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Minimizing the Weights dispersion in Cross-Efficiency Measurement in data envelopment analysis

A. Amirteimoori[*a], S.Kordrostami [b]

(a)*Department of Applied Mathematics, Islamic Azad University, Rasht, Iran*

(b)*Department of Applied Mathematics, Islamic Azad University, Lahidjan, Iran*

Abstract

Because of the piecewise linear nature of the data envelopment analysis (DEA) frontier, the optimal multipliers of the DEA models may not be unique. Choosing weights from alternative optimal solutions of dual multiplier models is one of the most frequently studied subjects in the context of DEA. In this paper, the authors have been inspired by the idea of Cooper et al. (2011) to propose a linear programming problem in which a specific decision making unit chooses the optimal solution of a linear program as the profile of weights for use in the cross-efficiency calculation. The approach proposed in this paper to determine input/output weights, prohibit the large differences in weights in cross efficiency evaluation. A real case on Chinese cities and special economic zones is given to illustrate the applicability of the proposed approach.

Keywords: Data envelopment analysis, Cross-efficiency, input/output weights.

1 Introduction

Data envelopment analysis (DEA) is a mathematical programming approach created to analyze the relative performance of a set of homogeneous decision making units (DMUs) which use similar types of multiple resources to generate similar kinds of multiple products. DEA has been used in many contexts including education systems, health care units, agricultural productions, military logistics and many other applications (See Nigam et al. (2012) and Sreekumar and Mahapatra (2011)).

Discussion of how to determine optimal weights in DEA models is an important and most frequently studied subject in the context of DEA. This subject has been studied from different perspectives by different authors. (See Lam and Bai (2011), Cooper et al. (2007) and Cooper et al. (2011)). Because of the piecewise linear nature of the DEA frontier, the optimal multipliers of the DEA models may not

[*] Corresponding author, email: teimoori@guilan.ac.ir , aamirteimoori@gmail.com

be unique. Specifically, extreme efficient units usually have alternative optimal solutions for their weights in the conventional DEA models. In these models, the existence of alternative optima has led to the use of secondary goals by some DEA researchers to choose a set of favorable weights for cross efficiency evaluations. (See Liang et al. (2008),Wu et al. (2009) and Wang and Chin (2010).). Suppose there are n DMUs and each DMU_j uses m inputs $x_{ij} : i = 1,...,m$ to produce s outputs $y_{rj} : r = 1,...,s$. Assume also that E is the set of all extreme efficient units and DMU_o is a given DMU in E. Cooper et al. (2011) stated that DMU_d must choose, among of its alternative optima in the CCR model, the optimal solution of the following model:

$$
\begin{aligned}
& Max \quad \varphi_d && (1) \\
& s.t. \\
& \quad \sum_{i=1}^{m} v_i^d x_{id} = 1, && (1.1) \\
& \quad \sum_{r=1}^{s} u_r^d y_{rd} = 1, && (1.2) \\
& \quad \sum_{r=1}^{s} u_r^d y_{rj} - \sum_{i=1}^{m} v_i^d x_{ij} \le 0, \quad j = 1,\cdots,n, && (1.3) \\
& \quad z_I \le v_i^d x_{id} \le h_I, \quad i = 1,\cdots,m, && (1.4) \\
& \quad z_O \le u_r^d y_{rd} \le h_O, \quad r = 1,\cdots,s, && (1.5) \\
& \quad \frac{z_I}{h_I} \ge \varphi_d, && (1.6) \\
& \quad \frac{z_O}{h_O} \ge \varphi_d, && (1.7) \\
& \quad u_r^d, v_i^d, h_O, z_O, h_I, z_I, \varphi_d \ge 0.
\end{aligned}
$$

As they stated, constraints (1.4) and (1.5) forces all the inputs virtual and all the outputs virtual to vary in between the bounds z_I and h_I, and the bounds z_O and h_O, respectively. In (1.6) and (1.7) we have $\varphi_d \le \frac{z_I}{h_I} \le 1$ and $\varphi_d \le \frac{z_O}{h_O} \le 1$, respectively. Maximizing φ_d means that they maximize the minimum of the two ratios and in this sense look for the weights profiles with the least dissimilar virtual inputs and outputs to allow DMU_d to be rated as efficient. In this model, h_I, z_I, h_O and z_O are decision variables and hence, this model is a nonlinear programming problem. Although they proved in proposition 1 that model (1) has a global optimum, however, the nonlinearity of this model could have some computational difficulties in practice. In the following section we make a moderate modification on the model proposed by Cooper et al. (2011) to reduce its computational complexity.

