Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol. 10, No. 4, Year 2023 Article ID IJDEA-00422, Pages 57-68
Research Article

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

DEA-TOPSIS with interval data

Z. Iravani[1], M. Ahadzadeh Namin[*2]

[1]Yadegar-e-Imam Khomeini (RAH) Shahre Rey Branch, Islamic Azad University, Tehran, Iran.
[2]Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.

Received 8 March 2022, Accepted 12 July 2022

Abstract
Since it plays a significant role in the best choice of industry and science, it is always chosen to find a more accurate job. One of these methods is the TOPSIS method.
In this article, after introducing the DEA and TOPSIS methods, we will introduce the TOPSIS-DEA method and then we will expand this method for interval criteria. In this method, there is no need to know the exact weight of the qualities, and this method deals with the best option without changing the qualitative qualities. This method is a suitable method for choosing the best option when the data is an interval. And it is faster than the classic method.

Keywords: Data Envelopment Analysis, Efficiency, Interval data, Ideal Point, TOPSIS.

---

[*] Corresponding author: Email: mahnzaahadzadehnamin@gmail.com

1. Introduction

Data Envelopment Analysis (DEA) is a technique based on mathematical programming to calculate the relative efficiency of a set of homogeneous decision-making units [1], in 1978 the result of research by Cooper, Charnes and Rhodes in an article titled [2] CCR, it was published. Due to the complexity of decision-making in the presence of multiple quantitative and qualitative criteria, a special branch of operations research, called multi-criteria decision-making (MCDA) [3], has been created and is developing rapidly. One of the methods for solving MCDA problems is the TOPSIS method [4]. The TOPSIS method, by Yun and Huang, was presented in 1981 [5], in this method, the goal is to be close to the ideal option and to be far from the anti-ideal option. In the classical TOPSIS method, the data matrix is first normalized, and after determining the ideal and anti-ideal points, as well as calculating the weight of the criteria and converting quantitative to qualitative criteria, it calculates the distance of each option to the ideal and anti-ideal and finally selects the option. is done [6]. In this article, after reviewing the DEA-TOPSIS method, which is a method based on TOPSIS with the structure of DEA problems [7, 8, 9, 10, 11, 12, 13], without the need to find the weight of each criterion and also without the need to calculate the distance of quantitative criteria from ideal and anti-ideal, Selects options.

2. A review of the subject literature

In classic TOPSIS, the selection of ideal and anti-ideal points is based on the software matrix (maximum or minimum $V_j(a^i)$ according to different criteria) [7]. In the DEA-TOPSIS method, instead of choosing ideal and anti-ideal points from the software table, we choose ideal and anti-ideal options for each criterion according to whether it is profit or cost [14]. If $m_j(a^i)$ is the value of the i-th option on the j-th and we index different criteria, we will have a simpler calculation. Suppose that $C = C^c \cup C^o$ so that C includes all criteria. C^c includes quantitative criteria and C^o includes qualitative and sequential criteria. We assume:

$$C^c = \{c_1^c, \cdots, c_j^c, \cdots, c_{q_c}^c\}, C^o = \{c_1^o, \cdots, c_j^o, \cdots, c_{q_o}^o\} \quad (1)$$

so that $q = q_c + q_o$ and q_c is the number of quantitative criteria and q_o is the number of qualitative and sequential criteria.

If $m_j^c(a^i)$ is the value of a^i on the quantitative criterion C_j^c, for each C_j^c, the distance between a^i, a^+ and a^- is as follows.

$$d_j^c(a^i)^+ = \frac{|m_j^c(a^+) - m_j^c(a^i)|}{\varepsilon_j^+} \quad (2)$$

The normalized distance between a^i and a^- on the criterion C_j^c is as follows.

$$d_j^c(a^i)^- = \frac{|m_j^c(a^-) - m_j^c(a^i)|}{\varepsilon_j^-} \quad (3)$$

It can be shown that the normalized distances (2) and (3) are between 0 and 1 [7].

Distance conditions when the criteria are qualitative and ordinal:

In DEA-TOPSIS, grading is used to measure quality criteria, suppose: $L = \{l_1, \cdots, l_r, \cdots, l_m\}$

The collection should be arranged so that l_1 is the best grade and l_m is the worst grade assigned to the quality criteria, also $m_j^o(a^i) = l_r$.

$d_j^o(a^i)^-$ and $d_j^o(a^i)^+$ respectively show the distance between a^i and a^+ and a^i and a^- on the criterion c_j^o.

As in the qualitative criteria, the normalized distance between 0 and 1 was set, here also we assume that the distance between a^+ and a^- on c_j^o criterion is equal to 1, that is: $d_j^o(a^+)^- = d_j^o(a^-)^+ = 1$.

