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Research Article

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Some Conditions for Characterizing Minimum Face in Non-Radial DEA Models with Undesirable Outputs

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Abstract

The problem of utilizing undesirable (bad) outputs in DEA models often need replacing the assumption of free disposability of outputs by weak disposability of outputs. The Kuosmanen technology is the only correct representation of the fully convex technology exhibiting weak disposability of bad and good outputs. Also, there are some specific features of non-radial data envelopment analysis (DEA) models for obtaining all projections of a decision making unit (DMU) on the boundary of production possibility set (PPS) or efficient frontier. Production technologies in DEA are modeled by polyhedral sets that envelop the observed DMUs. Because the efficient frontiers of DEA technologies are generally non-smooth and are characterized by different faces, thus, all projections of a DMU on efficient frontier can not belong to different faces that do not have common points. The rationale behind abovementioned statement is as follows: if all projections of a DMU belong to different faces then the interior points of PPS will become efficient that contradicts the principles of optimality conditions in linear programming models. Therefore all projections would belong to a unique face that is called *minimum face*. In this paper we propose a procedure to find minimum face and so all projections of a DMU on efficient frontiers in non-radial DEA models with undesirable outputs. This leads us to an interesting algorithm to obtain minimum face.

Keywords: Data Envelopment Analysis, Undesirable Outputs, Non-radial Model, Minimum Face.

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1.Introduction

In literature of DEA [1-3,7,9] the problem of undesirable outputs has been an important part of discussion. The problem of modeling undesirable (bad) outputs (such as air, water and ground emission) has been interesting among production economists. A simple approach is to consider the bad outputs as input [3]. However, Fare and Grosskopf [4,5] showed that considering undesirable outputs as inputs is inconsistent with the physical laws and the standard axioms of production theory. These authors introduced weak disposability of outputs. Weak disposability means that in order to abate emissions, it is possible to decrease the level of production activity by an abatement factor. Using non-uniform abatement factors has been investigated by Kuosmanen [7]. Also, various cases of convexity in the presence of weak disposability assumption have been considered in [11]. They established the relations between Kuosmanen, Shephard and Bogetoft technologies.

The problem of obtaining all projections of a unit (DMU) on the efficient frontier in non-radial DEA models has investigated by different authors [6,11]. It has been showed that all projections of a unit can not belong to different faces of efficient frontiers. Thus, a unique face has to be presented as a face that all projections of a unit belong to it. This unique face is called *minimum face* [11]. In order to find minimum face in non-radial DEA models [11] introduced a procedure based on the core concept of optimal solutions of envelopment and multiplier non-radial models.

In this paper we utilize the procedure introduced in [11] to obtain minimum face in the presence of weak disposability of outputs. This leads us to an algorithm for obtaining minimum face and all projections of a unit that belong to minimum face.

The rest of paper is as follows: section 2 provides main results of paper. A numerical example has been considered in section 3.

2. Main results

Consider a technology with m inputs, s good outputs and $(r-s)$ bad outputs. We assume that there are $n \geq 1$ observed DMUs and that each can be stated in the form $(X_j, Y_j^G, Y_j^B),\ j=1,...,n$, where $X_j \in R_+^m \setminus \{0\}, Y_j^G \in R_+^s \setminus \{0\}$ and $Y_j^B \in R_+^{(r-s)} \setminus \{0\}$ are its input, good output and bad output vectors, respectively. The production possibility set is defined as follows:

$$T = \left\{ (X, Y^G, Y^B) \middle| X \text{ can produce} (Y^G, Y^B) \right\} \quad (1)$$

Or alternatively, by output set in the following manner:

$$P(X) = \left\{ (Y^G, Y^B) \middle| (X, Y^G, Y^B) \in T \right\} \quad (2)$$

Weak disposability of bad outputs is defined as follows [10].

Definition 1. *(Weak Disposability of outputs)* If $(Y^G, Y^B) \in P(X)$ and $0 \leq \theta \leq 1$ implies $(\theta Y^G, \theta Y^B) \in P(X)$.