2 Secondary goal in weights selection

In this section, inspired by the idea of Cooper et al. (2011), a new linear programming problem has been proposed to determine optimal weights to an efficient DMU.

The existence of the constraints $\varphi_d \le \frac{z_I}{h_I}$ and $\varphi_d \le \frac{z_O}{h_O}$ leads to the nonlinearity problem in model (1).

In the modified model the nonlinear constraints $\frac{z_I}{h_I} \ge \varphi_d$ and $\frac{z_O}{h_O} \ge \varphi_d$ are replaced by $0 \le h_I - z_I \le \varphi_d$ and $0 \le h_O - z_O \le \varphi_d$, respectively. Then, we minimize φ_d such that z_I and z_O could be as close to h_I and h_O, respectively, as possible. In this case, we propose the following linear programming problem:

$$
\begin{array}{lll}
Min & \varphi_d & (2) \\
s.t. & & \\
& \sum_{i=1}^{m} v_i^d x_{id} = 1, & (2.1) \\
& \sum_{r=1}^{s} u_r^d y_{rd} = 1, & (2.2) \\
& \sum_{r=1}^{s} u_r^d y_{rj} - \sum_{i=1}^{m} v_i^d x_{ij} \le 0, \quad j = 1,\cdots,n, & (2.3) \\
& z_I \le v_i^d x_{id} \le h_I, \quad i = 1,\cdots,m, & (2.4) \\
& z_O \le u_r^d y_{rd} \le h_O, \quad r = 1,\cdots,s, & (2.5) \\
& 0 \le h_I - z_I \le \varphi_d, & (2.6) \\
& 0 \le h_O - z_O \le \varphi_d, & (2.7) \\
& u_r^d, v_i^d, z_I \ge \varepsilon, & (2.8)
\end{array}
$$

in which $\varepsilon \ge 0$ is a sufficiently small positive to ensure strict positivity of h_I, z_I, h_O and z_O (Note that $z_O \le h_O$ and $z_I \le h_I$.) The necessity of using $\varepsilon \ge 0$ is that it prevents the zero values to the weights. Minimizing φ_d means that we minimize the distance between h_I and z_I, h_O and z_O, and in this sense, the model looks for the weights with the least dissimilar virtual inputs and outputs to allow DMU_d to be efficient. Model (2) is a linear programming problem and it is easy to show that at optimality of (2), we have $\varphi_d^* = Max\{h_I^* - z_I^*, h_O^* - z_O^*\}$. At the first sight, it seems that the existence of the small positive number ε give raise difficulties in practice. However, it should be pointed out that the structure of this ε is different from the non-Archimedean constant $\varepsilon \geq 0$ in CCR and BCC models that forces the weights u_r and v_i to be positive.

The following theorem guarantees the feasibility and boundedness of model (2).

Theorem 1- The LP model (2) is feasible and bounded.

Proof: A feasible solution to LP model (2) can be determined as:

$$
\begin{aligned}
& u_r = u_r^* : r = 1, \cdots, s, \\
& v_i = v_i^* : r = 1, \cdots, m, \\
& z_I = \underset{1 \leq i \leq m}{Min} \{\sum_{i=1}^{m} v_i x_{id}\} \ , \quad h_I = \underset{1 \leq i \leq m}{Max} \{\sum_{i=1}^{m} v_i x_{id}\}, \\
& z_O = \underset{1 \leq r \leq s}{Min} \{\sum_{r=1}^{s} u_r y_{rd}\} \ , \quad h_O = \underset{1 \leq r \leq s}{Max} \{\sum_{r=1}^{s} u_r y_{rd}\}, \\
& \varphi = Max\{h_I - z_I \ , h_O - z_O\}
\end{aligned}
$$

in which u_r^* and u_r^* are optimal solutions to the standard multiplier form of the CCR model. The objective value to this solution is nonnegative and hence, the LP model (2) is bounded. This completes the proof.

