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An Extension of Enhanced Russell Measure to deal with Interval Scale Data in DEA

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Abstract

Data Envelopment Analysis (DEA) models with interval inputs and outputs have been rarely discussed in DEA literature. This paper, using the enhanced Russell measurement proposes an extended model which permits the presence of interval scale variables which can take both negative and positive values. The model is compared with most well-known DEA models of which include the CCR model, the BCC model and the additive model. An empirical data set is used to illustrate the model.

Keywords: data envelopment analysis; decision making; Translation invariance*;* interval scale; ratio scale.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric method for evaluating the relative efficiency of decision-making units (DMUs) on the basis of multiple inputs and outputs. The original DEA models use (positive) input and output variables which are measured on a ratio scale but these models do not apply to variables in which interval scale data can appear [2]. With the widespread use of interval scale variables, such as profit or the increase/decrease in bank accounts emphasis has shifted to the simultaneous consideration of ratio and interval scale data in DEA models. However, interval scale inputs/outputs cannot be used widely in DEA models. The problem with interval scale variables arises from the fact that rations of measurement on such a scale are meaningless. For handling interval scale data, Halme et al. (2000) proposed an approach which based on the idea of replacing each of the interval scale variables with the two ratio scale variables that give rise to them [6]. As acknowledged by Halme et al. as a drawback, since the number of variables rise as a consequence of the decomposition of the

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interval scale variables, some of the inefficient units may become efficient. But, there are some cases which the interval scale variables is not a consequence of using the subtraction of two ratio scale variables. For instance in the assessment of the quality of research of different departments of a university, one of the criteria may be ‘‘standard of Research at International Level” with a scale where one represents ‘‘Poor” and two represents ‘‘Fair”. The scale can be regarded as a proxy to the value scale, which is an interval scale. Thus we cannot say that a score of two (‘‘Fair”) implies that it is two times better than ‘‘Poor”.

In this paper, we propose an extended of enhanced Russell measurement (pastor et al, 1999) that yields a measure of efficiency and also is able to handle variables consisting of positive values for some and negative values for other sample DMUs. This paper unfolds as follows. In section 2 we discuss interval scale, ratio scale, translation invariant and the problem of interval scale data in DEA models. In section 3, we introduce preliminaries of DEA. In sections 4 using the enhanced Russell measurement we propose an extended model which permits the presence of interval scale variables which can take both negative and positive values. In section 5 we provide a numerical example to compare the results. Conclusions are summarized in section 6.

2. Interval scale, ratio scale and translation invariant

Definition 2.1 : (Interval Scale) Let P is a set of observed judgments and $\mathcal{R}$ real numbers. A mapping s: P $\longrightarrow \mathcal{R}$ such that $\forall i, j(i, j \epsilon P \Longrightarrow$ i is preferred to j $\Leftrightarrow s(i) > s(j))$.

is said to be interval scale if any transformation of the function values that preserves their numerical difference produces another function that shares the same one-to-one relation between comparisons among objects (using) and comparisons among corresponding function values (using –) [7].

So the set of permissible transformations for interval scales preserves relative differences. Specifically, a transformation, *f*, is permissible for interval scales if and only if there is a constant *c* such that $s(i) - s(j) = c\{f[s(i)] - f[s(j)]\}$.Thus, linear transformations in which we add the same constant to each value and/or multiply each value by a constant are permissible for interval scale data. On these scales, of course, it is meaningless to say, one value is twice or some other proportion greater than another [10] .An example of an interval scale is temperature, either measured on a Fahrenheit or Celsius scale. But it is important to understand the different levels of measurement when using and interpreting scales. For example, we cannot say that 100 degrees is twice as hot as 50 degrees. The interval scale data are invariant under any translation transformation. Thus, the zero point on an interval scale is arbitrary; that is, it does not present the total absence of the measured characteristic; therefore a negative interval scale data, for example–20 C, is a menningful quantity.

Definition 2.2 : (Ratio Scale) Let P is a set of observed judgments and $\mathcal{R}$ real numbers. A mapping s: P $\longrightarrow \mathcal{R}$ such that

$\forall$i, j(i, jϵP $\Longrightarrow$ i is preferred to j $\Leftrightarrow$ s(i) $>$ s(j)).

is said to be ratio scale if any transformation of the function values that preserves their numerical ratio produces another function that shares the same one-to-one relation between comparisons among objects (using) and comparisons among corresponding function values (using /) [7].

