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Ranking with fuzzy data using symmetrical weights as a secondary goal

E. Abdolahi*
Department of Mathematics, Kerman Branch, Islamic Azad University of Kerman, Iran

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Abstract
When we use the CCR model in the input-oriented with fuzzy data for ranking with the help of cross-efficiency, there is a possibility that the model will find a different optimal answer. This means that the ranking is not unique, that is, a decision-making unit may be assigned several ranks. Here, the judgment regarding the ranking faces a problem. To solve this problem, a secondary goal is determined for weight selection. According to that secondary goal, a suitable weight is selected from among the optimal solutions. In this article, the secondary goal of the concept of symmetrizing the weights plays a fundamental role in solving the mentioned problem. The model selects weights that are symmetrical, the act of choosing symmetrical weights causes many weights that are not useful to be removed from the set. The decision-making unit that selects symmetrical weights for all indicators, has a better performance than the decision-making unit that does not use symmetrical weights and covers its weak points with low weight and highlights its strong points with high weight. The model along with the mentioned secondary goal is used to evaluate decision-making units with fuzzy input and output, by choosing the optimal weight, a cross-efficiency table is formed. By using the cross-efficiency table, the efficiency of each unit is determined and ranked compared to other units. Units are done.

Keywords: Data envelopment analysis, Secondary goal, Cross-efficiency, Ranking.

* Corresponding author: Email: abdolahi.eskandar@gmail.com

1. Introduction

For the first time in 1957, the efficiency function for estimating efficiency of units with one input and one output was proposed by Farrell [1]. In 1978, Charles et al. introduced a linear programming technique called data envelopment analysis to determine the efficiency of decision-making units with multiple inputs and outputs (CCR model) [2]. In 1984, Banker et al. introduced the BCC model [3]. In the wide world of science and knowledge, we deal with imprecise, indeterminate and relative definitions and concepts, comparing imprecise data is a necessity, but in this comparison, we face a kind of uncertainty that is related to the lack of precise and firm demarcation of concepts. These concepts cannot be reasoned, deduced and decided with the logic that requires accurate and quantitative data. Fuzzy theory, by using certain models, is able to give mathematical formulation to many concepts, and systems that are imprecise and ambiguous, and provides the basis for inference and decision-making in conditions of uncertainty. In this article, we will use our efforts in this direction. The cross-efficiency rating model with fuzzy input and output uses the fuzzy number ranking function to rank the decision-making units. See [4-9]. In this article, Sexton's method (cross-efficiency) plays a role as a ranking model, and in addition, a new method called secondary goal is used to reduce the optimal solutions.

By using the secondary goal model in the concept of symmetric weights with fuzzy data, the goal is achieved. In the second part of this research, efficiency and ranking are mentioned. Finally, Sexton's method (cross-efficiency) is briefly described. The necessary fuzzy prerequisites are also stated. In the third part, which is the main goal of this article, ranking with fuzzy input and output using the cross-efficiency model and using the secondary goal model that was introduced, acknowledging that all the regions of the cross-efficiency table, the numbers are fuzzy and the average efficiency of all the decision-making units are also fuzzy numbers.

2. Background

2-1 Efficiency and ranking

By using data envelopment analysis (DEA) models to evaluate the relative efficiency of decision-making units (DMU), usually, there are more than one decision-making units that are introduced as efficient. Here we have to rank efficient decision-making units. One of the models that determine efficiency is the multiplier CCR model in the input-oriented. One of the methods that perform the ranking with the help of the multiplier CCR model is the Sexton method (cross-efficiency) [10].

The multiplier form of the CCR in the input-oriented is as follows [1].

$$
\begin{aligned}
e_0 = \max\ & wy_{\circ} && (1)\\
\text{s.t}\quad & wy_j - vx_j \le \circ,\ j = 1,\dots,n \\
& Vx_j = 1 \\
& w \ge \circ \ ,\ v \ge \circ
\end{aligned}
$$

Where $\{1,2,\dots,n\}$ is the set of indices of the decision-making units and $0 \in \{1,2,\dots,n\}$ the unit under evaluation. $X_j = (X_{1j},\dots,X_{mj})$ is the input vector of the jth unit, $Y_j = (Y_{1j},\dots,Y_{sj})$ is the output vector of the jth unit. It is assumed $Y_j \neq 0$, $Y_j \geq 0$, $X_j \neq 0$, $X_j \geq 0$.

