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Malmquist Productivity Index Based on Means of Weights for Ranking of Decision Making Units in Data Envelopment Analysis

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Abstract

The Malmquist Index is the prominent Index for measuring the productivity change of Decision Making Units (DMUs) in multiple time periods that use Data Envelopment Analysis (DEA) models with Variable Return to Scale (VRS) and Constant Return to Scale (CRS) technology. One of the drawbacks of DEA is the problem of lack of discrimination among efficient DMUs and hence yielding many numbers of DMUs as efficient. The main purpose of this paper is to overcome this inability. In this paper, we compute the Malmquist Index based on means of weights evaluation, and by using this method we can rank DMUs by logical criteria. For illustration numerical example is given.

Keywords: Data Envelopment Analysis (DEA), Decision Making Units (DMUs), Means of Weights, Malmquist Index.

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1. Introduction

Data Envelopment Analysis (DEA) is a mathematical programming technique that measures the relative efficiency of Decision Making Units (DMUs) with multiple inputs and outputs. Charnes (1978) first proposed DEA as an evaluation tool to measure and compare the relative efficiency of DMUs [1]. Their model assumed Constant Returns to Scale (CRS, the CCR model), the model with Variable Return to Scale (VRS, the BCC model) was developed [2]. The Malmquist Index is the most important Index for measuring the relative productivity change of DMUs in multiple time periods. For the first time, the Malmquist Index was introduced by Caves (1982) [3]; later DEA was used by Fare (1992), for measuring the Malmquist Index [4,5]. They used DEA model (CRS) and VRS for computing Malmquist Index. The rest of the paper is organized as follows: In sections 2, we describe Data Envelopment Analysis (DEA), in section 3; we compute efficiency of DMUs by using means of weight in different period and different model of DEA. In section 4, described new method for majoring Malmquist index. To illustrate numerical. example is mentioned in section 5. The last section summarizes and concludes [6].

2. Literature Review

2.1 Data Envelopment Analysis (DEA)

Assuming that there are n DMUs, each with m inputs and s outputs, the relative efficiency of a particular DMU_o ($o \in \{1, 2..., n\}$) is obtained by solving the following programing problem:

$$\begin{aligned}
&\theta^* = \beta_o = \text{Max} \sum_{r=1}^{s} u_r y_{ro} \\
&\text{S.t } \sum_{i=1}^{m} v_i x_{io} = 1 \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
&\quad\quad j = 1,2, \dots, n \\
&u_r \ge 0 \qquad r = 1, 2, , \dots, s \qquad\qquad (1) \\
&v_i \ge 0 \qquad i = 1, 2, , \dots, m
\end{aligned}$$

where j is the DMU index (j=1,...,n), r the output index, (r=1,2,...,s) and i the input index (i=1,2,...,m), y_{rj} the value of the rth output for the jth DMU, x_{ij} the value of the ith input for the jth DMU, u_r the weight given to the rth output, v_i the weight given to the i input. DMU_o is efficient if and only if $\theta^* = \beta_o = 1$. A relative efficiency score of 1 indicates that the DMU under consideration is efficient.

Consider the following Table (Table 1):

Table 1 (Weights of DMUs with m inputs and r outputs)

	Input weights				Output weights			
	v_1	v_2	…	v_m	u_1	u_2	…	u_s
1	v_{11}	v_{21}	…	v_{m1}	u_{11}	u_{21}	.	u_{s1}
.	.	.	. .		.	.	. .	
.	.	.	. .		.	.	. .	
.	.	.	. .		.	.	. .	
k	v_{1k}	v_{2k}	…	v_{mk}	u_{1k}	u_{2k}	…	u_{sk}
.	.	.	. .		.	.	. .	
.	.	.	. .		.	.	. .	
.	.	.	. .		.	.	. .	
n	v_{1n}	v_{2n}	…	v_{mn}	u_{1n}	u_{2n}	…	u_{sn}
mean	$\sum_{j=1}^{n} v_{1j} / n$	$\sum_{j=1}^{n} v_{2j} / n$	…	$\sum_{j=1}^{n} v_{mj} / n$	$\sum_{j=1}^{n} u_{1j} / n$	$\sum_{j=1}^{n} u_{2j} / n$	…	$\sum_{j=1}^{n} u_{sj} / n$