So, a transformation, *f*, is permissible for ratio scales if and only if there is a constant *c* such that $s(i)/s(j) = c\{f[s(i)]/f[s(j)]\}$. Thus, it is permissible to multiply ratio scale data by a constant, but we may not take logs or add a constant, and so the ratio scale data are not invariant under any translation transformation. Therefore, a defined zero on these scales may not be changed. Thus, the zero point in Kelvin degrees which is about -485 F means that there is no hot. Also the absolute or true zero point, means that observations can be compared as a ratio. Therefore, we can say that 100 K is twice as hot as50 K. Also the true zero point represent the total absence of the measured characteristic. In other words, the ratio scale measures characteristic being measured from true zero point, for example Kelvin degrees measures the hot characteristic from the true zero point, therefore, negativity of such a scale are meaningless.

Definition 2.3: (Translation Invariance) Given any problem, a DEA model is said to be tranlation invariant if translating the original inpout and/or output data values results in a new problem that has the same optimal solution for the envelopment form as the old one.[10].

The translation invariance property of a DEA model allows us to deale with negative data. On the other hand, translation transformations in which we add the same constant to each value by a constant are permissible for interval scale data. But, translation transformations are not permissible for ratio scale data. Therefore, we can use the translation invariance property of a DEA model only to deal with unrestricted-in-sign interval scale input and/or output variables.

3. The initial DEA models

Suppose we have *n* peer observed DMUs (decision-making units) where every $DMU_j (j = 1,2, \dots, n)$ produce multiple outputs $y_{rj}, (r = 1, \dots, s)$ by utilizing multiple inputs $x_{ij}, (i = 1, \dots, m)$. The input and output vectors of DMU_j are denoted by x_j and y_j, respectively, and we assume $x_j$and y_j are semi positive, i.e.,

$$x_j \geq 0, x_j \neq 0; \;\; y_j \geq 0, y_j \neq 0, \;\; \text{for } j = 1, \dots, n. \qquad (1)$$

We use by $(x_j,\ y_j)$ to descript DMU_j, and specially use $(x_o,\ y_o)$ $(o\epsilon\{1,2, \dots, n\})$ as the DMU under evaluation. Also we assume

$$\forall i \left(i\epsilon\{ 1, \dots, m\} \Rightarrow \exists j\big(j\epsilon\{1, \dots, n\} \,\&\, x_{ij} > 0 \,\big)\right),$$
$$\forall r \left(r\epsilon\{ 1, \dots, s\} \Rightarrow \exists j\big(j\epsilon\{1, \dots, n\} \,\&\, y_{rj} > 0 \,\big)\right). \quad (2)$$

Throughout this paper, vectors will be denoted by bold letters.

The efficiency of each DMU_o can be evaluated by the envelopment form of the input-oriented CCR model by solving the following linear program,

$$\begin{aligned} \min \quad & \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}\,; \quad i = 1, \dots, m, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r = 1, \dots, s, \\ & \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0; \quad j = 1, \dots, n\,, i = 1, \dots, m\,, r = 1, \dots, s, \end{aligned} \quad (3)$$

where θ is a scalar, and $\varepsilon > 0$ is the non-Archimedean element.

As shown by Charnes and Cooper [2], DMU_o is CCR-efficient if and only if for the optimal solution $(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of the linear programming problem (3) the following are satisfied:

$\theta^* = 1$, $s^{-*} = 0, s^{+*} = 0$.

Also, DMU_o is extreme CCR-efficient if only if $E_o = \{DMU_o\}$ where E_o is defined as

$E_o = \left\{DMU_j \middle| j\epsilon\{1, \dots, n\}\& \lambda_j^* > 0 \text{ in some optimal solution } (\theta^*, \lambda^*, s^{-*}, s^{+*}) \text{ of model (3)}\right\}$

$E_o \neq \emptyset$ and DMUs in E_o are CCR-efficient [10].