By solving model (1), the optimal solution is obtained. The efficiency of all units based on this weight is calculated as follows:

$$
\theta_{io} = \frac{w_o^* y_i}{v_o^* x_i} \quad ,i = 1,\dots,n \qquad (2)
$$

The sexton table (cross efficiency) is as follows.

Table (1) cross efficiency for non-phase input and output

	DMU_1	DMU_2	...	DMU_n	average
DMU_1	$\theta_{11} = \theta_1^*$	$\theta_{12} = \theta_2^*$	...	$\theta_{1n} = \theta_n^*$	$\frac{1}{n}\sum_{j=1}^{n}\theta_{1j}$
DMU_2	$\theta_{21} = \theta_1^*$	$\theta_{22} = \theta_2^*$	...	$\theta_{2n} = \theta_n^*$	$\frac{1}{n}\sum_{j=1}^{n}\theta_{2j}$
...	...	...	...	...	...
DMU_n	$\theta_{n1} = \theta_1^*$	$\theta_{n2} = \theta_2^*$	...	$\theta_{nn} = \theta_n^*$	$\frac{1}{n}\sum_{j=1}^{n}\theta_{nj}$

In addition to being a method for measuring the efficiency of decision-making units, Sexton's method also ranks decision-making units. Any decision-making unit with a better average has a better rank. One of the biggest advantages of Sexton's method (cross-efficiency) is that it ranks all decision-making units.

2.2 Fuzzy concepts

2.2.1 Membership function:

Suppose X is an infinite set, every fuzzy subset of X is defined by a function

$$\mu_{\tilde{A}(x)} : X \to [0,1]$$

It is defined as the membership function. $\mu_{\tilde{A}(x)}$ indicates the extent to which x belongs to the fuzzy set $\tilde{A}$.

2.2.2 Triangular fuzzy numbers

The triangular fuzzy number $\tilde{N} = (L, m, u)$ is also shown as $\tilde{N} = (x^1, x^m, x^u)$, where x^m is the central value of $(\mu_{\tilde{N}}(x^m) = 1)$, and X^1 is the pessimistic value or the lowest value of $((\mu_{\tilde{N}}(X^1) = 0)$, and X^u is the optimistic value or the highest value of $((\mu_{\tilde{N}}(X^u) = 0)$.

Membership function for triangular fuzzy number is as follows.

$$\mu_{i(x)} = \begin{cases} \dfrac{X-L}{m-L} & L \le X \le m \\ \dfrac{U-X}{U-m} & m \le X \le U \\ 0 & \text{Otherwis} \end{cases} \tag{3}$$

3. Secondary goal with fuzzy input and output for ranking

Suppose the input and output of the decision-making unit are fuzzy numbers, so we have:

$$\tilde{X}_j = (\tilde{X}_{1j}, ..., \tilde{X}_{mj}) \tag{4}$$

$$\tilde{Y}_j = (\tilde{Y}_{1j}, ..., \tilde{Y}_{sj}) \tag{5}$$

3.1 The fuzzy CCR model in the fractional form

$$e_o = \max \frac{w\tilde{y}_o}{v\tilde{x}_o} \tag{6}$$

$$\text{s.t} \quad \frac{w\tilde{y}_j}{v\tilde{x}_j} \le 1 \quad , j = 1,\dots,n$$

$$w \ge 1\varepsilon \quad , \quad v \ge 1\varepsilon$$

Model (6) in the linear fuzzy form is as follows:

$$\max w\tilde{y}_o \tag{7}$$

$$\text{s.t} \quad w\tilde{y}_i - v\tilde{x}_j \le 0 \ , j = 1,\dots,n$$

$$v\tilde{x}_o = 1$$

$$w \ge \circ \ , \ v \ge \circ$$

In this research, we follow how to form a cross-efficiency table (Sexton model) for fuzzy data. See [11-12]. Fuzzy numbers are considered triangular, and finally, they are ranked with the help of the ranking function of the decision-making units. To achieve this goal, we consider

$$\forall_r \forall_j \tilde{Y}_{rj} = \left(L_{rj}^y, m_{rj}^y, u_{rj}^y\right) \tag{8}$$

$$\forall_i \forall_j \tilde{X}_{ij} = \left(L_{ij}^x, m_{ij}^x, u_{ij}^x\right) \tag{9}$$