Following [7], we shall present the production possibility set in the following manner:

$$T_k = \left\{ (X, Y^G, Y^B) \middle| \begin{array}{l} \sum_{j=1}^{n} (\lambda_j + \mu_j) X_j \le X, \sum_{j=1}^{n} \lambda_j Y_j^G \ge Y, \sum_{j=1}^{n} \lambda_j Y_j^B = Y^B, \\ \sum_{j=1}^{n} (\lambda_j + \mu_j) = 1, \lambda_j \ge 0, j = 1, \ldots, n, \mu_j \ge 0, j = 1, \ldots, n \end{array} \right\} \quad (3)$$

The boundary points of T_K is called the efficient frontier of technology. If a point (DMU) is located on the efficient frontier then it is called "efficient". Utilizing (3), the non-radial DEA model in the presence of bad outputs can be written in the following manner:

$$\text{Max} \quad \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{i=1}^{s} s_i^+\right) \quad (4)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j y_{ij}^G - s_i^+ = y_{io}^G, \quad i = 1, \ldots, s, \quad (4.1)$$

$$\sum_{j=1}^{n} \lambda_j y_{ij}^B = y_{io}^B, \quad i = s+1, \ldots, r,$$

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) x_{ij} + s_i^- = x_{io}, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} (\lambda_j + \mu_j) = 1,$$

$$\lambda_j \ge 0, j = 1, \ldots, n, \mu_j \ge 0, \quad j = 1, \ldots, n,$$

$$s_i^+ \ge 0, \quad i = 1, \ldots, s,$$

$$s_i^- \ge 0, \quad i = 1, \ldots, m.$$

The dual of model (4) is as follows:

$$\text{Min} \quad \sum_{i=1}^{m} v_i x_{io} - \sum_{i=1}^{s} u_i^G y_{io}^G + \sum_{i=s+1}^{r} u_i^B y_{io}^B + u_0 \quad (5)$$

$$\text{s.t.} \quad \sum_{i=1}^{m} v_i x_{ij} - \sum_{i=1}^{s} u_i^G y_{ij}^G + \sum_{i=s+1}^{r} u_i^B y_{ij}^B + u_0 \ge 0,$$

$$j = 1, \ldots, n \quad (5\text{-}1)$$

$$\sum_{i=1}^{m} v_i x_{ij} + u_0 \ge 0, \; j = 1, \ldots, n,$$

$$u_i^G \ge \varepsilon, i = 1, \ldots, s,$$

$$v_i \ge \varepsilon, i = 1, \ldots, m,$$

$$u_i^B, i = s+1, \ldots, r, u_0 \; free.$$

Let $(\lambda^*, \mu^*, s^{+*}, s^{-*})$ and $(v^*, u^{G*}, u^{B*}, u_0^*)$ be the optimal solutions of model (4) and (5), respectively.

Theorm 1. (Podinovski et al., [8]) A technology such as T_K is a polyhedral (and therefore convex) set.

Definition 1. (Pareto-Koopmans efficiency) A unit $(X, Y^G, Y^B) \in T_K$ is Pareto-Koopmans efficient if there is no unit $(\bar{X}, \bar{Y}^G, \bar{Y}^B) \in T_K$ such that $\bar{X} \le X, Y^G \le \bar{Y}^G, Y^B = \bar{Y}^B$ and at least one of the components of inequalities is strict.

Definition 2. (Kuosmanen-additive efficiency) A unit $(X, Y^G, Y^B) \in T_K$ is Kousmanen-additive efficient if $s^{+*} = 0 \, and \; s^{-*} = 0$.

Theorm 2. Definition 1 and Definition 2 are equivalent.

Proof. The proof is simple and omitted.

The set Λ^* of projections of unit $(X_o, Y_o^G, Y_o^B) \in T_K$ on the frontier in model (4) can be written as follows:

$$\Lambda^* = \left\{ \begin{array}{l} \sum_{j=1}^{n} (\lambda_j + \mu_j) X_j \le X, \sum_{j=1}^{n} \lambda_j Y_j^G \ge Y, \sum_{j=1}^{n} \lambda_j Y_j^B = Y^B, \\ \sum_{j=1}^{n} (\lambda_j + \mu_j) = 1, \lambda_j \ge 0, j = 1, \ldots, n, \mu_j \ge 0, j = 1, \ldots, n \end{array} \right\} \quad (6)$$

Since, the Kousmanen technology T_K is a convex polyhedral set thus, the boundary points or frontier has been constructed by a finite number of faces. Therefore, the projections of unit $(X_o, Y_o^G, Y_o^B) \in T_K$ belong to a face of T_K. Also, the set Λ^* can not belong to different faces that do not have common points, otherwise interior points of T_K would belong to the solution set. Furthermore, a point of T_K is a vertex point, if it cannot be written as convex combination of any other points of T_K or there is at least one set of $m+r$ binding linearly independent hyperplanes at this point. The dimension of a face in T_K may vary from 0 up to $m+r-1$. Faces of 1-dimension are called edges and 0-dimension as vertices.

Theorem 3. In model (4), there exists a unique minimum face $\Gamma_{\min}$ that is a polyhedron and also, contains set Λ^*.