Now, suppose that DMU_d is a specific unit that does not belong to M. For such a DMU, we need to choose the optimal solution of the following linear programming problem as the optimal weights for use in the calculation of the cross-efficiencies:

$$
\begin{aligned}
& Max \quad \sum_{r=1}^{s} u_r^d y_{rd} && (3) \\
& s.t. \\
& \quad \sum_{i=1}^{m} v_i^d x_{id} = 1, && (3.1) \\
& \quad \sum_{r=1}^{s} u_r^d y_{rj} - \sum_{i=1}^{m} v_i^d x_{ij} \leq 0 \ , \quad j = 1, \cdots, n, && (3.3) \\
& \quad z_I \leq v_i^d x_{id} \leq h_I \ , \quad i = 1, \cdots, m, && (3.4) \\
& \quad z_O \leq u_r^d y_{rd} \leq h_O \ , \quad r = 1, \cdots, s, && (3.5) \\
& \quad 0 \leq h_I - z_I \leq \varphi^*, && (3.6) \\
& \quad 0 \leq h_O - z_O \leq \varphi^*, && (3.7) \\
& \quad u_r^d, v_i^d, z_O, z_I \geq \varepsilon, && (3.8)
\end{aligned}
$$

in which φ^* is defined as $\varphi^* = \underset{j \in M}{Max}\, \varphi_j^*$.

Now, cross efficiency of DMU_j using the weights that DMU_d has chosen, is defined as

$$E_j^{(d)} = \frac{\sum_{r=1}^{s} u_r^d y_{rj}}{\sum_{i=1}^{m} v_i^d x_{ij}} \qquad (4)$$

For $DMU_j : \ j = 1, \cdots, n$, the average of all $E_j^{(d)} : \ d = 1, \cdots, n$ is called the cross-efficiency score of DMU_j.

3 A simple example

In what follows, we illustrate the weights determination model in cross efficiency evaluation with a small-scale example consisting of five *DMU*s. The *DMU*s use two inputs to produce a single output whose value is normalized to one for each *DMU*. All of the five units are CCR-efficient.

Table 1

The data for simple example

DMU	x_1	x_2	y
A	1	10	1
B	2	5	1
C	4	2	1
D	6	1	1
E	12	0.5	1

Model (2) is applied to this data set for DMUs A, B, C, D and E. The result to each DMU is an efficient surface of the production set as follows:

