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Research Article

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The Overall Efficiency and Projection Point in Network DEA

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Abstract
Data Envelopment Analysis (DEA) is one of the best methods for measuring the efficiency and productivity of Decision Making Units (DMU). Evaluating the efficiency of DMUs which have two or several stages by using the conventional DEA models, is equal to consider them as black box. This method, omits the effect of intermediate measure on efficiency. Therefore, just the first network inputs and the last network outputs will be effected on network efficiency. After solving each DEA model, in order to improving the DMUs efficiency, introduce one efficient market to evaluated DMU is so important. Conventional DEA network models cannot ensure that the market of each evaluated DMU is efficient. In this paper, by considering DMUs that have two stages network process, in order to find projection frontier, point some models have been proposed. These models after finding the best intermediate product corresponded to each network, found the overall network efficiency and introduce one projection point for each DMU.

Keywords: Data envelopment analysis (DEA), Two-stage network, Efficiency, projection point.

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric method for the evaluation of the efficiency of a set of homogeneous decision making units (DMUs) with multiple inputs and multiple outputs. The CCR (Charnes et al. [1]) and BCC (Banker et al. [2]) models are two basic radial models under constant returns to scale (CRS) and variable returns to scale (VRS). Respectively, there has been a great deal of researches on DEA models [3,4,5]. In conventional DEA, the decision making units are treated as a black-box. In other words, DEA just considers initial inputs and final outputs for each DMU with no consideration of the internal operations, whereas, in the real world, by opening the black-box, many production units may consist of some units referred to as stage. Indeed, the challenging question is how to measure the efficiency for such multi-stage units [6,7]. In our study we focus on a two-stage structure, in which all outputs are given in stage one and the intermediate measures play the roles of the inputs for the stage two.

Kao and Hwang [8] showed that the two stages are related and defined the overall efficiency, as the geometric mean of the stages efficiency. But their approach only works under CRS. Chen et al. [7] defined the overall efficiency, as an arithmetic mean of the stages efficiency and present some models, named additive models that are able to measure the efficiency under both CRS and VRS technology. They also modified the models presented in Kao and Hwang [8] to achieve the frontier projection for a two-stage DMU. Chen et al. [9,10] also study the overall efficiency under both CRS and VRS. Chen [11] divided DEA models in two groups. The first group contains the models that yield from the standard multiplier DEA models upon the DEA ratio efficiency, and the second group contains the models that yield based upon production possibility set (PPS), named envelopment models. Chen claimed that these two kinds of models are not equivalent in the case of two- stage network under VRS.

The rest of this paper is structured as follows: In section 2 we review the models presented by Kao and Hwang [8] and Chen [9], for measuring the overall efficient value in a two-stage system. Section 3 production possibility set in network is introduced. In section 4 a method to introduce a projection point for each DMU in a two-stage system is introduced. Finally, Section 5 contains some conclusions.

2. Two stage network DEA efficiency

In this section we discuss the efficiency in network DEA. Consider a two-stage system consisting of n DMUs, each of which has two stages in its internal structure. Suppose DMU_j $(j=1,\dots,n)$ consumes the input vector $X_j=(x_{1j},\dots,x_{mj})$ in stage 1. The intermediate measure $Z_j=(z_{1j},\dots,z_{dj})$ is the output of the first stage that will be the input of the second stage without any alteration to produce the output vector $Y_j=(y_{1j},\dots,y_{sj})$

(see Fig 1).

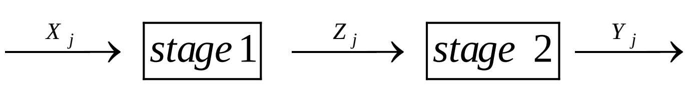

Fig 1. Two-stage process

Suppose $\theta_k^{\ 1}, \theta_k^{\ 2}$ are respectively the efficiency score of the first and second stage and θ_k is the k-th decision making unit overall efficiency score. Kao and Hwang [8] presented Models (1) for measuring the efficiency of DMU_k under CRS technology in the following format:

$$\begin{aligned} \theta_k^* &= Max\, \theta_1^k \times \theta_2^k \\ s.t \quad & \theta_1^k \le 1 \qquad (1) \\ & \theta_2^k \le 1 \\ & \theta^k \le 1 \end{aligned}$$

The fractional model corresponding Model (1) can alternatively be written as Model (2) applying the Charns-Cooper transformation:

$$\begin{aligned} \theta_k^* &= Min\,\theta \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta\, x_{ik} \quad i = 1,\dots,m \\ s.t \quad & \sum_{j=1}^{n} (\lambda_j - \delta_j) z_{dj} \ge 0 \quad d = 1,\dots,t \qquad (2) \\ & \sum_{j=1}^{n} \delta_j y_{rj} \ge y_{rk} \qquad r = 1,\dots,s \\ \lambda_j &\ge 0 \quad \varsigma_j \ge 0 \quad j = 1,\dots,n \end{aligned}$$

In fact, Model (2) (CCR network model) yields the overall efficiency corresponding to the k-th DMU under CRS technology.