If $m\_m_j^o(a^i) = l_r\_r$, then $d_j^o(a^i)^+$ and $d_j^o(a^i)^-$ in the conditions the following apply.

$$\frac{m-r}{m} \le d_j^o(a^i)^- \le \frac{m-r+1}{m} \quad (4)$$

$$\frac{r-1}{m} \le d_j^o(a^i)^+ \le \frac{r}{m}$$

If $w_j^c = \{w_1^c, \cdots, w_j^c, \cdots, w_q^c\}$, the weight related to quality criteria and $w_j^o = \{w_1^o, \cdots, w_j^o, \cdots, w_q^o\}$ weight information is related to quantitative criteria, the weighted distance from a^i to a^+ on C^c and C^o with soft P is as follows:

$$D(a^i)^+ = \quad (5)$$

$$\left\{\sum_{j=1}^{q_c} w_j^c.\left(d_j^c(a^i)^+\right)^p + \sum_{j=1}^{q_o} w_j^o.\left(d_j^o(a^i)^+\right)^p\right\}^{1/p}$$

$$D(a^i)^- = \quad (6)$$

$$\left\{\sum_{j=1}^{q_c} w_j^c.\left(d_j^c(a^i)^-\right)^p + \sum_{j=1}^{q_o} w_j^o.\left(d_j^o(a^i)^-\right)^p\right\}^{1/p}$$

In the classic TOPSIS method, the weight of each criterion is determined, while in many cases the goal of the decision-maker is met by observing the relationships between the weights. It is enough to determine the relationships between the weights [15].

As mentioned, in the classic TOPSIS method, before performing mathematical operations, first, the values $d_j^o(a^i)^-, d_j^o(a^i)^+, w^c, w^o$ should be calculated for each $a^i \in A$ and found $c_j^o \in C^O$ for each criterion. Here, two models are designed without the need to find the values $d_j^o(a^i)^-, d_j^o(a^i)^+, w^c, w^o$.

The first model is designed separately according to the distance to the ideal and the distance to the anti-ideal, and the final decision is made by combining these two models. The second model, which is called the combined model, uses these two distances simultaneously in one model.

A) Specific optimization model

Two A and B are designed as below.

$$P(a^i)^+ = \text{minimize } : D(a^i)^+ \quad (7)$$

st:

$\forall a^i \in A \quad , D(a^i)^+ \le D(a^-)^+ \le 1$

$\forall a^i \in A \quad , \text{if } m_j^o(a^i) = l_r,$

$\frac{r-1}{m} \le d_j^o(a^i)^+ \le \frac{r}{m}$

$\forall c_j^o \in C^o \quad , d_j^o(a^-)^+ = 1$

$\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1$

$\forall c_j^o \in C^o, w_j^o \ge \rho$ and $\forall c_j^c \in C^c, w_j^c \ge \rho$

In this model, the goal is to minimize the distance between each option and the ideal option. The first condition states that the distance between each option and the ideal option must be less than the distance between the ideal and anti-ideal options. This condition is established in normal TOPSIS. The second clause expresses the interval in which the distance between each option and the ideal option is located for each quality criterion. The third and fourth clauses are for normalization.

$$P(a^i)^- = \text{maximize } : D(a^i)^- \quad (8)$$

st:

$\forall a^i \in A \quad , D(a^i)^- \le D(a^+)^- \le 1$

$\forall a^i \in A \quad , \text{if } m_j^o(a^i) = l_r ,$

$\frac{m-r}{m} \le d_j^o(a^i)^- \le \frac{m-r+1}{m}$

$\forall c_j^o \in C^o \quad , d_j^o(a^+)^- = 1$

$\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1$

$\forall c_j^o \in C^o, w_j^o \ge \rho$ and $\forall c_j^c \in C^c, w_j^c \ge \rho$

In this model, the goal is to maximize the distance between each option and the anti-ideal option, the first condition states that the distance between each option and the anti-ideal option must be less than the distance between the anti-ideal and ideal options, which is true in normal TOPSIS.

The second clause expresses the interval in which the distance between each option and the ideal option is located for each quality criterion. The third and fourth clauses are for normalization.

Note that ρ is a real number is positive. Finally, we obtain the collective distance from the following relationship.

$$D(a^i) = \frac{D(a^i)^-}{D(a^i)^+ + D(a^i)^-} \quad (9)$$

The combined optimization model is obtained simultaneously based on the distance to a^+ and a^-.

$$P(a^i) \quad (10)$$

$$\text{maximize:} \frac{D(a^i)^-}{D(a^i)^+ + D(a^i)^-}$$

st:

$$\forall a^i \in A \quad, D(a^i)^+ \leq D(a^-)^+ \leq 1$$
$$\forall a^i \in A \quad, D(a^i)^- \leq D(a^+)^- \leq 1$$
$$\forall a^i \in A \quad, \text{if } m_j^o(a^i) = l_r , \frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m}$$
$$\text{and } \frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m}$$
$$\forall c_j^o \in C^o \quad, d_j^o(a^-)^+ = 1$$
$$\forall c_j^o \in C^o \quad, d_j^o(a^+)^- = 1$$
$$\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1$$
$$\forall c_j^o \in C^o, w_j^o \geq \rho \quad \text{and} \quad \forall c_j^c \in C^c, w_j^c \geq \rho$$

Note that $P(a^i)^+$ improves the estimation of the weight of quality criteria by minimizing the distance a^i to the ideal point, and $P(a^i)^-$ improves such performance in the case of the distance to the anti-ideal point. and $P(a^i)$ is designed in such a way that it executes the previous two processes in one step.