Theorem 2.1. DMU_o is extreme CCR-efficient iff

$$\begin{aligned} \min \quad & \theta - \varepsilon\left(\sum_{j\neq o} \lambda_j\right) \\ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}\,; \quad i = 1, \dots, m, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r = 1, \dots, s, \\ & \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0; \quad j = 1, \dots, n\,, i = 1, \dots, m\,, r = 1, \dots, s, \end{aligned} \quad (4)$$

has an optimal objective function value of one.

Proof. Let DMU_o not be extreme CCR-efficient. Then there exists an optimal solution $(\theta^*, \lambda^*, s^{*-}, s^{*+})$ to model (3) such that a $\lambda_j^* > 0$ $(j \neq o)$. Also, $\theta^* \leq 1$. Thus $\theta^* - \varepsilon \sum_{j\neq o} \lambda^*{}_j < 1$.

Let the optimal objective function value of model (4) be less one, and let $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ be an the model, then either $\tilde{\theta} < 1$ or $\tilde{\theta} = 1$ and $\sum_{j\neq o} \tilde{\lambda_j} > 0$. If $\tilde{\theta} < 1$, DMU_o isn't extreme BCC-efficient. If $\tilde{\theta} = 1$ and $\sum_{j\neq o} \tilde{\lambda_j} > 0$, then either $(\tilde{s}^-, \tilde{s}^+) \neq (0,0)$ or $(\tilde{s}^-, \tilde{s}^+) = (0,0)$ If $(\tilde{s}^-, \tilde{s}^+) \neq (0,0)$, since $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ is a feasible solution of model (3). Then DMU_o isn't CCR-efficient, thus DMU_o isn't extrem CCR-efficient. If $(\tilde{s}^-, \tilde{s}^+) = (0,0)$, then either $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ is an optimal solution of mode (3) or isn't. If $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ be an optimal solution of (3), then, since $\sum_{j=1}^{n} \tilde{\lambda}_j > 0$, thus DMU_o isn't extrem CCR-efficient. If $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ not be an optimal solution of (3), then there exists an optimal solution$(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of model (3) such that $\theta^* = 1$ with$(s^{*-}, s^{*+}) \neq (0,0)$. Thus DMU_o isn't extreme CCR-efficient.■

In model (3) the ratio of virtual effect inputs and observed inputs (positive data) play a central role in the calculations, thus input data must be measured on ratio scale; however in this model the ratio of virtual effect outputs and observed outputs hasn't any role in the calculations. Therefore we can apply the model (3) with interval scale output data. But, since an assumption of CRS is not possible in technologies where negative data can exist (see Portela et al., 2004), thus the model (3) cannot be used with the negative interval scale output data. Also, since negativity ratio scale variables is meaningless, therefore in model (3) none of the input and output variables cannot be negative. If some inputs are interval scale data, a modified ratio scale input-oriented CCR model is proposed as follows:

$$\begin{aligned}
\min \quad & \frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}}\theta_i - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
& \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = \theta_i x_{io}; \quad i\epsilon R_{inp}, \\
& \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}; \quad i\epsilon I_{inp}, \\
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro}; \quad r\epsilon R_{out}\cup I_{out}, \\
& \theta_i \geq 0, i = 1,\dots,|R_{inp}|,\ \lambda_j \geq 0, \forall j;\ s_i^- \geq 0, \forall i;\ s_r^+ \geq 0, \forall r.
\end{aligned} \tag{5}$$

Where

$$\begin{aligned}
R_{inp} &= \{i | i\epsilon\{1,\dots,m\}\ \&\ \forall j(j\epsilon\{1,\dots,n\} \Rightarrow\ x_{ij} \text{ is a ratio scale variable})\} \\
I_{inp} &= \{i | i\epsilon\{1,\dots,m\}\ \&\ \forall j(j\epsilon\{1,\dots,n\} \Rightarrow\ x_{ij} \text{ is an interval scale variable})\} \\
R_{out} &= \{r | r\epsilon\{1,\dots,s\}\ \&\ \forall j(j\epsilon\{1,\dots,n\} \Rightarrow y_{rj} \text{ is a ratio scale variable})\} \\
I_{out} &= \{r | r\epsilon\{1,\dots,s\}\ \&\ \forall j(j\epsilon\{1,\dots,n\} \Rightarrow y_{rj} \text{ is an interval scale variable})\},
\end{aligned} \tag{6}$$

and for simplification in using defined sets in (6) let $R_{inp} = \{1,\dots,|R_{inp}|\}, R_{out} = \{1,\dots,|R_{out}|\}$, where symbol of $|.|$ is the cardinality of sets. Since DMU_j ($j = 1,\dots,n$) are peer, then $R_{inp}\cup I_{inp} = \{1,\dots,m\}, R_{out}\cup I_{out} = \{1,\dots,s\}$.