All inputs and outputs are triangular fuzzy numbers.

3.2 Proposed secondary goal

Introducing a new secondary goal to reduce the optimal solutions of the multiple CCR model.

(10)

$$\min \frac{1}{n-1}\sum_{j=1}^{n} \alpha_{pj} \tag{10-1}$$

$$\min e^t ze \tag{10-2}$$

$$s.t \sum_{r=1}^{s} w_{rp} y_{rj} - \sum_{i=1}^{m} v_{ip} x_{ij} + \alpha_{pj} = 0, \ j = 1,\dots,n \tag{10-3}$$

$$\sum_{r=1}^{s} w_{rk} y_{rk} - \theta_k^* \sum_{i=1}^{m} v_{ik} x_{ik} = 0 \tag{10-4}$$

$$\sum_{i=1}^{m} v_{ip} x_{ip} = 1 \tag{10-5}$$

$$w_{tlp} y_{tlp} - w_{tlp} y_{tlp} \le z_{tlp}, \quad t = 1,\dots,s, l = 1,\dots,s \tag{10-6}$$

$$-w_{tlp} y_{tlp} + w_{tlp} y_{tlp} \le z_{tlp}, \ t = 1,\dots,s, l = 1,\dots,s \tag{10-7}$$

$$w \ge \varepsilon, v \ge \varepsilon, z \ge 0, \alpha \ge 0 \tag{10-8}$$

The above model does the following together.

1. It selects the answers from among the optimal solutions (constraint (10-4))
2. By reducing the auxiliary variables, he chooses weights that increase the efficiency, that is, give them a tendency (objective function (10-1) and constraint (10-3)).
3. By minimizing the difference between the L-th output share and the t-th output share, the weights are symmetric. (objective function (10-2) and constraints (10-6) and (10-7)).
4. The weights obtained by the model are the feasible solutions of the CCR model. (constraints (10-3), (10-5) and (10-8)).

So we have:

$$\min \frac{1}{n-1}\sum_{j=1}^{n} \tilde{\alpha}_{pj} \tag{11}$$

$$\min e^t \tilde{z} e$$

$$s.t \sum_{r=1}^{s} w_{rp} \tilde{y}_{rj} - \sum_{i=1}^{m} v_{ip} \tilde{x}_{ij} + \tilde{\alpha}_{pj} = 0, \ j = 1,\dots,n$$

$$\sum_{r=1}^{s} w_{rk} \tilde{y}_{rk} - \theta_k^* \sum_{i=1}^{m} v_{ik} \tilde{x}_{ik} = 0$$

$$\sum_{i=1}^{m} v_{ip} \tilde{x}_{ip} = 1$$

$$w_{tlp} \tilde{y}_{tlp} - w_{tlp} \tilde{y}_{tlp} \le \tilde{z}_{tlp}, t, l = 1,\dots,s$$

$$-w_{tlp} \tilde{y}_{tlp} + w_{tlp} \tilde{y}_{tlp} \le \tilde{z}_{tlp}$$

$$w \ge \varepsilon, v \ge \varepsilon, \tilde{z} \ge 0, \tilde{\alpha} \ge 0$$

So that:

$$\tilde{x}_{ij} = (l_{ij}^x, m_{ij}^x, u_{ij}^x) \tag{12}$$

$$\tilde{y}_{ij} = (l_{ij}^y, m_{ij}^y, u_{ij}^y) \tag{13}$$

$$\tilde{\alpha}_{pj} = (l_{pj}^\alpha, m_{pj}^\alpha, u_{pj}^\alpha) \tag{14}$$

$$\tilde{z}_{tl} = (l_{tl}^z, m_{tl}^z, u_{tl}^z) \tag{15}$$

The model is as follows:

$$\min \frac{1}{n-1}\sum_{j=1}^{n} (l_{pj}^{\alpha}, m_{pj}^{\alpha}, u_{pj}^{\alpha}) \tag{16}$$

$$\min e^{t}(l_{tl}^{z}, m_{tl}^{z}, u_{tl}^{z})e$$

$$s.t\sum_{r=1}^{s} w_{rp}(l_{rj}^{y}, m_{rj}^{y}, u_{rj}^{y}) - \sum_{i=1}^{m} v_{ip}(l_{ij}^{x}, m_{ij}^{x}, u_{ij}^{x}) + (l_{pj}^{\alpha}, m_{pj}^{\alpha}, u_{pj}^{\alpha}) = 0, \ j = 1,\dots,n$$

$$\sum_{r=1}^{s} w_{rk}(l_{rk}^{y}, m_{rk}^{y}, u_{rk}^{y}) - \theta_{k}^{*}\sum_{i=1}^{m} v_{ik}(l_{ik}^{x}, m_{ik}^{x}, u_{ik}^{x}) = 0$$

$$\sum_{i=1}^{m} v_{ip}(l_{ip}^{x}, m_{ip}^{x}, u_{ip}^{x}) = 1$$

$$w_{tp}(l_{tlp}^{y}, m_{tlp}^{y}, u_{ltp}^{y}) - w_{tlp}(l_{tlp}^{y}, m_{tlp}^{y}, u_{tlp}^{y}) \le (l_{tlp}^{z}, m_{tlp}^{z}, u_{tlp}^{z}) \qquad t,l = 1,\dots,s$$

$$-w_{tlp}(l_{ltp}^{y}, m_{tlp}^{y}, u_{tlp}^{y}) + w_{tlp}(l_{tlp}^{y}, m_{tlp}^{y}, u_{tlp}^{y}) \le (l_{tlp}^{z}, m_{tlp}^{z}, u_{tlp}^{z}) \qquad t,l = 1,\dots,s$$

$$w \ge \varepsilon \quad v \ge \varepsilon \quad u_{tlp}^{z} \ge m_{tlp}^{z} \ge l_{tlp}^{z} \ge 0 \quad u_{tlp}^{\alpha} \ge m_{tlp}^{\alpha} \ge l_{tlp}^{\alpha} \ge 0 \quad t,l = 1,\dots,s \quad o = 1,\dots,n$$

By performing the necessary calculations, the model is as follows:

$$\min\left(\frac{1}{n-1}\sum_{j=1}^{n} L_{pj}^{\alpha}, \frac{1}{n-1}\sum_{j=1}^{n} m_{pj}^{\alpha}, \frac{1}{n-1}\sum_{j=1}^{n} u_{pj}^{\alpha}\right) \tag{17}$$

$$\min(e^{t}L_{tl}^{z}e, e^{t}m_{tl}^{z}e, e^{t}\ u_{tl}^{z}e)$$

$$s.t\left(\sum_{r=1}^{s} w_{rp}L_{rj}^{y} - \sum_{i=1}^{m} v_{ip}u_{ij}^{x} + l_{pj}^{\alpha}, \sum_{r=1}^{s} w_{rp}m_{rj}^{y} - \sum_{l=1}^{m} v_{ip}m_{ij}^{x} + m_{pj}^{\alpha}, \sum_{r=1}^{s} w_{rp}u_{rj}^{y} - \sum_{r=1}^{s} v_{ip}l_{ij}^{x} + u_{pj}^{\alpha}\right) = (0,0,0), \ j = 1,..,n$$

$$\left(\sum_{r=1}^{s} w_{rk}L_{rk}^{y} - \theta_{k}^{*}\sum_{i=1}^{m} v_{ik}u_{ik}^{x}, \sum_{r=1}^{s} w_{rk}m_{rk}^{y} - \theta_{k}^{*}\sum_{i=1}^{l=1m} v_{ik}m_{ik}^{x}, \sum_{r=1}^{s} w_{rk}u_{rk}^{y} - \theta_{k}^{*}\sum_{i=1}^{m} v_{ik}l_{ik}^{x}\right) = (0,0,0)$$