$P(a^i)^+$, $P(a^i)^-$and $P(a^i)$ restrictions are a guarantee that the above models reach the maximum or minimum value at least once.

3. DEA-TOPSIS method for interval data

Since the data are not always definite and are used in many interval data problems, there is a great deal of research about interval DEA [16, 17, 18, 19, 20]. we extend the DEA-TOPSIS model for interval data. Here, we are going to state that the three presented models can be expanded for interval criteria, that is, each option (a^i) is located in an interval for each criterion. In other words, the interval (L_j, U_j) is for each C_j . Table 1).

In TOPSIS, the range of ideal and anti-ideal options is also in the form of an interval, and the goal is to be close to the ideal option and to be far from the anti-ideal option [21].

First, we introduce the following symbols for each criterion C_j :

Definition 1: a^{iU} and a^{iL} are the upper and lower limits of each option, respectively.

Definition 2: a^{+U} and a^{+L} are respectively the best upper bound and the best lower bound in each criterion.

Table 1: Interval criteria

		Options					
		A_1 … … … A_n					
standards	C_1			⋮			
	⋮ ⋮ ⋮	. . .	. .	$m_i(L_j,U_j)$			
	C_q						

Definition 3: a^{-U} and a^{-L} are respectively the worst upper bound and the worst lower bound in each criterion

With these definitions, we have: $\forall\, a^i \in A$

$$D\left(a^{iU}\right)^{+U} \le D\left(a^{-L}\right)^{+U} \quad (11)$$

$$D\left(a^{iU}\right)^{+L} \le D\left(a^{+U}\right)^{-L} \quad (12)$$

$$D\left(a^{iL}\right)^{-U} \le D\left(a^{-L}\right)^{+U} \quad (13)$$

$$D\left(a^{iL}\right)^{-L} \le D\left(a^{+U}\right)^{-L} \quad (14)$$

For each c_j^c, the distance between a^{iU} and a^{+U} and a^{iU} and a^{-U} is as follows.

$$\left|m_j^c\left(a^{-U}\right) - m_j^c\left(a^{iU}\right)\right|,\quad \left|m_j^c\left(a^{+U}\right) - m_j^c\left(a^{iU}\right)\right| \quad (15)$$

For each c_j^c, the distance between a^{iL} and a^{+L} and a^{iL} and a^{-L} is as follows.

$$\left|m_j^c(a^{-L}) - m_j^c\left(a^{iL}\right)\right|,\quad \left|m_j^c\left(a^{+L}\right) - m_j^c\left(a^{iL}\right)\right| \quad (16)$$

As mentioned, in this method, the ideal and anti-ideal options are selected first, and then the normalized distance is defined on each option according to the normalization function.

We define the following normalization function:

$$\varepsilon_j^{+U} = \sqrt{\sum_{i=1}^{n} \left(m_j^c(a^{+U}) - m_j^c(a^{iU})\right)^2 + \left(m_j^c(a^{-L}) - m_j^c(a^{+U})\right)^2} \quad (17)$$

$$\varepsilon_j^{+L} = \sqrt{\sum_{i=1}^{n} \left(m_j^c(a^{+L}) - m_j^c(a^{iL})\right)^2 + \left(m_j^c(a^{-L}) - m_j^c(a^{+U})\right)^2} \quad (18)$$

$$\varepsilon_j^{-L} = \sqrt{\sum_{i=1}^{n} \left(m_j^c(a^{-L}) - m_j^c(a^{iL})\right)^2 + \left(m_j^c(a^{+U}) - m_j^c(a^{-L})\right)^2} \quad (19)$$

$$\varepsilon_j^{-U} = \sqrt{\sum_{i=1}^{n} \left(m_j^c(a^{-U}) - m_j^c(a^{iU})\right)^2 + \left(m_j^c(a^{+U}) - m_j^c(a^{-L})\right)^2} \quad (20)$$

The normalized distances on the c_j^c criterion according to the above normalization function are as follows:

$$d_j^c\left(a^i\right)^{+U} = \frac{\left|m_j^c(a^{+U}) - m_j^c(a^{iU})\right|}{\varepsilon_j^{+U}} \quad (21)$$

$$d_j^c\left(a^i\right)^{+L} = \frac{\left|m_j^c(a^{+L}) - m_j^c(a^{iL})\right|}{\varepsilon_j^{+L}} \quad (22)$$

$$d_j^c\left(a^i\right)^{-L} = \frac{\left|m_j^c(a^{-L}) - m_j^c(a^{iL})\right|}{\varepsilon_j^{-L}} \quad (23)$$

$$d_j^c\left(a^i\right)^{-U} = \frac{\left|m_j^c(a^{-U}) - m_j^c(a^{iU})\right|}{\varepsilon_j^{-U}} \quad (24)$$

Theorem 1: It can be shown that the normalized distances (21), (22), (23) and (24) are between 0 and 1, that is:

$$\forall\, a^i \in A, \forall\, c_j \in C \quad (25)$$

$$0 \le \left\{ \begin{matrix} d_j^c\left(a^i\right)^{-L}, d_j^c\left(a^i\right)^{-U} \\ , d_j^c\left(a^i\right)^{+L}, d_j^c\left(a^i\right)^{+U} \end{matrix} \right\} \le 1$$

Proof: Because in the relationships mentioned for the normalization intervals, the numerator is greater than or equal to the denominator of the fraction, each of the mentioned intervals is between 0 and 1.

