Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol.7, No.4, Year 2019 Article ID IJDEA-00422, 6 pages
Research Article

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

# Super Efficiency Model in Front of MPSS

**F. Roozbeh[1], M. Ahad zadeh Namin[*2]**

(1) Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran.
(2) Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.

Received 09 May 2019, Accepted 28 August 2019

**Abstract**

By using the Double Frontiers Criteria, Hurwicz succeeded to achieve The Most Productive Scale Size of decision making units. This Double frontiers criteria was achieved by using two models of "CCR" in optimistic viewpoint in input form and the "CCR" model in pessimistic viewpoint in input form. In this paper, we intend to find a criteria for Double frontiers of super efficiency by using two models of super efficiency in input form and in both viewpoint of optimistic and pessimistic in such a way that conclude in Hurwicz Double frontiers of MPSS.

**Keywords:** double frontier, Most Productive Scale Size, super efficiency, optimistic and pessimistic.

*. Corresponding author: Email: MahnazAhadzadehnamin@gmail.com

## 1. Introduction

Data Envelopment Analysis which was initially proposed by Charnes et al. [1] is a methodology by employing mathematical programming for measuring the relative efficiency of decision making units (DMUs) with multiple inputs and multiple outputs. Banker et al. [2] constructed a link between DEA and the estimation of efficient production frontiers with an axiomatic framework. The CCR model is one of the most common DEA models first introduced by Charnes, Cooper and Rhodes in 1978 to measure the efficiency of a set of DMUs. This model is the extension of Farrell measure used for multiple inputs and outputs and it deals with the calculation of radial efficiency in Production Possibility Set (PPS) under Constant Returns to Scale (CRS) and it has two characteristic of input orientation (envelopment form), and output orientation (multiple form). Most Productive Scale Size (MPSS) is an important topic in DEA that links to returns to scale (RTS). The MPSS is closely related to the concept of constant returns to scale. In addition, the concept of MPSS was introduced into DEA by Banker [2]. Banker et al. [3] reviewed the development of MPSS as one part of the literature review of RTS. However, all the papers about the MPSS in DEA are measured from the optimistic point of view. From the pessimistic point of view can also be measured the performances of decision making units (DMUs), that may be more interesting. The literature about the pessimistic measurement in DEA can be found in Wang et al [4], Wang and Chin [4], [5] and also Wang and Lan [6]. Since, the results of MPSS application in different evaluation system, might end to different results, hence by applying Double frontiers and Hurwicz's Criterion, the performance of each unit is assessed in both optimistic and pessimistic point of view. The double frontiers efficiency measurement integrates both optimistic and pessimistic efficiencies of each DMU and is therefore more comprehensive than either of them. Supporting hyperplane, also known as compatible $T_c$, are hyper planes which their gradient is calculated based on multiplier form of CCR Model and has created a cone that is in fact "Binding hyper planes".

On $T_v$ concept is different, Jahanshahloo et.al. In 1993, Anderson and Peterson presented the AP Model for the ranking of efficient units which in order to rank the efficient units (DMU), eliminates it from the process, and separates it from the rest of the decision making units and runs the model for the rest of decision making units. This means the efficient unit is only used in the objective function and limitations of efficient unit are eliminated from the structural limitations and in the end, after the rest of the units are calculated, they can be compared with each other. One of the complications faced in this model, is the envelopment form when it comes to a special structure of data (for example, if there is more space between numbers or data) or the instability of the model which by eliminating some of the DMUs, a more quantitative $\theta^*$ is obtainable.

## 2. MPSS criteria to Hurwicz double frontiers

Suppose n (j=1,…,n) units are given under evaluation. Consider two models as under:

$$\theta^{*opt} = Min\ \theta \tag{1}$$

$$\begin{aligned} s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} \qquad i = 1,...,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} \qquad r = 1,...,s \\ & \lambda_j \ge 0 \end{aligned}$$

And

$$\theta^{*pes} = Max\ \theta \tag{2}$$

$$\begin{aligned} s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \geq \theta x_{io} && i=1,\dots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{ro} && r=1,\dots,s \\ & \lambda_j \geq 0 \end{aligned}$$

Model (1) is the CCR model from the optimistic viewpoint and input orientation, and model (2) is the CCR model from the pessimistic viewpoint and output orientation [7].

If $\theta_j^{*opt} = 1$ then $DMU_j$ is *MPSS* from the optimistic viewpoint.