Assume

$\bar{v}_i = \frac{1}{n}\sum_{j=1}^{n} v_{ij} \quad i = 1,2,\dots,m \qquad (3)$

and

$\bar{u}_r = \frac{1}{n}\sum_{j=1}^{n} v_{rj} \quad r = 1,2,\dots,s \qquad (4)$

The efficiency of DMU_j by using means of weights is:

$\bar{\theta}_j = \frac{\sum_{r=1}^{s}\bar{u}_r y_{rj}}{\sum_{i=1}^{m}\bar{v}_i x_{ij}} \quad j = 1,2,\dots,n \qquad (5)$

Now, by using $\bar{\theta}_j$ we compute Malmquist index in next section.

3. Research finding

3. 1 Computing of efficiency by using means of weights in different period and different models of DEA

We can compute $\bar{\theta}^{\,t(CRS)}_{k(t)}$, $\bar{\theta}^{t(VRS)}_{,k(t)}$ (DMUk in period t and frontier period=t), Likewise Previous Section, where $x^t{}_{ij}$, $y^t{}_{ij}$, $v^t{}_{ij}$, $u^t{}_{ij}$ are substituted x_{ij} , y_{rj} , v_{ik} , u_{rk} .

$\left(\bar{\theta}^{\,t(CRS)}_{k(t)},\ \bar{\theta}^{t(VRS)}_{k(t)}(DMU_k\ in\ period\ t + 1\ and\ frontier\ period = t+1)\right)$ DEA model of CRS technology in input orientation, DMUk in period t and frontier period = t+1

$Phase(1)$:

$Max\ \bar{\theta}^{\,t(CRS)}_{k(t)} = \sum_{r=1}^{s} u_{rk}^{t+1}\, x_{rk}^{t}$

$s.t.$

$\sum_{i=1}^{m} v_{ik}^{t+1}\, x_{ik}^{t} = 1$

$\sum_{r=1}^{s} u_{rk}^{t+1}\, y_{rj}^{t+1} - \sum_{i=1}^{m} v_{ik}^{t+1}\, x_{ij}^{t+1} \le 0$

$j = 1,2,\dots,n \qquad (6)$

$v_{ik}^{t+1} \ge 0 \qquad i = 1,2,\dots,m$

$u_{rk}^{t+1} \ge 0 \qquad r = 1,2,\dots,s$

Model (6) is solved n times, each time for one DMU.

Therefore $\bar{v}_i{}^{t+1} = \frac{\sum_{j=1}^{n} v_{ij}^{t+1}}{n}$

$i = 1,2,\dots,m \qquad (7)$

$\bar{u}_r{}^{t+1} = \frac{\sum_{j=1}^{n} u_{rj}^{t+1}}{n} \qquad (8)$

Phase (2): Efficiency of DMU_j in period t and frontier period = t+1 by using means of weights is:

$\bar{\theta}^{\,t(CRS)}_{j(t)} = \frac{\sum_{r=1}^{s}\bar{u}_r^{t+1}\, y_{rj}^{t}}{\sum_{i=1}^{m}\bar{v}_i^{t+1} x_{ij}^{t}}$

$j = 1,2,\dots,n \qquad (9)$

DEA model of CRS technology in input orientation DMU_k in period t+1 and frontier =t+1

$Phase(1)$:

$Max\ \bar{\theta}^{\,t(CRS)}_{k(t)} = \sum_{r=1}^{s} u_{rk}^{t}\, x_{rk}^{t+1}$

$subject\ to$:

$\sum_{i=1}^{m} v_{ik}^{t}\, x_{ik}^{t+1} = 1$

$\sum_{r=1}^{s} u_{rk}^{t}\, y_{rj}^{t} - \sum_{i=1}^{m} v_{ik}^{t}\, x_{ij}^{t} \le 0$

$j = 1,2,\dots,n \qquad (10)$

$v_{ik}^{t} \ge 0 \qquad i = 1,2,\dots,m$

$u_{rk}^{t} \ge 0 \qquad r = 1,2,\dots,s$

Model (10) is solved n times, each time for one DMU.

Therefore $\bar{v}_i{}^{t} = \frac{\sum_{j=1}^{n} v_{ij}^{t}}{n}$

$i = 1,2,\dots,m \qquad (11)$

$\bar{u}_r{}^{t} = \frac{\sum_{j=1}^{n} u_{rj}^{t}}{n} \qquad (12)$

Phase (2): Efficiency of DMU_j in period t+1 and frontier period = t by using means of weights is:

$\bar{\theta}^{t(CRS)}_{j(t+1)} = \frac{\sum_{r=1}^{s}\bar{u}_r^{t}\, y_{rj}^{t+1}}{\sum_{i=1}^{m}\bar{v}_i^{t} x_{ij}^{t+1}}$

$j = 1,2,\dots,n \qquad (13)$

Now, by using $\bar{\theta}^{\,t(CRS)}_{j(t)}$, $\bar{\theta}^{\,t+1(CRS)}_{j(t)}$, $\bar{\theta}^{\,t(CRS)}_{j(t+1)}$, $\bar{\theta}^{t(VRS)}_{j(t)}$ we can compute Malmquist index.

DEA model of VRS technology in input orientation DMU_k in period t and frontier period = t

$Phase(1)$:

$Max\ \bar{\theta}^{\,t(CRS)}_{k(t)} = \sum_{r=1}^{s} u_{rk}^{t}\, y_{rk}^{t} + u_{0k}^{t}$

$subject\ to$:

$\sum_{i=1}^{m} v_{ik}^{t}\, x_{ik}^{t+1} = 1$

$\sum_{r=1}^{s} u_{rk}^{t}\, y_{rj}^{t} - \sum_{i=1}^{m} v_{ik}^{t}\, x_{ij}^{t} + u_{0k}^{t} \le 0$

$j = 1,2,\dots,n \qquad (14)$

$v_{ik}^{t} \geq 0 \qquad i = 1,2,\dots,m$

$u_{rk}^{t} \geq 0 \qquad r = 1,2,\dots,s$

$u_0^t \quad free$

Model (10) is solved n times, each time for one DMU.

Therefore

$$\bar{v}_i{}^t = \frac{\sum_{j=1}^{n} v_{ij}^t}{n} \quad i = 1,2,\dots,m \qquad (15)$$

$$\bar{u}_r{}^t = \frac{\sum_{j=1}^{n} u_{0j}^t}{n} \qquad (16)$$

$$\bar{u}_0{}^t = \frac{\sum_{j=1}^{n} u_{0j}^t}{n} \qquad (17)$$

Phase (2): Efficiency of DMU_j in period t+1 and frontier period = t by using means of weights is:

$$\bar{\theta}_{j(t)}^{t(VRS)} = \frac{\sum_{r=1}^{s} \bar{u}_r^t\, y_{rj}^t + \bar{u}_0^t}{\sum_{i=1}^{m} \bar{v}_i^t x_{ij}^t} \quad j = 1,2,\dots,n \qquad (18)$$

Likewise we can compute $\bar{\theta}_{k(t+1)}^{t+1(CRS)}$, $,\bar{\theta}_{j(t)}^{t+1(VRS)}$, $\bar{\theta}_{j(t+1)}^{t(VRS)}$.