$$H_A = \{ (x, y) : y - 0.5x_1 - 0.05x_2 = 0 \} \cap T_c$$

$$H_B = \{ (x, y) : y - 0.25x_1 - 0.1x_2 = 0 \} \cap T_c$$

$$H_C = \{ (x, y) : y - 0.125x_1 - 0.25x_2 = 0 \} \cap T_v$$

$$H_D = \{ (x, y) : y - 0.0833x_1 - 0.5x_2 = 0 \} \cap T_v$$

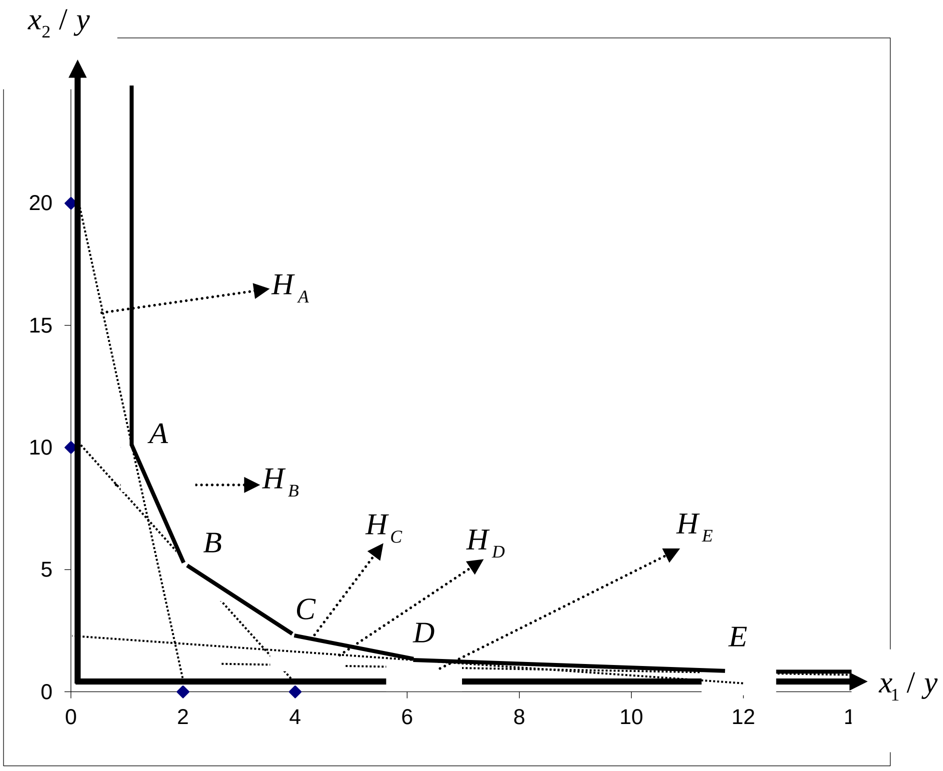
$$H_E = \{ (x, y) : y - 0.0417x_1 - x_2 = 0 \} \cap T_v$$

Figure 1 shows the production set in two-dimensional space along with the surfaces obtained from model 2.

Table 2

Matrix of cross efficiency scores

DMU	A	B	C	D	E
A	1	0.8	0.4762	0.3279	0.1660
B	0.8	1	0.8333	0.6250	0.3279
C	0.3810	0.6667	1	1	0.6154
D	0.1967	0.3750	0.75	1	0.8
E	0.0996	0.1967	0.4615	0.8	1
Mean	0.49546	0.60768	0.7042	0.75058	0.58186



4 Chinese cities

To illustrate the applicability of the proposed approach, we use a data set consists of 13 open coastal Chinese cities and 5 Chinese special economic zones in 1989 (This example has been taken from Zhu (1998)). There are three outputs with two inputs. Inputs include investment in fixed assets by state-owned enterprises (x_1) and foreign funds (x_2) and outputs include total industrial output value (y_1), total value of retail sales (y_2) and handling capacity of coastal ports (y_3). The input/output data set are listed in Table 3. The CCR model of Charnes et al. (1978) is used to determine the relative efficiency of each cities. The efficiency scores of all cities are listed in the last column of Table 3, and as the Table indicates, three cities/Zones are CCR-efficient. The model cities are hence $M = \{ Qinhuangdao, Weihai, Wenzhou \}$. The optimal absolute weights provided by the proposed models are listed in Table 4. It should be pointed out that we have calculated $\varphi_2 = 0.5893$, $\varphi_6 = 0.4376$ and $\varphi_{10} = 0.3494$. So, in model 3 we assumed $\varphi^* = 0.5893$. The matrix of cross efficiency and cross efficiency scores are presented in Table 3. The last row in Table 3 shows the cross efficiency score of DMU_j as the average of the cross efficiencies in Table 3. Different ranking methods have been applied to this data set and the results are listed in Table 6. Columns two and three report the cross efficiencies and rankings of the Liang et al. (2008) method, respectively. The fourth and fifth columns show the AP scores and rankings, respectively. We have also applied the super slack-based measure (SBM) of Tone (2002) to compare the results. Columns six and seven show the super SBM and rankings, respectively. As the Table shows, Qinhuangdao is the top-ranked city in all ranking methods. All methods have also determined Shenzhen as the low-ranked city. Regarding remaining cities/zones, there are no substantial and significant differences between ranking orders in different ranking procedures.