Let

$$\left(\lambda^*, \left(\theta_1^*,\dots,\theta_{|R_{inp}|}^*\right), s^{-*}, s^{+*}\right) \tag{7}$$

be an optimal solution of model (5), by (1) and (2), $\theta_i^* > 0$ $(i = 1,\dots,|R_{inp}|)$. Also since $\left(\left(\theta_1,\dots,\theta_{|R_{inp}|}\right), \lambda, s^-, s^+\right)$, with $\theta_i = 1$ $(i = 1,\dots,|R_{inp}|)$, $\lambda_j = 0$ $(j \neq o)$, $\lambda_o = 1$, $s_i^- = 0 (i = 1,\dots,m)$, and $s_r^+ = 0 (r = 1,\dots,s)$ is a feasible solution to model of (1), thus $\theta_i^* \leq 1$ $(i = 1,\dots,|R_{inp}|)$. Therefore $0 < \frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}}\theta_i^* \leq 1$. $\frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}}\theta_i^*$ represents the ratio scale input-oriented CCR-efficiency measure of DMU_o.

Model (5), at first evaluates the radial efficiencies of ratio scale input data $\theta_i^* (i\epsilon R_{inp})$, then it takes account of the interval scale input excesses and output shortfalls that are represent by non-zero slacks. Thus model (5) detects all inefficiencies of inputs and outputs, if any; therefore , the model indicates DMU_o with the interval scale input and/or output data is CCR-efficient. Also, assuming that (7) be an

optimal solution of model (5), the production possibility $\left(\sum_{j=1}^{n}\lambda_j^* x_j, \sum_{j=1}^{n}\lambda_j^* y_j\right)$ is BCC-efficient. However, model (5), does not indicate DMU_o that it is extreme BCC- efficient. A modified version of the model is proposed as follows, also indicate extreme CCR-efficiency and provide a performance measure DMU_o with interval scale inputs and/or outputs,

$$\begin{aligned}
\min\quad & \frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}}\theta_i - \varepsilon\left(\sum_{j\neq o}\lambda_j\right)\\
& \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = \theta_i x_{io}; \quad i\epsilon R_{inp},\\
& \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}; \quad r\epsilon I_{inp},\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro}; \quad r=1,\dots,s,\\
& \theta_i \geq 0, i=1,\dots,|R_{inp}|; \quad \lambda_j \geq 0, \forall j; \ s_i^- \geq 0, \forall i; \ s_r^+ \geq 0, \forall r.
\end{aligned} \tag{8}$$

Models (5) and (8) are not translation invariant with respect to either outputs or inputs; however these models with convexity constraint $\sum_{j=1}^{n}\lambda_j = 1$ are translation invariant with respect to interval scale outputs. Therefore, model (5) with convexity constraint $\sum_{j=1}^{n}\lambda_j = 1$ can be used with non-negative interval scale inputs and negative interval scale outputs. Also, model (5) with constraint$\sum_{j=1}^{n}\lambda_j = 1$ both indicates DMU_o with interval input and/or output data is BCC-efficient and provide a performance measure that we call it *the ratio scale input-oriented measure of BCC-efficiency of* DMU_o. Model (8) with constraint $\sum_{j=1}^{n}\lambda_j = 1$ also indicates DMU_o with the scale interval input and/or output data is extreme CCR-efficient and provide a performance measure.