- Distance conditions when the criteria are qualitative and ordinal

Here we consider the quality criteria with the same ranking method mentioned in the second part. Suppose:

$L = \{l_1, \cdots, l_r, \cdots, l_m\}$

be the ranking set so that l_1 is the best grade and l_m is the worst grade assigned to the quality criteria.

Also $m_j^o\left(a^i\right) = l_r$, that is:

a^i is in the rth degree on the criterion c_j^o.

We have:

$$m_j^o(a^-) = l_m \;,\; m_j^o(a^+) = l_1 \quad (26)$$

$d_j^o\left(a^i\right)^-$ and $d_j^o\left(a^i\right)^+$ respectively show the distance between a^i and a^+ and a^i and a^- on the criterion c_j^o.

As in the qualitative criteria, the normalized distance between 0 and 1 was set, here also we assume that the distance between a^+ and a^- on the c_j^o criterion is equal to 1, that is: $d_j^o(a^+)^- = d_j^o(a^-)^+ = 1$

If $m_j^o(a^i) = l_r$, then $d_j^o(a^i)^+$ and $d_j^o(a^i)^-$ apply in the following conditions.

$$\frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m} \quad (27)$$

$$\frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m} \quad (28)$$

If $w_j^c = \{w_1^c, \cdots, w_j^c, \cdots, w_q^c\}$ the weight related to quality criteria and $w_j^o = \{w_1^o, \cdots, w_j^o, \cdots, w_q^o\}$ weight information is related to quantitative criteria, the weighted distance on C^c and C^o with soft P is as follows:

$$D(a^i)^{+U} = \left\{\sum_{j=1}^{q_c} w_j^c.\left(d_j^c(a^i)^{+U}\right)^p + \sum_{j=1}^{q_o} w_j^o.\left(d_j^o(a^i)^{+}\right)^p\right\}^{1/p} \quad (29)$$

$$D(a^i)^{+L} = \left\{\sum_{j=1}^{q_c} w_j^c.\left(d_j^c(a^i)^{+L}\right)^p + \sum_{j=1}^{q_o} w_j^o.\left(d_j^o(a^i)^{+}\right)^p\right\}^{1/p} \quad (30)$$

$$D(a^i)^{-U} = \left\{\sum_{j=1}^{q_c} w_j^c.\left(d_j^c(a^i)^{-U}\right)^p + \sum_{j=1}^{q_o} w_j^o.\left(d_j^o(a^i)^{-}\right)^p\right\}^{1/p} \quad (31)$$

$$D(a^i)^{-L} = \left\{\sum_{j=1}^{q_c} w_j^c.\left(d_j^c(a^i)^{-L}\right)^p + \sum_{j=1}^{q_o} w_j^o.\left(d_j^o(a^i)^{-}\right)^p\right\}^{1/p} \quad (32)$$

At this stage, we want the decision-maker to state the relationships he expects regarding the weight of the criteria. Then, using the constraints of these variables, optimization models are formed.

4. Optimization models for interval data

(i) Four specially designed models

These four models are dedicated to finding the best option in comparison with the best and worst conditions.

A. distance of each option to the upper limit of the ideal option:

$$P(a^i)^+ \quad (33)$$

$$\begin{aligned}
&\text{minimize } : D(a^i)^{+U} \\
&st: \\
&\forall a^i \in A \quad , D(a^i)^{+U} \leq D(a^{-L})^{+U} \leq 1 \\
&\forall a^i \in A \quad , \text{if } \ m_j^o(a^i) = l_r \\
&\text{then} \quad , \frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m} \\
&\forall c_j^o \in C^o \quad , d_j^o(a^-)^+ = 1 \\
&\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1 \\
&\forall c_j^o \in C^o, w_j^o \geq \rho \\
&\text{and } \ \forall c_j^c \in C^c, w_j^c \geq \rho
\end{aligned}$$

B. The distance between each option and the anti-ideal option:

$$P(a^i)^- \quad (34)$$

$$\begin{aligned}
&\text{maximize } : D(a^i)^{-U} \\
&\text{st}: \\
&\forall a^i \in A \quad , D(a^i)^{-U} \leq D(a^{-L})^{+U} \leq 1 \\
&\forall a^i \in A \quad , \text{if } \ m_j^o(a^i) = l_r \\
&\text{then} \quad , \frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m} \\
&\forall c_j^o \in C^o \quad , d_j^o(a^+)^- = 1 \\
&\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1 \\
&\forall c_j^o \in C^o, w_j^o \geq \rho \\
&\text{and } \ \forall c_j^c \in C^c, w_j^c \geq \rho
\end{aligned}$$