If $\theta_j^{*pes} = Max_j\ \{\theta_j^{*pes}\} = \theta_{jMax}^{*pes}$ then $DMU_j$ is *MPSS* from the pessimistic viewpoint.

To find the MPSS, Hurwicz has defined the Double frontiers Criteria as below:

$$\zeta_j = (1-\lambda_j)\frac{\theta_j^{*pes}}{\theta_{jMax}^{*pes}} + \lambda_j \theta_j^{*opt}$$

Wherein $0 \leq \lambda_j \leq 1$.

If $\zeta_j = 1$ then $DMU_j$ is *MPSS* compared with Double Frontiers.

When $\theta_j^{*pes} = \theta_{jMax}^{*pes}$ and $\theta_j^{*opt} = 1$ then $\zeta_j = 1$.

## 3. Super efficiency for MPSS in double frontiers

In this section we are going to use super efficiency models and find a yardstick enable us to reach the same results as Hurwicz achieved in MPSS with Double Frontiers.

For this purpose the below algorithm is used:

**Step (1):** Solve the Super efficiency Model in optimistic viewpoint in input orientation as Anderson and Peterson introduced for each DMU.

$$\theta_{\sup}^{*opt} = Min\ \ \theta \tag{3}$$

$$\begin{aligned} s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij} \leq \theta x_{io} && i=1,\dots,m \\ & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj} \geq y_{ro} && r=1,\dots,s \\ & \lambda_j \geq 0 && j=1,\dots,n \end{aligned}$$

The $\theta_{\sup j}^{*opt} \geq 1$ units is named as super-efficient optimistic unit.

**Step (2):** Solve the super-efficient model from pessimistic viewpoint in input orientation.

$$\theta_{\sup}^{*pes} = Max\ \ \phi \tag{4}$$

$$\begin{aligned} s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij} \geq \phi x_{io} && i=1,\dots,m \\ & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj} \leq y_{ro} && r=1,\dots,s \\ & \lambda_j \geq 0 && j=1,\dots,n \end{aligned}$$

The units $\theta_{\sup j}^{*\text{pes}} = \theta_{\substack{\text{supj} \\ \text{Max}}}^{*pes}$ are named as super-efficient pessimistic unit.

Step (3): The following criteria is defined for finding the DMU of super efficiency double frontier:

$$\zeta_j^{\sup} = (1-\lambda_j)\frac{\theta_{\sup}^{*opt}}{\theta_{Max}^{*opt}} + \lambda_j \theta^{*pes}$$

Wherein $0 \leq \lambda_j \leq 1$

The DMU which has the highest $\zeta_j^{\sup}$ is known as MPSS unit.

**Theorem:** when $0 < \zeta_j^{\sup} < \theta_{\sup}^{*pes}$

**Proof:** if $\lambda_j = 0$ then $\zeta_j^{\sup} = \dfrac{\theta_{\sup j}^{*opt}}{\theta_{\substack{\sup j \\ Max}}^{*opt}}$ in this situation if $\theta^{*opt}_{\underset{Max}{\sup j}} = \theta^{*opt}_{\sup j}$

Then $\zeta_j = 1$ otherwise $0 < \zeta < 2$.

If $\lambda_j = 1$ then $\zeta_j = \theta^{*pes}_{\sup} > 0$.

If $0 < \lambda_j < 1$ considering $\theta^{*opt}_{\sup} > 0$ then $0 < (1-\lambda_j)\dfrac{\theta^{*pes}}{\theta^{*pes}_{\underset{Max}{\sup j}}} < 1$.

Consider two following situation:

A): $\theta^{*pes}_{\sup} \geq 1$ in this situation $0 < \lambda_j \theta^{*pes}_{\sup} < \theta^{*pes}_{\sup}$ therefore $0 < \zeta_j < 1 + \theta^{*pes}$

B): $0 < \theta^{*pes}_{\sup} < 1$ in this situation $0 < \lambda_j \theta^{*pes}_{\sup} < 1$ therefore $0 < \zeta_j < 2$.

## 4. Numerical example

In this section a numerical example is given to prove that a right Algorithm applied.