$$\left(\sum_{i=1}^{m} v_{ip}L_{ip}^{x}, \sum_{i=1}^{m} v_{ip}m_{ip}^{x}, \sum_{i=1}^{m} v_{ip}u_{ip}^{y}\right) = (1,1,1)$$

$$(w_{tlp}l_{tlp}^{y} - w_{tlp}u_{tlp}^{y}, w_{tlp}m_{tlp}^{y} - w_{tlp}m_{tlp}^{y}, w_{tlp}u_{tlp}^{y} - w_{tlp}l_{tlp}^{y}) \le (l_{tlp}^{z}, m_{tlp}^{z}, u_{tlp}^{z}), \quad l,t = 1,\dots s$$

$$(-w_{tlp}l_{tlp}^{y} + w_{tlp}u_{tlp}^{y}, -w_{tlp}m_{tlp}^{y} + w_{tlp}m_{tlp}^{y}, -w_{tlp}u_{tlp}^{y} + w_{tlp}l_{tlp}^{y}) \le (l_{tlp}^{z}, m_{tlp}^{z}, u_{tlp}^{z}), \quad l,t = 1,\dots s$$

$$w \ge \varepsilon, v \ge \varepsilon, u_{tlp}^{z} \ge m_{tlp}^{z} \ge l_{tlp}^{z} \ge 0, u_{pj}^{\alpha} \ge m_{pj}^{\alpha} \ge l_{pj}^{\alpha} \ge 0, \ j = 1,..,n, t = 1,..,s.$$

By performing the necessary calculations, the model is as follows:

$$\min\left(\frac{1}{3n-1}\sum_{\substack{j=1\\ j\neq p}}^{n} L_{pj}^{\alpha}+\frac{1}{3n-1}\sum_{\substack{j=1\\ j\neq p}}^{n} m_{pj}^{\alpha}+\frac{1}{3n-1}\sum_{\substack{j=1\\ j\neq p}}^{n} u_{pj}^{\alpha}\right) \tag{18}$$

$$\min\left(\frac{1}{3}e^{t}l_{tl}^{z}e+\frac{1}{3}e^{t}m_{tl}^{z}e+\frac{1}{3}e^{t}u_{tl}^{z}e\right)$$

$$s.t\sum_{r=1}^{s} w_{rp}L_{rj}^{y}-\sum_{i=1}^{m} v_{ip}u_{ij}^{x}\leq -l_{pj}^{\alpha}, \qquad j=1,..,n$$

$$\sum_{r=1}^{s} w_{rp}m_{rj}^{y}-\sum_{i=1}^{m} v_{ip}m_{ij}^{x}\leq -m_{pj}^{\alpha}, \qquad j=1,..,n$$

$$\sum_{ri=1}^{s} w_{rp}u_{rj}^{y}-\sum_{i=1}^{m} v_{ip}l_{ij}^{x}\leq -u_{pj}^{\alpha}, \qquad j=1,..,n$$

$$\sum_{r=1}^{s} w_{rk}l_{rk}^{y}-\theta_{k}^{*}\sum_{i=1}^{m} v_{ik}u_{ik}^{x}\leq 0$$

$$\sum_{r=1}^{s} w_{rk}m_{rk}^{y}-\theta_{k}^{*}\sum_{i=1}^{m} v_{ik}m_{ik}^{x}\leq 0$$

$$\sum_{r=1}^{s} w_{rk}u_{rk}^{y}-\theta_{k}^{*}\sum_{i=1}^{m} v_{ik}l_{ik}^{x}\leq 0$$