In model (6) we assume that $R_{inp} \neq \emptyset$. Now, if $R_{inp} = \emptyset$, then the following model is obtained:

$$\begin{aligned}
\max\quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\\
& \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}; \quad i=1,\dots,m,\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j y_{rj} + s_r^+ = y_{ro}; \quad r=1,\dots,s,\\
& \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0; \quad \forall j,i,r.
\end{aligned} \tag{9}$$

which is the model additive model proposed by Charnes et al. (1985) and it can be used with nonnegative scale interval inputs and/or outputs. Model (9) with convexity constraint $\sum_{j=1}^{n}\lambda_j = 1$ is the additive model with the variable returns to scale assumption as follows:

$$\begin{aligned}
\max\quad & \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)\\
& \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}; \ i=1,\dots,m,\\
& \sum_{j=1}^{n}\lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r=1,\dots,s,\\
\text{s.t.}\quad & \sum_{j=1}^{n}\lambda_j = 1,\\
& \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0; \quad \forall i,j,k.
\end{aligned} \tag{10}$$

Model (10) which is translation invariant as demonstrated by Ali and Seiford can be used with the interval scale and/or negative interval scale input and output data. Also model (10) indicates DMU_o is BCC- efficient. In other words, the optimal solution model of (10) is zero if and only if DMU_o be BCC-

efficient [4]. The model (10), however, does not provide an efficiency measurement for DMUs which are not BCC-efficient.

4. DEA models with enhanced Russell measurement (ERM)

Enhanced Russell measurement proposed by Pastor et al (1999) under the constant returns-to-scale assumption is as follows:

$$\begin{aligned} min \quad & \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r} \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i x_{io}; \quad i=1,\dots,m, \\ s.t. \quad & \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r y_{ro}; \quad r=1,\dots,s, \\ & \lambda_j \ge 0, \theta_i \le 1, \varphi_r \ge 1; \quad \forall i,j,k. \end{aligned} \tag{11}$$

where inputs and outputs of each DMU are positive. In model (11) both the ratio of virtual effect inputs and observed inputs and the ratio of virtual effect outputs and observed outputs plays a central role in the calculations, thus input and output data must be measured on ratio scale. Therefore model (11) cannot be used with interval scale data. If there are some interval scale input and/or output data, then a modified version of ERM for dealing with non-negative interval scale inputs and/or outputs is proposed as follows:

$$\begin{aligned} min \quad & \frac{\frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}}\theta_i}{\frac{1}{|R_{out}|}\sum_{r\epsilon R_{out}}\varphi_r} \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i x_{io}; \quad i\epsilon R_{inp}, \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{io}; \quad r\epsilon R_{out}, \\ \text{s. t.} \quad & \sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi_r y_{ro}; \quad r\epsilon R_{out}, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro}; \quad r\epsilon R_{out}, \\ & \lambda_j \ge 0, \forall j;\ \theta_i \le 1, i\epsilon R_{inp};\ \varphi_r \ge 1, r\epsilon R_{out}, \end{aligned} \tag{12}$$

where the ratio scale input and/or output data of all DMUs are positive. Model (12) with convexity constraint $\sum_{j=1}^{n}\lambda_j = 1$ is translation with respect to interval scale inputs and outputs, thus it can be applied with unrestricted-in-sign interval scale input and/or output data. In model (12) we have $\forall i\left(i\epsilon R_{inp} \Rightarrow \theta_i \le 1\right)$ & $\forall r(r\epsilon R_{out} \Rightarrow \varphi_r \ge 1)$, thus the model provide a performance measure that we call it *ratio scale data-oriented ERM- efficiency measure of* DMU_o. Although model of (12) is a nonlinear programming, the model is an ordinary linear fractional programming problem that solution can be found through a linear fractional programming problem. Therefore, using the following Charnes and Cooper (1962) transformations, let

$$\begin{aligned}
&\lambda = \left(\frac{1}{|R_{out}|}\sum_{\mathrm{r}\epsilon \mathrm{R}_{\mathrm{out}}} \varphi_r > 0\right)^{-1} \\
&\alpha_i = \lambda\theta_i; \qquad i\epsilon R_{inp}, \\
&\beta_r = \lambda\varphi_r; \qquad r\epsilon R_{out}, \\
&\mu_j = \lambda\lambda_j; \qquad j = 1, \dots, n.
\end{aligned} \tag{13}$$