Consider a real efficiency measurement problem of the industrial economy of China, where 31 provinces, municipalities and autonomous regions (the DMUs) of country are evaluated in terms of efficiencies of industrial economy in 2007 [7]. The data of inputs and outputs are presented in Table 1, where Original value of fixed assets, Current assets and Gross industrial output value are measured in 100 million RMB (Chinese monetary unit) and calculated at current in prices, Number of staff and workers at year end is expressed in units of 10 thousand person; $DMU_4$ and $DMU_{27}$ are two different Chinese Provinces with the same English name.

Now Table 2 is considering for suggestion method:

**Table 1. Data of textile industry of China in year 2007**

| DMU | Input | | | Output |
|---|---|---|---|---|
| | Original value of fixed assets | Current assets | Number of staff and workers at year end | Gross industrial output value |
| Beijing | 31.68 | 57.77 | 2.94 | 70.85 |
| Tianjin | 74.64 | 57.56 | 4.48 | 77.97 |
| Hebei | 199.04 | 151.3 | 21.78 | 583.74 |
| Shanxi | 18.21 | 16.19 | 2.36 | 26.23 |
| Inner Mongolia | 69.7 | 127.96 | 5.18 | 253.57 |
| Liaoning | 103.04 | 87.05 | 9.24 | 207.1 |
| Jilin | 14.25 | 10.29 | 1.89 | 42.73 |
| Heilongjiang | 20.5 | 34.13 | 3.23 | 35.11 |
| Shanghai | 164.42 | 201 | 14.6 | 364.74 |
| Jiangsu | 1478.86 | 1661.91 | 119.24 | 4313.94 |
| Zhejiang | 1586.27 | 2111.26 | 116.56 | 4190.08 |
| Anhui | 126.54 | 86 | 13.95 | 240.66 |
| Fujian | 312.26 | 282 | 23.81 | 709.49 |
| Jiangxi | 55.68 | 50.06 | 10.8 | 227.13 |
| Shandong | 1246.87 | 900.36 | 107.46 | 3681.48 |
| Henan | 229.05 | 228.7 | 28.54 | 784.83 |
| Hubei | 196.97 | 164.11 | 24.79 | 462.06 |
| Hunan | 57.96 | 54.77 | 10 | 214.98 |
| Guangdong | 702.6 | 583.12 | 62.66 | 1485.23 |
| Guangxi | 22.03 | 28.53 | 4.57 | 64.23 |
| Hainan | 11.06 | 7.07 | 0.7 | 10.01 |
| Chongqing | 21.42 | 23.14 | 4.2 | 76.97 |
| Sichuan | 85.41 | 73.17 | 12.76 | 311.3 |
| Guizhou | 1.71 | 1.81 | 0.5 | 4.72 |
| Yunnan | 10.58 | 7.5 | 1.31 | 9.82 |
| Tibet | 0.06 | 0.19 | 0.01 | 0.21 |
| Shanxi | 69.45 | 37.99 | 8.53 | 77.9 |
| Gansu | 8.59 | 7.51 | 0.95 | 12.23 |
| Qinghai | 3.01 | 2.89 | 0.29 | 4.01 |
| Ningxia | 10.23 | 33.7 | 0.8 | 55.59 |
| Xinjiang | 122.55 | 98.46 | 8.15 | 134.4 |