$$\sum_{i=1}^{m} v_{ip}l_{ip}^{x}=1$$

$$\sum_{i=1}^{m} v_{ip}m_{ip}^{x}=1$$

$$\sum_{i=1}^{m} v_{ip}u_{ip}^{x}=1$$

$$w_{tlp}l_{tlp}^{y}-w_{tlp}u_{tlp}^{y}\leq l_{tlp}^{z}, \qquad l,t=1,\ldots,s$$

$$w_{tlp}m_{tlp}^{y}-w_{tlp}m_{tlp}^{y}\leq m_{tlp}^{z}, \qquad l,t=1,\ldots,s$$

$$w_{tp}u_{tp}^{y}-w_{lp}l_{lp}^{y}\leq u_{tl}^{z}, \qquad l,t=1,\ldots,s$$

$$-w_{tlp}l_{tlp}^{y}+w_{tlp}u_{tlp}^{y},\leq l_{tlp}^{z}, \qquad l,t=1,\ldots,s$$

$$-w_{tlp}m_{tlp}^{y}+w_{tlp}m_{tlp}^{y}\leq m_{tl}^{z}, \qquad l,t=1,\ldots,s$$

$$-w_{tlp}u_{tlp}^{y}+w_{tlp}l_{tlp}^{y}\leq u_{tlp}^{z}, \qquad l,t=1,\ldots,s$$

$$w\geq\varepsilon, v\geq\varepsilon, u_{tlp}^{z}\geq m_{tlp}^{z}\geq l_{tlp}^{z}\geq 0,$$

$$u_{pj}^{\alpha}\geq m_{pj}^{\alpha}\geq l_{pj}^{\alpha}\geq 0, \qquad j=1,\ldots,n$$

In the above model, it is a parameter. The importance of the objective functions is considered, and it is determined by the manager that $\sum \lambda_j = 1$. Suppose (w_p^*, v_p^*) is the optimal solution of model (8). Therefore, we have

$$\theta_{jp} = \frac{w_p^* \tilde{y}_j}{v_p^* \tilde{x}_j} \quad , j = 1, \dots, n \quad , p = 1, \dots, n \qquad (19)$$

$$\approx \frac{(\sum_{r=1}^{s} w_{rp}^* L_{rj}^y, \sum_{r=1}^{s} w_{rp}^* m_{rj}^y, \sum_{r=1}^{s} w_{rj}^* u_{rj}^y)}{(\sum_{i=1}^{m} v_{ip}^* L_{ij}^x, \sum_{i=1}^{m} v_{ip}^* m_{ij}^x, \sum_{i=1}^{m} v_{ip}^* u_{ij}^x)}$$

$$\approx (L_{jp}^{\theta}, m_{jp}^{\theta}, u_{jp}^{\theta})$$

$(L_{io}^{\theta}, m_{io}^{\theta}, u_{io}^{\theta}) =$ Darya (J, P) secondary target cross-efficiency table with symmetric weights.

In which $p = 1, \dots, n, j = 1, \dots, n$ Using the above explanation, the table of cross-efficiency of the secondary goal with symmetrical weights is as follows.

Table (2) secondary goal cross efficiency for fuzzy input and output

	DMU_1	....	DMU_k	....	DMU_n	average
DMU_1	$[L_{11}^{\theta}, m_{11}^{\theta}, U_{11}^{\theta}]$	....	$[L_{1k}^{\theta}, m_{1k}^{\theta}, u_{1k}^{\theta}]$	....	$[L_{1n}^{\theta}, m_{1n}^{\theta}, u_{1n}^{\theta}]$	$[\frac{1}{n}\sum_{j=1}^{n} L_{1j}^{\theta}, \frac{1}{n}\sum_{j=1}^{n} m_{1j}^{\theta}, \frac{1}{n}\sum_{j=1}^{n} u_{1j}^{\theta}]$
. . . .	. .	. . .		. . .		. . . .
DMU_p	$[L_{p1}^{\theta}, m_{p1}^{\theta}, U_{p1}^{\theta}]$	....	$[L_{pk}^{\theta}, m_{pk}^{\theta}, U_{pk}^{\theta}]$	....	$[L_{pn}^{\theta}, m_{pn}^{\theta}, u_{pn}^{\theta}]$	$[\frac{1}{n}\sum_{j=1}^{n} L_{pj}^{\theta}, \frac{1}{n}\sum_{j=1}^{n} m_{pj}^{\theta}, \frac{1}{n}\sum_{j=n}^{n} u_{pj}^{\theta}]$
. . . .	. .	. . .		. . .		. . . .
DMU_n	$[L_{n1}^{\theta}, m_{n1}^{\theta}, U_{n1}^{\theta}]$	....	$[L_{nk}^{\theta}, m_{nk}^{\theta}, U_{nk}^{\theta}]$	....	$[L_{nn}^{\theta}, m_{nn}^{\theta}, u_{nn}^{\theta}]$	$[\frac{1}{n}\sum_{j=1}^{n} L_{nj}^{\theta}, \frac{1}{n}\sum_{j=1}^{n} m_{nj}^{\theta}, \frac{1}{n}\sum_{j=1}^{n} u_{nj}^{\theta}]$