3. Production possibility set in network

In this section we intruded the production possibility set in network DEA. Following the conventional DEA approach at the individual process level, we can define the pps of first and second stage as follow;

$$pps_I = \left\{ \begin{pmatrix} X \\ Z \end{pmatrix} \middle| \sum_{j=1}^{n} \lambda_j X_j \le X, \sum_{j=1}^{n} \lambda_j Z_j \ge Z, \lambda_j \ge 0, j = 1,2,\dots,n \right\}$$

$$pps_{II} = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \middle| \sum_{j=1}^{n} \delta_j Z_j \le Z, \sum_{j=1}^{n} \delta_j Y_j \ge Y, \delta_J \ge 0, j = 1,2,\dots,n \right\}$$

We can mixed pps_I and pps_{II} as follow;

$$pps^* = \left\{ \begin{pmatrix} X \\ Z \\ Y \end{pmatrix} \middle| \begin{array}{l} \sum_{j=1}^{n} \lambda_j X_j \le X, \sum_{j=1}^{n} \lambda_j Z_j \ge Z, \ \sum_{j=1}^{n} \delta_j Z_j \le Z, \\ \sum_{j=1}^{n} \delta_j Y_j \ge Y, \delta_J \ge 0, \lambda_j \ge 0, j = 1,2,\dots,n \end{array} \right\}$$

Now with combining the constrains we can rewrite pps* as follow

$$pps = \left\{ \begin{pmatrix} X \\ Z \\ Y \end{pmatrix} \middle| \begin{array}{l} \sum_{j=1}^{n} \lambda_j X_j \le X, \sum_{j=1}^{n} (\lambda_j - \delta_j) Z_j \ge Z - Z = 0, \\ \sum_{j=1}^{n} \delta_j Y_j \ge Y, \delta_J \ge 0, \lambda_j \ge 0, j = 1,2,\dots,n \end{array} \right\}$$

It's clear that two sets listed above as pps and pps* are not equivalent and pps* is a subset of pps, so the new frontier is higher than the actual frontier.

Kao and Hwang [8] used above pps for measuring the efficiency of DMUs. In this paper we will use it as production possibility set too.

Kao Hwang [8]. used model 2 for calculating the overall efficiency of DMU_k.

$$\begin{aligned} \theta_{1k}^* &= Min\,\theta \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta\, x_{ik} \quad i = 1,\dots,m \\ s.t \quad & \sum_{j=1}^{n} (\lambda_j - \delta_j) z_{dj} \ge 0 \quad d = 1,\dots,t \qquad (2) \\ & \sum_{j=1}^{n} \delta_j y_{rj} \ge y_{rk} \qquad r = 1,\dots,s \\ \lambda_j &\ge 0 \quad \delta_j \ge 0 \quad j = 1,\dots,n \end{aligned}$$

Model 2 find the smallest amount of first stage inputs for network k, which can product output Y_k, but it can't find the optimal intermediate measure that is corresponded to this network. For evaluating the efficiency of DMU_k model

2 uses the difference of intermediate measure as follow:

$$Min\,\theta$$
$$s.t\begin{pmatrix}\theta x_k \\ z - z = 0 \\ y_k\end{pmatrix} \in pps$$

The first restriction in model 2 is balance restriction. Kao and Hwang [8] used :

$$\sum_{j=1}^{n}(\lambda_j - \delta_j)\,z_{dj} \geq 0 \quad d = 1,\ldots,t$$

as a balance restriction, that guarantee the amount product is at least equal to the amount consumed.in this paper we will change the balance restriction in order to find the optimal intermediate for each network. Also we will introduce one approach to find the efficient bench mark corresponded to each DMU.

4. Efficient Benchmark

In this section we intruded an efficient bench mark corresponded to each DMU. The technical efficiency of DMU_k can be computed using the following radial input oriented DEA model

$$\begin{aligned} & Min\,\theta \\ & s.t\ (\theta x_k, z_k) \in pps_I \\ & \quad (\theta z_k, y_k) \in pps_{II} \end{aligned} \tag{3}$$

Model 3 is equivalent to following model

$$\begin{aligned} & \theta^*_{2k} = Min\,\theta \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \leq \theta x_{ik} \quad i = 1,\ldots,m \\ s.t\ & \sum_{j=1}^{n}\lambda_j z_{dj} \geq z_{dk} \quad d = 1,\ldots,t \\ & \sum_{j=1}^{n}\delta_j z_{dj} \leq \theta z_{dk} \quad d = 1,\ldots,t \\ & \sum_{j=1}^{n}\delta_j y_{rj} \geq y_{rk} \quad r = 1,\ldots,s \\ & \lambda_j \geq 0 \quad \delta_j \geq 0 \quad j = 1,\ldots,n \end{aligned} \tag{4}$$

