, efficiencies of 10 DMUs are examined in the above table and among them, DMU3 and DMU5 are efficient in both upper and lower bounds in the first phase, and DMU1, DMU4, DMU5 and DMU6 are efficient in both upper and lower bounds in the second phase. However, a two-phase network is efficient when it is efficient both in the first and second phases. Here, only DMU5 is efficient in both lower and upper bounds in the first and second phases. It is easily observed that a DMU is efficient in the interval model when it is efficient in both upper and lower bounds in both phases. Classification of the above DMUs is as follows:

$$E^{++} = \{5\}$$

$$E^{+} = \varnothing$$

$$E^{-} = \{1,2,3,4,6,7,8,9,10\}$$

$$E_1^{++} = \{3,5\}$$

$$E_1^{+} = \varnothing$$

$$E_1^{-} = \{1,2,4,6,7,8,9,10\}$$

$$E_2^{++} = \{1,4,5,8\}$$

$$E_2^{+} = \{7,9,10\}$$

$$E_2^{-} = \{2,3,6\}$$

Table 1.1. Data and results of the example

DMU	$\tilde{X}$	Z	$\tilde{Y}$	First phase efficiency	Second phase efficiency	Total efficiency
1	(100,111)	30	(200,212)	(0.8070,0.8220)	(1,1)	(0.8890,0.9390)
2	(103,107)	32	(210,217)	(0.8430,0.8670)	(0.9330,0.9533)	(0.8800,0.9230)
3	(98,102)	46	(217,223)	(1,1)	(0.8740,0.8990)	(0.9220,0.9620)
4	(118,122)	31	(22,227)	(0.7400,0.7600)	(1,1)	(0.8520,0.8920)
5	(88,91)	40	(240,250)	(1,1)	(1,1)	(1,1)
6	(147,145)	38	(227,235)	(0.6329,0.6529)	(0.9321,0.9731)	(0.7550,0.8100)
7	(124,127)	44	(232,245)	0.7630,0.7830)	(0.9370,1)	(0.8520,0.8992)
8	(128,131)	41	(240,250)	(0.6050,0.7190)	(1,1)	(0.8220,0.8830)
9	(113,117)	35	(227,235)	(0.7520,0.8020)	(0.9520,1)	(0.8530,0.9001)
10	(103,107)	37	(230,240)	(0.8122,0.8750)	(0.9532,1)	(0.9010,0.9310)

6. Conclusions

A two-phase system is efficient when both phases of the two-phase system are efficient. So using this technique, any of the decision making units may not be efficient.

Examining a two-phase system, it is concluded that any change in the size of the middle values and their increase for increasing the first phase efficiency results in the decrease in the second phase efficiency, and decreasing those values to increase the second phase efficiency results in the decrease in the first phase efficiency. Hence, the decrease in the efficiency of any phase results in the decrease in the efficiency of the whole system. So, middle sized values are kept constant to prevent from the decrease in the system's efficiency.

Therefore, to increase the efficiency, it is proposed that BCC input-oriented model is used in the first phase and BCC output-oriented model is used in the second phase while the middle values are kept constant. According to the model by Kao and Hwang, we obtained models in the upper and lower bounds in this article, and we calculated the efficiency interval for when data are as intervals by presenting theorems.

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