Then, an optimal solution of the following linear programming problem

$$\begin{aligned}
\min \quad & \frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}} \alpha_i \\
& \sum_{j=1}^{n} \mu_j x_{ij} \le \alpha_i x_{io}; \qquad i\epsilon R_{inp}, \\
& \sum_{j=1}^{n} \mu_j x_{ij} \le \lambda x_{io}; \qquad i\epsilon I_{inp}, \\
& \sum_{j=1}^{n} \mu_j y_{rj} \ge \beta_r y_{ro}; \qquad r\epsilon R_{out}, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \mu_j y_{rj} \ge \lambda y_{ro}; \qquad r\epsilon I_{out}, \\
& \frac{1}{|R_{out}|}\sum_{r\epsilon R_{out}} \beta_r = 1, \\
& \lambda > 0;\ \mu_j \ge 0, \forall j;\ \ 0 < \alpha_i \le \lambda, i\epsilon R_{inp};\ \ \beta_r \ge \lambda, r\epsilon R_{out},
\end{aligned} \tag{14}$$

gives rise to an optimal solution of (12). To be precise, let

$$\left(\lambda^*, \mu^* = (\mu_1^*, \dots, \mu_n^*), \left(\alpha_1^*, \dots, \alpha_{|R_{inp}|}^*\right), \left(\beta_1^*, \dots, \beta_{|R_{out}|}^*\right)\right) \tag{15}$$

be an optimal solution of (14). Then, by (1) and (2) and with by$\mu^* \ge 0$,$\lambda^* > 0$. Thus through the change of variables (13) we can obtain the following optimal solution for model (12).

$$\left((\lambda_1^*, \dots, \lambda_n^*), \left(\theta_1^*, \dots, \theta_{|R_{inp}|}^*\right), \left(\varphi_1^*, \dots, \varphi_{|R_{out}|}^*\right)\right) \tag{16}$$

Also the associated optima are equal (see Charnes and Cooper, 1962). Moreover, since $\lambda^* > 0$, we have

$$\frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}} \alpha_i^* = \frac{\frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}} \alpha_i^*}{\frac{1}{|R_{out}|}\sum_{r\epsilon R_{out}} \beta_r^*} = \frac{\frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}} \theta_i^*}{\frac{1}{|R_{out}|}\sum_{r\epsilon R_{out}} \varphi_r^*}$$

Thus, if we are only interested in these efficiency values and not in the efficient projection of the DMUs being evaluated, we do not even need to transform the optimal solution of (14) through (13). If constraint $\sum_{j=1}^{n} \mu_j = \lambda$ that associated to the convexity constraint in model of (12) if VRS over the reference technology were assumed, is imposed to model (14), then we can apply the obtained model for dealing with unrestricted-in-sign interval scale input and/or output data. Let (15) be a feasible of model (14), then a necessary condition for optimality in the model is

$$\begin{aligned}
&\sum_{j=1}^{n} \mu_j^* x_{ij} = \alpha_i^* x_{io}; \qquad i\epsilon R_{inp}, \\
&\sum_{j=1}^{n} \mu_j^* y_{rj} = \beta_r^* y_{ro}; \qquad r\epsilon R_{out}.
\end{aligned} \tag{17}$$

Therefore, by transformations of (13), $\frac{(\lambda^* - \alpha_i^*)x_{io}}{\lambda^*}$ and $\frac{(\beta_r^* - \lambda^*)y_{ro}}{\lambda^*}$ are surplus of x_{io} $(i\epsilon R_{inp})$ and shortfall of $y_{ro}(r\epsilon R_{out})$, respectively. Model (14) takes account of all ratio scale input and output inefficiencies for providing of also the efficiency measurement and the efficiency projections DMU_o,

but the mode does not dealing with interval scale input and output inefficiencies. To address this problem, by (13), a modified version of model (14) is proposed as follows:

$$
\begin{aligned}
\min \quad & \frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}} \alpha_i \; + \varepsilon \left(\sum_{i\epsilon I_{inp}} t_i^- + \sum_{r\epsilon I_{out}} t_r^+\right) \\
& \sum_{j=1}^{n} \mu_j x_{ij} = \alpha_i \, x_{io}; \qquad i\epsilon R_{inp}, \\
& \sum_{j=1}^{n} \mu_j x_{ij} + t_i^- = \lambda x_{io}; \qquad i\epsilon I_{inp}, \\
& \sum_{j=1}^{n} \mu_j y_{rj} = \beta_r y_{ro}; \qquad r\epsilon R_{out}, \\
\text{s. t.} \quad & \sum_{j=1}^{n} \mu_j y_{rj} - t_r^+ = \lambda y_{ro}; \qquad r\epsilon I_{out}, \\
& \frac{1}{|R_{out}|}\sum_{r\epsilon R_{out}} \beta_r = 1, \\
& \lambda \geq 0; \; \mu_j \geq 0, \forall j; \;\; 0 < \alpha_{\,i} \leq \lambda, i\epsilon R_{inp}; \; \beta_r \geq \lambda, r\epsilon R_{out}, \\
& t_i^- \geq 0, i\epsilon I_{inp}; \;\; t_r^+ \geq 0, r\epsilon \, I_{out}.
\end{aligned}
\tag{18}
$$