Proof. Krivonozhko et al. [6].

Now, for obtaining $\Gamma_{\min}$ and Λ^* determine the following index sets:

$$I_\lambda^* = \{ j \mid \lambda_j^* > 0, j = 1,...,n \},$$

$$I_\mu^* = \{ j \mid \mu_j^* > 0, j = 1,...,n \}$$

$$I_x^- = \{ i \mid s_i^{-*} > 0, i = 1,...,m \},$$

$$I_y^+ = \{ i \mid s_i^{+*} > 0, i = 1,...s \}$$

Also, the index sets associated with the dual variables:

$$J_\lambda^* = \{ j \mid v^* X_j - u^{*G} Y_j^G + u^{*B} Y_j^B + u_0^* = 0, j = 1,...,n \}$$

$$J_\mu^* = \{ j \mid v^* X_j + u_0^* = 0, j = 1,...,n \},$$

$$J_v^* = \{ i \mid v_i^* = \varepsilon, i = 1,..., m \},$$

$$J_{u^G}^* = \{ i \mid u_i^{G*} = \varepsilon, i = 1,..., s \}$$

It is obvious that

$$I_\lambda^* \cup I_\mu^* \cup I_x^- \cup I_y^+ \subseteq J_\lambda^* \cup J_\mu^* \cup J_v^* \cup J_{u^G}^* .$$

Theorem 4. In model (5) at least one of the constraints of (5.1) is binding.

Proof. If all constraints of (5.1) are strict, then $\lambda_j^* = 0, j = 1,...,n$ that contradicts the constraints (4.1).

Now, for finding all points belonging to the face $\Gamma_{\min}$, we solve the following problems:

$$\begin{aligned} \phi_l = \max \quad & \lambda_l \\ s.t. \quad & \sum_{j\in J_\lambda^*} \lambda_j X_j + \sum_{j\in J_\mu^*} \mu_j X_j + \sum_{i\in J_v^*} e_i s_i^- = X_o, \quad (7) \\ & \sum_{j\in J_\lambda^*} \lambda_j Y_j^G - \sum_{i\in J_{u^G}^*} e_i s_i^- = Y_o^G, \\ & \sum_{j\in J_\lambda^*} \lambda_j Y_j^B = Y_o^B, \\ & \sum_{j\in J_\lambda^*} \lambda_j + \sum_{j\in J_\mu^*} \mu_j = 1, \\ & \lambda_j \ge 0, j \in J_\lambda^*, \mu_j \ge 0, j \in J_\mu^*, \\ & s_i^- \ge 0, i \in J_v^*, s_i^+ \ge 0, i \in J_{u^G}^* . \end{aligned}$$

And,

$$\begin{aligned} \eta_l = \max \quad & \mu_l \\ s.t. \quad & \sum_{j\in J_\lambda^*} \lambda_j X_j + \sum_{j\in J_\mu^*} \mu_j X_j + \sum_{i\in J_v^*} e_i s_i^- = X_o, \quad (8) \\ & \sum_{j\in J_\lambda^*} \lambda_j Y_j^G - \sum_{i\in J_{u^G}^*} e_i s_i^- = Y_o^G, \\ & \sum_{j\in J_\lambda^*} \lambda_j Y_j^B = Y_o^B, \\ & \sum_{j\in J_\lambda^*} \lambda_j + \sum_{j\in J_\mu^*} \mu_j = 1, \\ & \lambda_j \ge 0, j \in J_\lambda^*, \mu_j \ge 0, j \in J_\mu^*, \\ & s_i^- \ge 0, i \in J_v^*, s_i^+ \ge 0, i \in J_{u^G}^* . \end{aligned}$$

Note that problem (7) includes only those variables for which the corresponding dual constraints hold with strict equality for optimal dual variables of model (5).

Now, the procedure that finds all units belonging to the $\Gamma_{\min}$ and Λ^* is as follows:

1. Initialize set $J_o = \Phi, JH = J^*_\lambda, JM = J^*_\mu, J_1 = \Phi, J_2 = \Phi$ If the set JH in not empty, go to the next step. If JH is empty go to the step 4.

2.Choose index $l \in JH$, if the set JH is empty go to the step 4. Solve problem (7). If $\phi_l > 0$, then determine $J_o = J_o \cup l$. If $\phi_l = 1$ then $J_1 = J_1 \cup l$. Delete index l from the JH. Go to the beginning of step 2.

3. Choose index $l \in JM$. If the set JM is empty go to the step 4. Solve problem (8). If $\eta_l > 0$, then determine $J_2 = J_2 \cup l$. Delete index l from JM. Go to the beginning of step 3.