3.2 New Method for computing Malmquist Index based on Means Weights in different models of DEA:

According computing of $\bar{\theta}_{k(t)}^{t(CRS)}$, $\bar{\theta}_{k(t)}^{t(VRS)}$ in previous section. Consider the following equations:

$$EC_{\bar{„}} = \frac{\bar{\theta}_{(t+1)}^{t+1(CRS)}}{\bar{\theta}_{(t)}^{t(CRS)}} \qquad (19)$$

$$PEC_{\bar{„}} = \frac{\bar{\theta},_{(t+1)}^{t+1(VRS)}}{,\bar{\theta}_{(t)}^{t(VRS)}} \qquad (20)$$

$$TC_{\bar{„}} = \left[\frac{\bar{\theta}_{(t)}^{t(CRS)}}{,\bar{\theta}_{(t)}^{t+1(CRS)}} \times \frac{\bar{\theta}_{(t+1)}^{t(CRS)}}{\bar{\theta}_{(t+1)}^{t+1(CRS)}}\right]^{\frac{1}{2}} \qquad (21)$$

$$SEC_{\bar{„}} = \left[\frac{\bar{\theta}_{(t)}^{t(VRS)}}{\bar{\theta}_{(t)}^{t(CRS)}} \times \frac{\bar{\theta}_{(t+1)}^{t+1(CRS)}}{\bar{\theta}_{(t+1)}^{t+1(VRS)}}\right] \qquad (22)$$

Where $EC_{\bar{„}}$ Efficiency Change is based on $\bar{„}$, $PEC_{\bar{„}}$ is pure Efficiency Change based. on $\bar{„}$, $TC_{\bar{„}}$ is Technology Change based on $\bar{„}$ and $SEC_{\bar{„}}$ is scale Efficiency Change based on $\bar{„}$. The Malmquist Index and its FGLR and FGNZ decompositions are as follows (for more details, see [6]. By similar way we can compute Malmquist Index.

$$Malmquist\ Index\ based\ \text{on}\ \bar{\theta}\ (MI_{\bar{„}}) = EC_{\bar{„}} \times TC_{\bar{„}} \qquad (23)$$

$$Malmquist\ Index\ based\ \text{on}\ \bar{\theta}\ (MI_{\bar{„}}) = PEC_{\bar{„}} \times SEC_{\bar{„}} \times TC_{\bar{„}} \qquad (24)$$

$if\ MI_{\bar{„}} > 1\ , it\ shows\ DMU\ had\ progress.$

$if\ MI_{\bar{„}} < 1\ , it\ shows\ DMU\ had\ regress.$

$if\ MI_{\bar{„}} = 1\ , it\ shows\ DMU\ had\ not\ changing.$

We define Malmquist Index Disparity and Expanded Malmquist Index Disparity:

$$MID = \frac{MI_{„} - MI_{\bar{„}}}{MI_{„}} \times 100 \qquad (25)$$

4. Case Study

Consider Table (1), in this Table, we have six DMUs with one input and two outputs at two periods. Assume that all DMUs agree as being true the following judgments at two periods.

Table1. Data in period t and t+1

Unit in period t	X_1	Y_1	Y_2	Unit in period t+1	X_1	Y_1	Y_2
DMU_1	100	200	1000	DMU_1	100	1100	700
DMU_2	100	1200	600	DMU_2	100	1300	600
DMU_3	100	1600	100	DMU_3	100	1500	400
DMU_4	300	300	2850	DMU_4	300	900	2400
DMU_5	300	3600	1200	DMU_5	300	4200	2100
DMU_6	300	2100	2100	DMU_6	300	900	2700