Model (18) takes account of all input and output inefficiencies. Thus DMU_o is BCC-efficient if only if the optimal objective function value of model of (14), be equal one. Also, assuming

$$
\left(\lambda^*, \left(\theta_1^*, \dots, \theta_{|R_{inp}|}^*\right), \left(\varphi_1^*, \dots, \varphi_{|R_{out}|}^*\right), \left(s_{|R_{inp}|+1}^{-*}, \dots, s_m^{-*}\right), \left(s_{|R_{out}|+1}^{+*}, \dots, s_s^{+*}\right)\right) \tag{19}
$$

be an optimal solution of (14), production possible $\left(\sum_{j=1}^{n} \lambda_j^* x_j \, , \sum_{j=1}^{n} \lambda_j^* y_j\right)$, where $\lambda_j^* = \frac{\mu_j^*}{\lambda^*}$ is BCC-efficient. Model (14) with constraint $\sum_{j=1}^{n} \mu_j = \lambda$, take account of all input and output inefficiencies of DMU_o in the presence of unrestricted-in-sign interval scale input and/or output data. Alternatively, also to indicate extreme CCR-efficiency and to provide efficiency measure of DMU_o with interval scale inputs and/or outputs, we propose, by model (14) and model (2), a modified model as follows:

$$
\begin{aligned}
\min \quad & \frac{1}{|R_{inp}|}\sum_{i\epsilon R_{inp}} \alpha_i \; - \varepsilon\left(\sum_{j\neq o} \mu_j\right) \\
& \sum_{j=1}^{n} \mu_j x_{ij} = \alpha_i \, x_{io}; \qquad i\epsilon R_{inp}, \\
& \sum_{j=1}^{n} \mu_j x_{ij} + t_i^- = \lambda x_{io}; \qquad i\epsilon I_{inp}, \\
& \sum_{j=1}^{n} \mu_j y_{rj} = \beta_r y_{ro}; \qquad r\epsilon R_{out}, \\
\text{s. t.} \quad & \sum_{j=1}^{n} \mu_j y_{rj} - t_r^+ = \lambda y_{ro}; \qquad r\epsilon I_{out}, \\
& \frac{1}{|R_{out}|}\sum_{r\epsilon R_{out}} \beta_r = 1, \\
& \lambda \geq 0; \mu_j \geq 0, \forall j; \; 0 < \alpha_{\,i} \leq \lambda, i\epsilon R_{inp}; \beta_r \geq \lambda, r\epsilon R_{out}, \\
& t_i^- \geq 0, i\epsilon I_{inp}; \; t_r^+ \geq 0, r\epsilon \, I_{out}.
\end{aligned}
\tag{20}
$$

Model (20) with constraint $\sum_{j=1}^{n} \mu_j = \lambda$ can be used for dealing with unrestricted in sign interval scale input and/or output data.

5. Illustrative example

In this section, we use the data recorded in table 1 to illustrate how approaches introduced in Section 3 perform. These correspond to 20 DMUs, whose efficiency is assessed using three inputs that the first and second inputs are measured on ratio scale; the third input is measured on interval scale, and three outputs that the first and second outputs are measured on ratio scale; the third output is measured on interval scale.