4. Set J_o determines the set of units belonging to the face $\Gamma_{\min}$. Set J_1 determines the set of units belonging to the set Λ^*.

The procedure is completed.

After running the procedure, the minimum face $\Gamma_{\min}$ can be written as follows:

$$\Gamma_{\min} = \left\{ (X, Y^G, Y^B) \left| \begin{array}{c} X = \sum_{j \in J_0} \lambda_j X_j + \sum_{j \in J_2} \mu_j X_j, \\ Y^G = \sum_{j \in J_0} \lambda_j Y_j^G, \quad Y^B = \sum_{j \in J_0} \lambda_j Y_j^B, \\ \sum_{j \in J_0} \lambda_j + \sum_{j \in J_2} \mu_j = 1, \\ \lambda_j \geq 0, \quad j \in J_0, \quad \mu_j \geq 0, \quad j \in J_2 \end{array} \right. \right\} \tag{9}$$

Also, we can define set Λ^* as follows:

$$\Lambda^* = \left\{ (X, Y^G, Y^B) \left| \begin{array}{c} (X, Y^G, Y^B) \in \Gamma_{\min}, X \leq X_0, Y^G \geq Y_0^G, \\ Y^B = Y_0^B, \quad \varepsilon\left(\sum_{i=1}^{r} y_i^G - \sum_{i=1}^{m} x_i\right) = \\ (v^* - \varepsilon)X_0 - (u^{G^*} - \varepsilon)Y_0 + u^{B^*} Y_0^B + u_0^* \end{array} \right. \right\} \tag{10}$$

3. Numerical example

Consider a set of three DMUs consisting of DMU_1, DMU_2 and DMU_3 , in which each DMU utilizes one input to produce one good and one bad output as in Table 1. The production technology T_K is depicted in Fig. 1. As illustrated in Fig. 1, the DMU_1 and DMU_2 are extreme efficient (vertex) units. If we solve problems (4) and (5) for DMU_3 then the optimal solutions will be as follows:

$$\lambda_1^* = \frac{1}{2}, \lambda_2^* = \frac{1}{2}, \lambda_3^* = 0, \mu_1^* = 0,$$

$$\mu_2^* = 0, \mu_3^* = 0, s^{-*} = \frac{5}{2}, s^{G*} = 0,$$

$$v^* = \frac{1}{100}, u^{G*} = \frac{1}{100}, u^{B*} = \frac{1}{300}, u_0^* = \frac{2}{300}.$$

Thus, we will have the following index sets:

$$I_\lambda^* = \{1, 2\}, I_\mu^* = \Phi, I_x^- = \{1\},$$

$$I_y^+ = \Phi, J_\lambda^* = \{1, 2\}, J_\mu^* = \Phi,$$

$$J_v^* = \{1\}, J_{u^G}^* = \{1\}$$

By solving models (8) and (9), the optimal solutions $\phi_1^* = \frac{1}{2}, \phi_2^* = \frac{1}{2}$ can be obtained. Therefore the minimum face of DMU_3 is a convex combination of DMU_1 and DMU_2. The set Λ^* can be obtained by (10).

Table 1. Data of Example 1.

	DMU_1	DMU_2	DMU_3
Good output	3	5	5
Bad output	4	1	5/2
Input	1	4	5

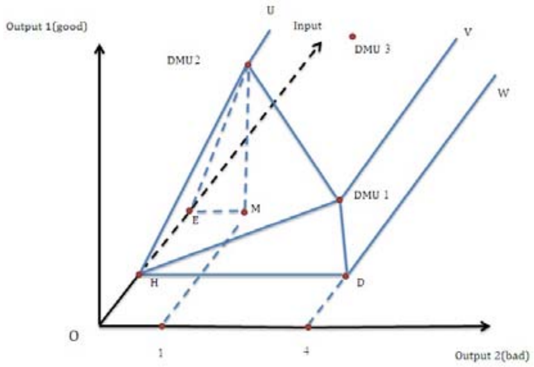

Figure 1. The PPS of Example 1.

4. Conclusion

In this paper, the problem of obtaining all projections of a unit on efficient frontier in the presence of weak disposability has been investigated. We utilized the core concept of weak disposability assumption in production technology and tired to find the minimum face in this case. Some additional linear programming has been provided to obtain such a minimum face. Also, a procedure has been proposed to find all units that construct minimum face. In actual fact, we made use of the concept of weak disposability to find minimum face. We hope that the concept of this paper will be a great value and significance to the readers.

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