Table 3

Data and CCR efficiencies

J	*Cities/Zones*	x_1	x_2	y_1	y_2	y_3	*Efficiency*
1	Dalian	2874.8	16738	160.89	80800	5092	0.4691
2	Qinhuangdao	946.3	691	21.14	18172	6563	1
3	Tianjin	6854	43024	375.25	144530	2437	0.2779
4	Qingdao	2305.1	10815	176.68	70318	3145	0.5022
5	Yantai	1010.3	2099	102.12	55419	1225	0.6311
6	Weihai	282.3	757	59.17	27422	246	1
7	Shanghai	17478.6	116900	1029.09	351390	14604	0.3580
8	Lianyungang	661.8	2024	30.07	23550	1126	0.4959
9	Ningbo	1544.2	3218	160.58	59406	2230	0.6577
10	Wenzhou	428.4	574	53.69	47504	430	1
11	Guangzhou	6228.1	29842	258.09	151356	4649	0.3010
12	Zhanjiang	697.7	3394	38.02	45336	1555	0.7866
13	Beihai	106.4	367	7.07	8236	121	0.7514
14	Shenzhen	4539.3	45809	116.46	56135	956	0.1382
15	Zhuhai	957.8	16947	29.2	17554	231	0.1867
16	Shantou	1209.2	15741	65.36	62341	618	0.4704
17	Xiamen	972.4	23822	54.52	25203	513	0.3059
18	Hainan	2192	10943	25.24	40267	895	0.1953

Table 4

Cross-efficiency evaluation: Absolute weights

Cities/Zones	u_1	u_2	u_3	v_1	v_2
1	0.00053145	0.00000173	0.00003465	0.00027642	0.00001227
2	0.00647515	0.00000753	0.00011065	0.000217	0.00115
3	0.00050418	0.0000001	0.00001532	0.00011594	0.00000477
4	0.00164958	0.0000001	0.00004612	0.00034474	0.00001899
5	0.00246312	0.00000456	0.00010316	0.00078655	0.00009783
6	0.01151632	0.00000826	0.00037447	0.00281491	0.00027127
7	0.00016842	0.0000001	0.00000592	0.00004546	0.00000176
8	0.00303643	0.00000723	0.00015402	0.00120074	0.00010146
9	0.00355205	0.0000001	0.00003914	0.00019068	0.00021925
10	0.00764787	0.00001241	0.0000001	0.00185493	0.00035775
11	0.00026372	0.00000079	0.00001608	0.00012759	0.00000688
12	0.0000001	0.0000097	0.00014373	0.00113896	0.0000605
13	0.0000001	0.00006547	0.00095449	0.00746852	0.00055954
14	0.0002869	0.00000111	0.00002171	0.00017506	0.00000448
15	0.00122176	0.00000532	0.00010223	0.00082966	0.00001212
16	0.0000001	0.00000535	0.00008132	0.00065717	0.00001305
17	0.00351742	0.0000001	0.00010713	0.0008172	0.00000862
18	0.0000001	0.00000308	0.00004571	0.00036252	0.00001877

$\varphi_2 = 0.5893,\ \ \varphi_6 = 0.4376,\ \ \varphi_{10} = 0.3494$

Table 5

Matrix of cross - efficiency and cross-efficiency scores.