We define the upper bound for each option as follows:

$$D(a^i)^U = \frac{D(a^i)^{-U}}{D(a^i)^{-U} + D(a^i)^{+U}} \quad (35)$$

C. The distance of each option to the lower limit of the ideal option:

$$P(a^i)^+ \quad (36)$$

$$\begin{aligned}
&\text{minimize } : D(a^i)^{+L} \\
&st: \\
&\forall a^i \in A \quad , D(a^i)^{+L} \leq D(a^{+U})^{-L} \leq 1 \\
&\forall a^i \in A \quad , \text{if } \ m_j^o(a^i) = l_r \\
&\text{then} \quad , \frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m} \\
&\forall c_j^o \in C^o \quad , d_j^o(a^-)^+ = 1 \\
&\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1 \\
&\forall c_j^o \in C^o, w_j^o \geq \rho \ \text{ and } \ \forall c_j^c \in C^c, w_j^c \geq \rho
\end{aligned}$$

D. The distance of each option to the bottom corner of the anti-ideal option:

$$P(a^i)^- \quad (37)$$

maximize : $D(a^i)^{-L}$
st:
$\forall a^i \in A \quad , D(a^i)^{-L} \leq D(a^{+U})^{-L} \leq 1$
$\forall a^i \in A \quad$, if $m_j^o(a^i) = l_r$ then $, \frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m}$
$\forall c_j^o \in C^o \quad , d_j^o(a^+)^- = 1$
$\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1$
$\forall c_j^o \in C^o, w_j^o \geq \rho$ and $\forall c_j^c \in C^c, w_j^c \geq \rho$

We define the lower bound for each option as follows:

$$D(a^i)^L = \frac{D(a^i)^{-L}}{D(a^i)^{-L} + D(a^i)^{+L}} \tag{38}$$

To choose the right option, we define $R(a^i)$ as follows:

$$R(a^i) = \frac{D(a^i)^L + D(a^i)^U}{2} \tag{39}$$

The higher the $R(a^i)$, the more appropriate the option to choose.

(ii) Two integrated optimization models

In these two models, we directly calculate the distance to the upper limit and the distance to the lower limit for each option.

A1 - the distance of each option to the upper limit:

$p(a^i)$ (40)

maximize : $D^\circ(a^i)^U = \frac{D(a^i)^{-U}}{D(a^i)^{-U} + D(a^i)^{+U}}$
st:
$\forall a^i \in A \quad , D(a^i)^{+U} \leq D(a^{-L})^{+U} \leq 1$
$\forall a^i \in A \quad , D(a^i)^{-U} \leq D(a^{-L})^{+U} \leq 1$
$\forall a^i \in A \quad$, if $m_j^o(a^i) = l_r$ then $, \frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m}$
and $, \frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m}$
$\forall c_j^o \in C^o \quad , d_j^o(a^-)^+ = 1$
$\forall c_j^o \in C^o \quad , d_j^o(a^+)^- = 1$
$\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1$
$\forall c_j^o \in C^o, w_j^o \geq \rho$ and $\forall c_j^c \in C^c, w_j^c \geq \rho$

A2 - the distance of each option up to the lower limit:

$P(a^i)$ (41)

maximize : $D^\circ(a^i)^L = \frac{D(a^i)^{-L}}{D(a^i)^{-L} + D(a^i)^{+L}}$
st:
$\forall a^i \in A \quad , D(a^i)^{+L} \leq D(a^{+U})^{-L} \leq 1$
$\forall a^i \in A \quad , D(a^i)^{-L} \leq D(a^{+U})^{-L} \leq 1$
$\forall a^i \in A \quad$, if $m_j^o(a^i) = l_r$
then $, \frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m}$
and $, \frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m}$
$\forall c_j^o \in C^o \quad , d_j^o(a^-)^+ = 1$
$\forall c_j^o \in C^o \quad , d_j^o(a^+)^- = 1$
$\sum_{j=1}^{q_c} w_j^c + \sum_{j=1}^{q_o} w_j^o = 1$
$\forall c_j^o \in C^o, w_j^o \geq \rho$ and $\forall c_j^c \in C^c, w_j^c \geq \rho$

At this stage, to choose the appropriate option, we define $R^\circ(a^i)$ as follows:

$$R^\circ(a^i) = \frac{D^\circ(a^i)^U + D^\circ(a^i)^L}{2} \tag{42}$$

The higher the value of $R^\circ(a^i)$, the more suitable the option is to choose.