Here the fuzzy averages are compared, any decision-making unit with a larger average is more efficient. According to the application of the secondary goal, the optimal solutions are either eliminated or at least reduced. If we look at the issue optimistically, different optimal answers will be eliminated, and if we look at the issue pessimistically, different optimal answers will decrease. With an optimistic view, the cross-efficiency table considers the secondary goal uniquely, and the decision-making units have a unique rank. and are comparable. To rank the decision-making units according to the nature of the issue, we use the following ranking function. Let's assume $B=(b_1,b_2,b_3), A=(a_1,a_2,a_3)$ are triangular fuzzy numbers.
We explain

$$D(A)=a_1+\frac{1}{4(a_3-a_2)} \tag{20}$$

$$D(B)=b_1+\frac{1}{4(b_3-b_2)} \tag{21}$$

The following relationships are established .

$$A<B \Leftrightarrow D(A)<D(B) \tag{22}$$

$$A\le B \Leftrightarrow D(A)\le D(B) \tag{23}$$

$$A=B \Leftrightarrow D(A)=D(B) \tag{24}$$

By using the aforementioned ranking function and the relationships obtained from that column related to the fuzzy average of the table, they are compared, and through this comparison, the decision-making units are ranked. Each line of the cross-efficiency table, its corresponding fuzzy number in the fuzzy average column, is converted to a non-fuzzy number using relationships (21-23), compared to the numbers of other rows, if this number is larger, it is more efficient. This comparison is done to rank all the decision-making units.

4. Practical example

Ten branches of a commercial bank were studied and the required information was obtained from these ten bank branches, model (8) was solved and finally the following results were obtained.

Table 3 Lower bounds of decision maker inputs

Bank branches	Personnel score i_1	Creative demands i_2	interest paid i_3
DMU_1	5.2	166965005	347912609
DMU_2	6.5	1364254263	321089157
DMU_3	5.13	102154263	439622053
DMU_4	7.58	1023094065	247470622
DMU_5	3.89	244442242	28332000
DMU_6	4.44	150114017	175107405
DMU_7	2.69	416603512	55843067
DMU_8	2.26	1025368685	5079356
DMU_9	3.77	125611949	321956067
DMU_{10}	5.6	1220212885	5870000

Table 4 Midpoint of input of decision making units

Bank branches	**Personnel score i_1**	**Creative demands i_2**	**interest paid i_3**
DMU_1	11.505	3204527225	6267251735
DMU_2	17.77571429	5420093121	13974801379
DMU_3	14.77714286	4062997330	2430817731
DMU_4	14.93928571	5811247992	3264590272
DMU_5	14.79285714	3728154627	8935005782
DMU_6	11.85928571	4825940830	5148882350
DMU_7	10.82642857	4557896111	7333046577
DMU_8	10.1457142	3925577955	4346972893
DMU_9	15.90214286	5262472780	7062857806
DMU_{10}	13.45428571	4451249734	8880600175