In Table 2, the first and second columns, respectively, record the efficiency measure provided by model (3) and model (5). The third column contains efficiency measure provided by the phase I procedure of model (8). The fourth column contains efficiency measure provided by model (8). The fifth and the sixth, respectively, record the efficiency measure provided by model (14) and model (18). Applying model (3) to the data reveals 8 efficient units. The most inefficient unit detected by model (3) is Unit 2, with 0.3214259. The efficiency measure of Unit 2 detected by of model (8) and model (14), respectively, is 0.1632614 and 0.1069703. Also values of Eff (3), for inefficient unites go from 0.3214259, for Unit 2, to 0.9999992, for Unit 7. Values of Eff (8-2), for inefficient unites go from 0.1428554, for Unit 3, to 0.9999992, for Unit 7 which is ratio scale input-oriented CCR-efficient. Values of Eff (18), for inefficient unites go from 0.1069690, for Unit 2, to 0.9549058, for Unit 9. This shows that the discriminating capability model (18) is much stronger than that of model (3), and model (8), also discriminating capability model (8) is much stronger than that of model (3). Model (18) diagnoses radial efficient units correctly. On the other hand, the efficiency scores for units which are not efficient will differ with efficiency scores obtained from model (3).

Table1. Data Set

	Input1	Input2	Input3	Output1	Output2	Output3
Unit 1	7	1	4	1	2.5	3
Unit 2	3	7	4	2.5	1	3
Unit 3	6	6	3	2.5	2	3
Unit 4	3	1	3	4	5.5	7
Unit 5	6	0.5	3	5	3.5	6
Unit.6	4	0.5	3	2	6	6
Unit.7	1.5	2.5	3	6	4	7
Unit.8	0.5	4	4	1.5	5	6
Unit.9	2.75	1.75	4	8	3	6
Unit.10	1	3	3	8	3	7
Unit.11	2	2	3	5.5	5	7
Unit.12	2.5	1.5	3	7	3	6
Unit.13	4.5	1.5	6	4	2	4
Unit.14	2	4	7	1.5	2	4
Unit.15	4	3	6	6.5	3.5	3.5
Unit.16	2	5	4	5	3.5	4
Unit.17	1.5	6	4	4.5	4.5	5
Unit.18	0.5	4	3	3.5	5.5	6
Unit.19	3.5	0.75	3	7.5	2.5	6
Unit.20	6	3.5	4	3.5	3.5	6

Table2. Example Results

	Eff (3)	Eff (5)	Eff (8-1)	Eff (8-2)	Eff (14)	Eff (18)
Unit 1	0.3461525	0.3461534	0.2678571	0.2678541	0.1470588	0.1470579
Unit 2	0.3214259	0.3214281	0.1632653	0.1632614	0.1069703	0.1069690
Unit 3	0.4285680	0.4285710	0.1428571	0.1428554	0.1418440	0.1418423
Unit 4	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit 5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.6	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.7	0.9999992	0.9999990	1.0000000	0.9999992	0.9411765	0.9411765
Unit.8	0.9999965	0.9999990	1.0000000	0.9999965	0.5825243	0.5825237
Unit.9	0.9765503	0.9765507	0.9741641	0.9741627	0.9549072	0.9549058
Unit.10	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.11	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.12	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.13	0.4309243	0.4309246	0.4093567	0.4093524	0.3527337	0.3527309
Unit.14	0.3809487	0.3809518	0.3658537	0.3658457	0.1829268	0.1829243
Unit.15	0.5607670	0.5607692	0.5594298	0.5594240	0.5004307	0.5004255
Unit.16	0.6052253	0.6052271	0.4541045	0.4541015	0.4476744	0.4476712
Unit.17	0.6847133	0.6847146	0.4925373	0.4925352	0.4801402	0.4801376
Unit.18	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.19	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.20	0.6428523	0.6428563	0.3378237	0.3378211	0.2795755	0.2795744

6. Conclusion

In this paper we presented a systematic investigation of the problem of interval scale data in DEA. The problem with interval scale variables arises from the fact that rations of measurement on such a scale are meaningless. Consequently, a DEA model can be used to deal with interval scale input and/or output variables in which the ratio of virtual effect inputs and/or outputs and observed inputs and/or outputs does not have any role in the calculations. Also, we pointed out that only the interval scale variables are invariant under translation transformations; therefore we can use the translation invariance property of a DEA model to deal only with interval scale input and/or output variables. We have paid attention to the issue that since only the interval scale data are invariant under translation transformations, thus negativity of such a scale is meaningful, but negativity of the ratio scale variables are meaningless. We also discussed an extended model using the enhanced Russell measurement which permits the presence of interval scale variables which can take both negative and positive values.

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