Theorem 2: For each value of the parameters, the calculated $D(a^i)$ is between $D(a^i)^L$ and $D(a^i)^U$. means:

$$\frac{D(a^i)^{-L}}{D(a^i)^{-L} + D(a^i)^{+L}} = D(a^i)^L \leq D(a^i)$$
$$\leq D(a^i)^U = \frac{D(a^i)^{-U}}{D(a^i)^{-U} + D(a^i)^{+U}} \tag{43}$$

Proof: The following relations are established:

$$D(a^i)^{+L} \leq D(a^i)^+ \leq D(a^i)^{+U} \tag{44}$$
$$D(a^i)^{-L} \leq D(a^i)^- \leq D(a^i)^{-U} \tag{45}$$

By adding the unequal sides of (44) and (45), we will have:

$$D(a^i)^{+L} + D(a^i)^{-L} \leq D(a^i)^+ + D(a^i)^- \leq D(a^i)^{+U} + D(a^i)^{-U} \tag{46}$$

Since each of the sides of the inequality (46) is opposite to zero, by dividing the relation (45) by (43), we have:

$$\frac{D(a^i)^{-L}}{D(a^i)^{+L} + D(a^i)^{-L}} \leq \frac{D(a^i)^-}{D(a^i)^+ + D(a^i)^-}$$

$$\leq \frac{D(a^i)^{-U}}{D(a^i)^{+U}+D(a^i)^{-U}} \quad (47)$$

This is the same relationship we are discussing, so the argument ends.

Theorem 3: For each parameter value in the corresponding intervals $D(a^i)$ is calculated between $D^{\circ}(a^i)^L$ and $D^{\circ}(a^i)^U$ has it. means:

$$D^{\circ}(a^i)^L \leq D(a^i) \leq D^{\circ}(a^i)^U \quad (48)$$

Proof: We must show:

$$S^L \subseteq S \subseteq S^U \quad (49)$$

which shows the zoning of each of the unequal sides (49). We want to show that if the constraints of the model (19) are valid, then the constraints of model (41) are valid, and if the constraints of the model (40) are valid, then the constraints of the model (19) are valid. It should be mentioned that the following restrictions apply to all models.

$$\frac{r-1}{m} \leq d_j^o(a^i)^+ \leq \frac{r}{m}$$

$$\frac{m-r}{m} \leq d_j^o(a^i)^- \leq \frac{m-r+1}{m}$$

For this purpose, we must show if the relations:

$$D(a^i)^- \leq D(a^+)^- \leq 1$$

$$D(a^i)^+ \leq D(a^-)^+ \leq 1$$

are established, then the following relationships are also established.

$$D(a^i)^{-L} \leq D(a^{+U})^{-L} \leq 1$$

$$D(a^i)^{+L} \leq D(a^{+U})^{-L} \leq 1$$

First, suppose $D(a^i)^- \leq D(a^+)^- \leq 1$, then:

$$D(a^{+U})^{-L} \leq D(a^+)^- \Rightarrow D(a^{+U})^{-L} - 1 \leq D(a^+)^- - 1 \leq 0$$

$$\Rightarrow D(a^{+U})^{-L} - 1 \leq 0$$

So, the relationship $D(a^i)^{-L} \leq D(a^{+U})^{-L} \leq 1$ is established. Now we assume $D(a^i)^+ \leq D(a^-)^+ \leq 1$, so:

$$D(a^{+U})^{-L} \leq D(a^+)^- = D(a^-)^+ \Rightarrow$$

$$D(a^{+U})^{-L} - 1 \leq D(a^-)^+ - 1 \leq 0$$

$$\Rightarrow D(a^{+U})^{-L} - 1 \leq 0$$

So, the relationship $D(a^i)^{-L} \leq D(a^{+U})^{-L} \leq 1$ has been established. Here we can conclude:

$$S^L \subseteq S \quad (50)$$

Now we show that if the relations:

$$D(a^i)^{+U} \leq D(a^{-L})^{+U} \leq 1$$

$$D(a^i)^{-U} \leq D(a^{-L})^{+U} \leq 1$$

are established, then the following relationships are also established:

$$D(a^i)^+ \leq D(a^-)^+ \leq 1$$

$$D(a^i)^- \leq D(a^+)^- \leq 1$$

First, we assume that $D(a^i)^{-U} \leq D(a^{-L})^{+U} \leq 1$, then:

$$D(a^-)^+ \leq D(a^{-L})^{+U} \Rightarrow D(a^-)^+ - 1 \leq D(a^{-L})^{+U} - 1 \leq 0$$

$$\Rightarrow D(a^-)^+ - 1 \leq 0$$

So, the relation $D(a^i)^+ \leq D(a^-)^+ \leq 1$ is established.

Now we assume that $D(a^i)^{+U} \leq D(a^{-L})^{+U} \leq 1$, so:

$$D(a^+)^- \leq D(a^{+U})^{-L} = D(a^{-L})^{+U} \Rightarrow$$

$$D(a^+)^- - 1 \leq D(a^{-L})^{+U} - 1 \leq 0$$

$$D(a^+)^- - 1 \leq 0$$

So, the relationship .$D(a^i)^- \leq D(a^+)^- \leq 1$ is established. From here we can conclude:

$$S \subseteq S^U \quad (51)$$

that obtained relations (50) and (51) result in relation (49).