**Table 2. Three different measurement and super efficiency in front of MPSS considering of Hurwicz Criteria**

| DMU | $\theta^{*opt}_{\sup}$ Effici ency | MPSS | $\theta^{*pes}_{\sup}$ Efficiency | MPSS | super efficiency in Double fronties Efficiency | MPSS |
|---|---|---|---|---|---|---|
| **Beijing** | 0.54 | | 1.19 | | 0.60051 | |
| **Tianjin** | 0.45 | | 1.07 | | 0.539592 | |
| **Hebei** | 0.92 | | 3.02 | | 1.519388 | |
| **Shanxi** | 0.41 | | 1.34 | | 0.674184 | |
| **Inner Mongolia** | 0.95 | | 1.93 | | 0.974694 | |
| **Liaoning** | 0.65 | | 1.93 | | 0.971633 | |
| **Jilin** | 0.95 | | 3.02 | | 1.519694 | |
| **Heilongjiang** | 0.38 | | 0.88 | | 0.443878 | |
| **Shanghai** | 0.64 | | 1.64 | | 0.826531 | |
| **Jiangsu** | 0.9 | | 2.3 | | 1.159184 | |
| **Zhejiang** | 0.81 | | 1.83 | | 0.923265 | |
| **Anhui** | 0.65 | | 2.06 | | 1.036633 | |
| **Fujian** | 0.79 | | 2.09 | | 1.053061 | |
| **Jiangxi** | 1.11 | yes | 2.81 | | 1.416327 | |
| **Shandong** | 1.19 | yes | 3.14 | | 1.582143 | |
| **Henan** | 0.96 | | 2.94 | | 1.479796 | |
| **Hubei** | 0.69 | | 2.28 | | 1.147041 | |
| **Hunan** | 0.94 | | 2.87 | | 1.444592 | |
| **Guangdong** | 0.68 | | 2.06 | | 1.036939 | |
| **Guangxi** | 0.68 | | 1.79 | | 0.901939 | |
| **Hainan** | 0.4 | | 0.98 | | 0.494082 | |
| **Chongqing** | 0.86 | | 2.44 | | 1.228776 | |
| **Sichuan** | 1 | yes | 3.25 | | 1.635204 | yes |
| **Guizhou** | 0.66 | | 1.26 | | 0.636735 | |
| **Yunnan** | 0.3 | | 0.78 | | 0.393061 | |
| **Tibet** | 0.65 | | 1.07 | | 0.541633 | |
| **Shanxi** | 0.45 | | 1.21 | | 0.609592 | |
| **Gansu** | 0.43 | | 1.34 | | 0.674388 | |
| **Qinghai** | 0.41 | | 1.17 | | 0.589184 | |
| **Ningxia** | 1.49 | yes | 1.6 | | 0.815204 | |
| **Xinjiang** | 0.43 | | 1.09 | | 0.549388 | |

Have been observe in Table 3, there are four DMUs, $DMU_{14}$, $DMU_{15}$, $DMU_{23}$ and $DMU_{30}$ attain the MPSS from the angle of super-efficient optimistic point of view and $DMU_{23}$has the best super efficiencies from the pessimistic viewpoint that called the super-efficient pessimistic and $DMU_{23}$ represent the MPSS under the double frontiers suggestion criteria, therefore it is a super- efficient.

## 5. Conclusions

In this paper, it has been estimated the most productive scale size (MPSS) by using the pessimistic, optimistic DEA models and a double frontiers approach which will be considered Hurwicz' criteria. Also it has been proved that the decision making unit (DMU) that obtain the maximum optimal value under the pessimistic DEA model represents the MPSS. And from the optimistic viewpoint, that the DMUs have efficiency equals unity, has been known as a MPSS and by using a double frontiers measurement with the Hurwicz' criteria, the DMUs is chosen as a MPSS. In the suggestion super-efficiency in double frontiers Model by considering the Anderson and Peterson Model the same result has been obtained.

**References:**

[1] Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429–444.

[2] Banker, R.D., Charnes, A., Cooper, W.W., 1984. Some models for estimating technical and scale efficiencies in data envelopment analysis. Management Science 30, 1078–1092 Banker, R.D., $1984a$.

[3] Banker, R.D., Cooper, W.W., Thrall, R.M., Seiford, L.M., Zhu, J., 2004. Returns to scale in different DEA models. European Journal of Operational Research 154, 345–362.

[4] Wang, Y.M., Chin, K.S., 2007. Discriminating DEA efficient candidates by considering their least relative total scores. Journal of Computational and Applied Mathematics 206, 209–215.

[5] Wang, Y.M., Chin, K.S., 2009. A new approach for the selection of advanced manufacturing technologies: DEA with double frontiers. International Journal of Production Research 47, 6663–6679.

[6] Wang, Y.M., Lan, Y.X., 2011. Measuring Malmquist productivity index: a new approach based on double frontiers data envelopment analysis. Mathematical and Computer Modelling 54, 2760–2771.

[7] Wang, Y.M., Lan, Y.X., 2013. Estimating most productive scale size with frontiers data envelopment analysis. *Economic Modelling*, 33, 182–186.

[8] Andersen, P., & Petersen N. C. (1993). A Procedure for Ranking Efficient Units in Data Envelopment Analysis. Management Science, 39(10), 1261-1264.

[9] Estimating most productive scale size using Data envelopment analysis. *European Journal of Operational Research*, 17, 35–44.

[10] Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429–444.

[11] Jahanshahloo, G.R., Hosseinzadehlotfi, F., Nikomaram, H., Data Envelopment Analysis and its Applications.