J	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0.4	1	0.22	0.42	0.63	1	0.27	0.46	0.57	1	0.27	0.65	0.66	0.11	0.11	0.31	0.16	0.15
2	0.11	1	0.07	0.16	0.46	0.66	0.08	0.2	0.43	1	0.09	0.19	0.27	0.02	0.02	0.05	0.02	0.04
3	0.41	1	0.24	0.45	0.6	1	0.3	0.4	0.62	0.73	0.25	0.49	0.44	0.11	0.1	0.23	0.17	0.1
4	0.39	1	0.24	0.44	0.59	1	0.29	0.39	0.63	0.71	0.24	0.46	0.41	0.1	0.09	0.2	0.15	0.09
5	0.33	1	0.19	0.38	0.63	1	0.22	0.41	0.59	1	0.23	0.52	0.56	0.08	0.07	0.2	0.1	0.12
6	0.35	1	0.21	0.4	0.61	1	0.25	0.4	0.61	0.86	0.23	0.48	0.49	0.09	0.08	0.2	0.11	0.11
7	0.41	1	0.24	0.45	0.6	1	0.3	0.42	0.61	0.8	0.26	0.53	0.5	0.11	0.11	0.25	0.17	0.11
8	0.36	1	0.2	0.4	0.63	1	0.24	0.44	0.58	1	0.25	0.58	0.6	0.09	0.09	0.24	0.12	0.14
9	0.18	1	0.13	0.27	0.63	1	0.15	0.27	0.66	1	0.14	0.22	0.3	0.04	0.03	0.07	0.04	0.04
10	0.2	0.19	0.17	0.27	0.56	1	0.17	0.27	0.49	1	0.17	0.34	0.48	0.06	0.06	0.16	0.07	0.09
11	0.39	1	0.22	0.41	0.63	1	0.26	0.45	0.57	1	0.26	0.63	0.64	0.1	0.11	0.29	0.15	0.15
12	0.35	1	0.17	0.35	0.56	0.82	0.2	0.45	0.46	1	0.24	0.66	0.68	0.09	0.1	0.3	0.13	0.16
13	0.33	1	0.16	0.33	0.55	0.8	0.19	0.43	0.45	1	0.23	0.63	0.66	0.08	0.08	0.26	0.1	0.16
14	0.43	1	0.23	0.44	0.63	1	0.28	0.47	0.56	1	0.28	0.69	0.68	0.12	0.14	0.36	0.2	0.17
15	0.44	1	0.24	0.45	0.63	1	0.29	0.48	0.56	1	0.29	0.72	0.7	0.13	0.15	0.4	0.23	0.17
16	0.4	1	0.19	0.38	0.57	0.85	0.24	0.47	0.47	1	0.27	0.73	0.72	0.11	0.13	0.38	0.19	0.18
17	0.45	1	0.27	0.49	0.59	1	0.34	0.41	0.63	0.68	0.27	0.51	0.43	0.13	0.14	0.27	0.25	0.1
18	0.36	1	0.17	0.35	0.56	0.82	0.21	0.45	0.46	1	0.24	0.67	0.68	0.09	0.1	0.3	0.13	0.17

Table 6

Different ranking results

Unit	Liang et al. Score	Liang et al. Rank	AP Score	AP-Rank	Super-SBM	SBM-Rank	New cross eff. Score	New rank
1	0.4461	10	0.4685	11	0.27970	10	0.349444	10
2	1	1	12.6072	1	1.44309	1	0.955	1
3	0.2436	15	0.2742	15	0.09187	15	0.197778	14
4	0.4522	9	0.5005	8	0.30578	9	0.38	9
5	0.6050	6	0.6311	7	0.47351	5	0.592222	4
6	0.9722	2	1.6698	3	1.15476	3	0.941667	2
7	0.3047	12	0.3413	12	0.18228	11	0.237778	12
8	0.4545	8	0.4959	9	0.33657	8	0.403889	8
9	0.5693	7	0.6553	6	0.48098	4	0.552778	5
10	0.8870	3	2.2848	2	1.31655	2	0.932222	3
11	0.2788	14	0.3004	14	0.16928	12	0.233889	13
12	0.6576	4	0.7864	4	0.41334	6	0.538889	7
13	0.6086	5	0.7514	5	0.38881	7	0.55	6
14	0.1288	18	0.1377	18	0.04599	18	0.092222	18
15	0.1665	16	0.1864	17	0.05249	17	0.095	17
16	0.3877	11	0.4697	10	0.13918	13	0.248333	11
17	0.2819	13	0.3052	13	0.11725	14	0.138333	15
18	0.1513	17	0.1951	16	0.08609	16	0.125	16

4 Conclusions

Cross efficiency evaluation in DEA is used to discriminate between DMUs and it eliminates unrealistic and unfavorable weights in conventional DEA models. The existence of unrealistic and unfavorable weights in DEA models may lead to incorrect assessment in efficiency analysis. In this paper, we have extended the multiplier form of CCR model for use in cross-efficiency evaluations with the aim of preventing unrealistic weighting schemes. The paper has modeled a linear programming model to choose a set of favorable weights for use in the cross-efficiency calculation.

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