Table2. Result of DMUs in period 1 and frontier period=1

Unit	Efficiency	V_1	U_1	U_2	$\bar{,,}_j$
DMU_1	1.000	0.0100	0.0004	0.0009	0.8651
DMU_2	1.000	0.0100	0.0004	0.0009	0.9667
DMU_3	1.000	0.0100	0.0006	0.0005	0.7414
DMU_4	0.9500	0.0033	0.0000	0.0003	0.7845
DMU_5	0.9048	0.0033	0.0002	0.0002	0.8102
DMU_6	0.9074	0.0033	0.0001	0.0003	0.8377
Average	-	0.0067	0.0003	0.0005	-

Table3. Result on DMUs in period 2 and frontier period=2

Unit	Efficiency	V_1	U_1	U_2	$\bar{,,}_j$
DMU_1	0.9429	0.0100	0.0002	0.0010	1.0034
DMU_2	0.9184	0.0100	0.0006	0.0002	1.0081
DMU_3	1.0000	0.0100	0.0006	0.0002	0.9346
DMU_4	0.8952	0.0033	0.0001	0.0003	0.7501
DMU_5	1.0000	0.0033	0.0001	0.0003	1.1278
DMU_6	1.0000	0.0033	0.0000	0.0004	0.8283
Average	-	0.0067	0.0003	0.0005	-

Table4. Result of DMUs in period 1 and frontier period=2

Unit	Efficiency	V_1	U_1	U_2	$\bar{,,}_j$
DMU_1	1.1111	1.1111	0.0000	0.0011	0.7928
DMU_2	0.8571	0.8571	0.0002	0.0010	0.7794
DMU_3	1.0667	1.0667	0.0007	0.0000	0.5235
DMU_4	1.0556	1.0556	0.0000	0.0004	0.7279
DMU_5	0.3163	0.3163	0.0002	0.0001	0.6320
DMU_6	0.8667	0.8667	0.0001	0.0003	0.7124
Average	-	-	0.0002	0.0003	-

Table5. Result on DMUs in period 2 and frontier period=1

Unit	Efficiency	V_1	U_1	U_2	$\bar{,,}_j$
DMU_1	1.0556	0.0100	0.0004	0.0009	0.8249
DMU_2	1.0595	0.0100	0.0006	0.0005	0.8075
DMU_3	1.0833	0.0100	0.0006	0.0005	0.7164
DMU_4	0.8519	0.0033	0.0001	0.0003	0.6736
DMU_5	1.1667	0.0033	0.0001	0.0003	0.9093
DMU_6	0.9444	0.0033	0.0001	0.0003	0.7473
Average	-	0.0067	0.0002	0.0005	-

Table6. Result of Malmquist Index for DMUs based on $\bar{„}$

Unit	$EC_{\bar{„}}$	$TC_{\bar{„}}$	$MI_{\bar{„}}$	$SEC_{\bar{„}}$
DMU_1	0.9429	1.0038	0.9464	0.9429
DMU_2	0.9184	1.1602	1.0655	0.9184
DMU_3	1.000	1.0078	1.0078	1.000
DMU_4	0.9424	0.9254	0.8721	1.0469
DMU_5	1.1053	1.1371	1.2568	1.0521
DMU_6	1.1020	0.9944	1.0959	1.1020

Table7. Result of Malmquist Index for DMUs based on $\bar{„}$

Unit	$EC_{\bar{„}}$	$TC_{\bar{„}}$	$MI_{\bar{„}}$	$Rank$
DMU_1	1.1599	0.9470	1.0984	3
DMU_2	1.0429	0.9967	1.0394	4
DMU_3	1.2606	1.0418	1.3132	2
DMU_4	0.9561	0.9837	0.9405	6
DMU_5	1.3919	1.0165	1.4148	1
DMU_6	0.9888	1.0299	1.0183	5

5. Conclusion

For obtaining relative Efficiency of DMUs, we use means of weights, and by using this method we could compute Malmquist Index. The result seems to be quite satisfactory by comparing the AP method. By using a new method (means of weights) we can rank DMUs by logical criteria, that you can see the result from the performance of this method in a numerical example.

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