Table 5 Upper bounds of decision maker inputs

Bank branches	**Personnel score i_1**	**Creative demands i_2**	**interest paid i_3**
DMU_1	21.61	5826283949	30033076818
DMU_2	56.23	15713640424	948702165089
DMU_3	38.59	11969476089	9958384916
DMU_4	31.43	30435770419	17958774018
DMU_5	26.41	11450174945	82231846069
DMU_6	19.31	15105382313	29728030018
DMU_7	25.01	11461622508	36434239083
DMU_8	22.8	12424040548	19571582641
DMU_9	31.9	18113748298	57849361275
DMU_{10}	33.24	12448647772	91314625872

Table 6 The lower bound of the outputs of the decision-making units

Bank branches	Sum of deposits o_1	Other deposits o_2	Facilities o_3	profit received o_4	Fee received o_5
DMU_1	7098487595	28948043462	848671179	38368691	5162000
DMU_2	7949322656	23631050649	566162650	42883893	4405000
DMU_3	14995970790	38631972139	864134766	36411899	7300000
DMU_4	18514914833	29432773959	895838606	77239155	18275000
DMU_5	17332785899	19275628277	18843288	1431607	3925000
DMU_6	20385936597	21779585799	4024246	56474	11799913
DMU_7	7452604318	9327588934	569445	391780	1106670
DMU_8	6146639414	28111168328	221934579	25128335	650000
DMU_9	27850207872	20439226024	1367871203	25582456	17733000
DMU_{10}	10096103316	28454116922	517976795	27773398	3500000

Table 7 The middle point of the outputs of the decision-making units

Bank branches	Sum of deposits o_1	Other deposits o_2	Facilities o_3	profit received o_4	Fee received o_5
DMU_1	52171726468	55729852938	3957105421	158252484.1	209362113
DMU_2	92582493774	91501177806	5900502096	431321754.4	1907797026
DMU_3	80053680037	72004902720	5333368785	3094866691.8	1347623292
DMU_4	80184379532	93783934491	5620799538	516295053.6	2014151023
DMU_5	89211750661	84640654618	4911634657	168888314.4	401399286.7
DMU_6	80255164183	72319356101	5875395233	351730379.7	1530606378
DMU_7	69565403708	64608470345	4198035611	196957135.3	632104134.3
DMU_8	50638122921	59838074926	3005501778	207056005.1	676723365
DMU_9	81830300111	88236906513	5600393718	223554326.6	542498353.9
DMU_{10}	86094189632	85908459121	5128502987	354636065.6	2959881112

Table 8 The upper limit of the outputs of the decision-making units

Bank branches	Sum of deposits o_1	Other deposits o_2	Facilities o_3	profit received o_4	Fee received o_5
DMU_1	122497919047	121751849411	6914738665	305467932	747392532
DMU_2	378785621113	329970969669	22330988268	2306896375	15347945440
DMU_3	144962760424	167941180111	14534600915	10572334112	5720495923
DMU_4	320467175205	353170012187	24239290627	1683454030	17734125040
DMU_5	286782666394	390987804421	19805543528	457407119	1939809308
DMU_6	311973279912	211466703060	35304269319	2688388239	11142908153
DMU_7	266434914435	124678248906	15460389283	792206863	3956028971
DMU_8	188565753715	140587061901	8069534218	570261939	6055939136
DMU_9	232643001693	259004083181	20717137949	517953877	1841137172
DMU_{10}	4437205575872	355691625890	21626923578	2316300360	30782513082

In this evaluation, Branch 2 has been assigned the 1st rank and Branch 10 has been assigned the 10th rank. Considering the inputs and outputs and comparing them with reality, it is consistent.

5. Conclusion

Ranking with fuzzy input and output in cross-efficiency requires an optimal solution of the multiplicative form of the CCR model in the nature of the input. Despite the optimal solutions of different models, different tables were obtained from different optimal ones, from which different ratings were obtained. That is, a decision-making unit had different ranks. To avoid this problem, we must find an optimal solution among other optimal solutions. The realization of this task is done by considering a secondary goal. This secondary goal selects the optimal solution from among the optimal solutions. In this article, the secondary goal has contributed to the realization of the issue. Based on this research, the cross-efficiency table related to the mentioned secondary goal of formation, the decision-making units were evaluated with the help of the ranking function, and a practical example was presented.

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