Now if $D^{\circ}(a^i)^L$, $D^{\circ}(a^i)^U$ the optimal answer of models (40) and (41), we have:

$$D^{\circ}(a^i)^L \leq D(a^i) \leq D^{\circ}(a^i)^U$$

Which is the ideal of the case. So, the proof ends.

5. Example

According to Table 2, we want to choose the options with the presented method. In this table, the first two criteria are quantitative and the third criterion is qualitative. The relation of the criterion's weight is as: $w_1^c \geq w_2^c \geq w_1^o$. All three criteria are profit type.

First, we rank the options based on quality criteria. This rating is possible with the following conditions:

if $a^i \leq 400000 \Rightarrow a^i \in l_4$
if $400000 < a^i \leq 800000 \Rightarrow a^i \in l_3$
if $800000 < a^i \leq 1400000 \Rightarrow a^i \in l_2$
if $a^i > 1400000 \Rightarrow a^i \in l_1$

It can be seen that we have the following ranking:

$\{1\} \in l_1, \{2,3\} \in l_2, \{6,7,8\} \in l_3, \{4,5\} \in l_4$

Also, it is enough to consider $P(a^{i+})$ and $P(a^{i-})$ as below.

$w_1^c \geq w_2^c, w_2^c \geq w_1^o$, $w_1^o \geq 0.00001$

The normalized distance of each option to the upper and lower limits of the anti-ideal is shown in Table 3. And the normalized distance of each option to the ideal upper and lower limits is shown in Table 4. And finally, the results related to the ranking of options are presented in Table 5.

Table 2: Data table

	C_1^c		C_2^c		C_1^o
a^1	500	961	26364	28254	3126798
a^2	873	1775	3791	50308	1061260
a^3	95	196	22964	26846	1213541
a^4	848	1752	492	1213	395241
a^5	58	120	18053	18061	165818
a^6	464	955	40539	48643	416416
a^7	155	342	33797	44933	410427
a^8	1752	3629	1437	1519	768593

Table 3: Normalized distance to anti-ideal

	C_1^c		C_2^c		C_1^o
	dL⁻	d U⁻	dL⁻	dU⁻	
a^1	0.1063	0.0478	0.1780	0.2610	(0.75, 1)
a^2	0.1960	0.1386	0.0227	0.4739	(0.5, 0.75)
a^3	0.0089	0.0063	0.1546	0.2474	(0.5, 0.75)
a^4	0.19	0.1367	0.0	0.0	(0.0, 0.25)
a^5	0.0	0.0	0.1208	0.1626	(0.0, 0.25)
a^6	0.0977	0.0699	0.2755	0.4578	(0.25, 0.5)
a^7	0.023	0.0018	0.2291	0.422	(0.25, 0.5)
a^8	0.4075	0.2939	0.0065	0.0029	(0.25, 0.5)

Table 4: Normalized to the ideal distance

	C_1^c		C_2^c		C_1^o
	dL	dU⁺	dL⁺	d U⁺	

a^1	0.1838	0.3214	0.1581	0.2277	(0.0, 0.25)
a^2	0.1290	0.2234	0.41	0.0	(0.25, 0.5)
a^3	0.2433	0.4136	0.1782	0.2422	(0.25, 0.5)
a^4	0.1327	0.2261	0.4468	0.5069	(0.75, 1)
a^5	0.2487	0.4228	0.2508	0.3329	(0.75, 1)
a^6	0.1891	0.3221	0.0	0.0172	(0.5, 0.75)
a^7	0.2345	0.3960	0.0752	0.0555	(0.5, 0.75)
a^8	0.0	0.0	0.4362	0.5037	(0.5, 0.75)

Table 5: Ranking of options

	$D(a^i)^L$	$D(a^i)^U$	$R(a^i)$	Ranking	$D^\circ(\mathrm{a}^\mathrm{i})^\mathrm{L}$	$D^\circ(\mathrm{a}^\mathrm{i})^\mathrm{L}$	$R^\circ(a^i)$	Ranking
a^1	0.8387067	0. 7796446	0.80917565	2	0.9999808	0.9999717	0.99997625	1
a^2	0.7499663	0.8571123	0.7499663	3	0.7499663	0.8303689	0.7901676	2
a^3	0.7779076	0.8454175	0.81166255	1	0.7499877	0.2499888	0.7499857	3
a^4	0.3600240	0.3358223	0.34792315	8	0.2499888	0.2499865	0.24998765	8
a^5	0.3750499	0.332465	0.35375745	7	0.2703373	0.2758893	0.2731133	7
a^6	0.6159108	0.5990501	0.60748045	4	0.60798787	0.649332	0.62865535	4
a^7	0.6493968	0.6428545	0.64612565	5	0.5589863	0.6240069	0.5914966	5
a^8	0.6851716	0.6590869	0.67212925	6	0.4998784	0.4999739	0.49997615	6

Finally, to rank the options, first, we solve the models (33), (34), (36) and (37) for each option, then we put the obtained values in the relations (35) and (38) and we also solve models (40) and (41) for each option, the results of these calculations are shown in table (5).

The above table shows that the first 3 options have different ratings by two methods, but the other options have the same rating in both methods.

6. Conclusions and suggestions

Since choice plays an important role in people's lives, people always seek to find quick and reliable ways to choose. The method that was presented for interval criteria based on DEA-TOPSIS, selects the option without needing to know the exact weight of data and criteria and without the need to convert qualitative criteria into quantitative ones.

Since the criteria are always definite and interval and not specific, it is possible to expand DEA-TOPSIS for options with a combination of criteria.

The DEA-TOPSIS method can be used for fuzzy criteria.

It is possible to generalize the DEA-TOPSIS method for interval quality criteria.

References

[1] Jahanshahlou G.R, F. Hosseinzadeh Lotfi, H. Niko Meram, 2017, data Envelopment analysis and its applications, Islamic Azad University-Tehran Science and Research Unit.

[2] Charnes A, Cooper W.W and Rhodes E: Measuting the efficiency of decision-making units. *European Journal of Operational Research*, (4) 2, (1978) 429-444.

[3] Asgharpour M.J, 1377, multi-criteria decision making, University of Tehran.

[4] Ataei M., 2018, multi-criteria decision making, Shahrood University of Technology.

[5] Howang C.L, K Yoon: Multiple Attribute Decision Making. Springer- verlag, berlin (1981).

[6] Shih H.S, Shyur H.J and Lee E.S: An extension of topsis for group decision making. *Mathematical and computer modeling* 45 (2007) 801-813.

[7] Chen Y, Li. K, Xu .H and Liu. S: A DEA TOPSIS Method for Multiple Criteria Decision Analysis in Emergency Management. *J Syst Sci Syst Eng, (2009) 489-507.*

[8] Namazi, Mehdi, and Emran Mohammadi. "Natural resource dependence and economic growth: A TOPSIS/DEA analysis of innovation efficiency." *Resources Policy* 59 (2018): 544-552.

[9] Rakhshan, Seyed Ali. "Efficiency ranking of decision making units in data envelopment analysis by using TOPSIS-DEA method." *Journal of the Operational Research Society* 68 (2017): 906-918.

[10] Liu, Jianfen, and Shengpu Li. "Research on the ranking of university education based on Grey-TOPSIS-DEA method." *International Journal of Emerging Technologies in Learning* 10, no. 8 (2015).

[11] KONUK, F. (2018). Financial and performance analysis of food companies: application of TOPSIS and DEA. *Manas Sosyal Araştırmalar Dergisi*, *7*(3).

[12] Rosić, Miroslav, Dalibor Pešić, Dragoslav Kukić, Boris Antić, and Milan Božović. "Method for selection of optimal road safety composite index with examples from DEA and TOPSIS method." *Accident Analysis & Prevention* 98 (2017): 277-286.

[13] Meng, Di, Cheng Shao, and Li Zhu. "A new energy efficiency evaluation solution for ethylene cracking furnace based on integrated TOPSIS-DEA model." *International Journal of Energy Research* 44, no. 2 (2020): 732-755.

[14] Seiford L.M and Zhu J: Context-dependent data envelopment analysis-measuring attractiveness and progress. *Omega* 31 (2003) 397-408.

[15] Eum Y.S, Park K.S and Soung S.H: Establishing dominance and potential with emprecise weight and value. *Computer and Operations Research*, 28 (2001) 397-409.

[16] Lotfi, F. Hosseinzadeh, M. Navabakhs, A. Tehranian, M. Rostamy-Malkhalifeh, and R. Shahverdi. "Ranking bank branches with interval data the application of DEA." In *International mathematical forum*, vol. 2, no. 9, pp. 429-440. 2007.

[17] Emrouznejad, Ali, Mohsen Rostamy-Malkhalifeh, Adel Hatami-Marbini, and Madjid Tavana. "General and multiplicative non-parametric corporate performance models with interval ratio data." *Applied Mathematical Modelling* 36, no. 11 (2012): 5506-5514.

[18] Ghaemi, Nasab F., and MALKHALIFEH M. ROSTAMY. "Extension of TOPSIS for group decision-making based on the

type-2 fuzzy positive and negative ideal solutions." (2010): 199-213.

[19] Seyed Esmaeili, Fatemeh Sadat, Mohsen Rostamy-Malkhalifeh, and Farhad Hosseinzadeh Lotfi. "Two-stage network DEA model under interval data." *Mathematical Analysis and Convex Optimization* 1, no. 2 (2020): 103-108.

[20] Kianfar, Kianoosh, Mahnaz Ahadzadeh Namin, Akbar Alam Tabriz, Esmaeil Najafi, and Farhad Hosseinzadeh Lotfi. "Hybrid cluster and data envelopment analysis with interval data." *Scientia Iranica* 25, no. 5 (2018): 2904-2911.

[21] Jahanshahloo G.R, Hosseinzadeh Lotfi. F and Izadikhah .M: An algorithmic method to extend TOPSIS for decision-making problems with interval data. *Applied Mathematics and Computation* 175 (2006) 1375-1384.