Now with combining model 4constrains, we will have model 5 as follow:

$$\begin{aligned} & \theta^*_k = Min\,\theta \\ & \sum_{j=1}^{n}(\lambda_j - \delta_j)\,z_{dj} \geq (1-\theta)\,z_{dk} \quad d = 1,\ldots,t \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \leq \theta x_{ik} \quad i = 1,\ldots,m \\ & \sum_{j=1}^{n}\delta_j y_{rj} \geq y_{rk} \quad r = 1,\ldots,s \\ & \delta_j, \lambda_j \geq 0 \quad j = 1,\ldots,n \end{aligned} \tag{5}$$

Model 4 is feasible because $\theta = \lambda_k = \delta_k = 1, (\lambda_j = 0, \delta_j = 0 (j = 1,\ldots,n, j \neq k))$ is a feasible solution so $\theta^* \leq 1$ .For finding the overall efficiency corresponded to each network, at first we should find the optimal intermediate measure then according to these optimal intermediate we will find the smallest amount of first stage input that can product second stage output.

In model 4 the balance restriction shows that the input and output role for intermediate measure, is intended. Balance restriction in model 4 with maximizing $\theta \leq 1$ guarantee that input and output role for intermediate measure is considered. Let θ^* be the optimal value of model 4, so $\theta^* z_k$ will be the optimal intermediate measure for DMU_k. Model 4has dual of the following form:

$$\begin{aligned} & \theta^*_k = MAX\ \sum_{r=1}^{s}\mu_r\, y_{rk} + \sum_{d=1}^{t}\pi_d z_{dk} \\ & \sum_{d=1}^{t}\pi_d z_{dj} - \sum_{i=1}^{m}\omega_i\, x_{ij} \leq 0 \quad j = 1,\ldots,n \\ & \sum_{r=1}^{s}\mu_r\, y_{rj} - \sum_{d=1}^{t}\pi_d z_{dj} \leq 0 \quad j = 1,\ldots,n \\ & \sum_{i=1}^{m}\omega_i x_{ik} + \sum_{d=1}^{t}\pi_d z_{dk} = 1 \quad j = 1,\ldots,n \\ & \mu_r, \pi_d, \omega_i \geq 0 \quad r = 1,\ldots,s \text{ و } d = 1,\ldots,t \text{ و } i = 1,\ldots,m \end{aligned} \tag{6}$$

Theorem1: if θ^*_{2k} , θ^*_k be the optimal value of models 4 and 5 respectively then $\theta^*_{2k} \geq \theta^*_k$.

Proof: as mentioned before, the production possibility set (pps) corresponded to model 5 is smaller than production possibility set corresponded to model 4 so the optimal value of model 5 will be smaller than the optimal value of model 4. Therefore $\theta*_{2k} \geq \theta*_{k}$.

From theorem 1 can be concluded, it's possible for model 2 to report all DMUs as inefficient. It means we won't have efficient DMU. this problem is shown in table 1, in this example the efficiency of 4 DMUs with 2 input for first stage, 2 intermediate measure and two output are calculate with model 3.

Table1:Result of efficiency

DMU	x_1	x_2	z_1	z_2	y_1	y_2	$overall$
1	2	5	3	5	3	2	0.59375
2	1	3	0	1	2	7	0.43750
3	5	4	1	0	2	5	0.29412
4	5	7	2	2	3	2	0.36184

After solving each DEA model, in order to improving the DMUs efficiency, introduce one efficient market to evaluated DMU is so important. Conventional DEA network models cannot ensure that the market of each evaluated DMU is efficient. For example the bench mark of DMU_k with model 2 is $\begin{pmatrix} \theta_1^* x_k \\ z_k \\ y_k \end{pmatrix}$ and with model 5 is $\begin{pmatrix} \theta^* x_k \\ \theta^* z_k \\ y_k \end{pmatrix}$. It's clear that, $\begin{pmatrix} \theta^* x_k \\ \theta^* z_k \\ y_k \end{pmatrix}$ dominate DMU_k, but it can be inefficient. Model 5 found the optimal amount for intermediate measure. As it clear from model 5, this model found the optimal amount of first stage inputs that can product z_k. For finding the efficient bench mark for DMU_k we should found the minimum $\theta^* x_k$ that can product $\theta^* z_k$. In other word, mode5 report the optimal intermediate measure, now with fixing this optimal value for intermediate measure by solving following model we can find the optimal firs stage input.

$$\hat{\theta} = \min \theta$$

$$\begin{pmatrix} \theta\theta^* x_k \\ o \\ y_k \end{pmatrix} \in pps \qquad (7)$$

Let $\hat{\theta}$ be the optimal solution for model 7, the overall efficiency and the bench mark for DMU_k will be $\theta_k^{overall} = \theta * \hat{\theta}$ and $\begin{pmatrix} \hat{\theta}\theta^* x_k \\ \theta^* z_k \\ y_k \end{pmatrix}$ respectively. Next theorem guarantee that this bench mark is efficient.

Theorem 2: Network, $\begin{pmatrix} \hat{\theta}\theta^* x_k \\ \theta^* z_k \\ y_k \end{pmatrix}$ is an efficient network.

Proof: let $\bar{\theta}_1$ and $\bar{\theta}_2$ be the optimal solution for evaluating the efficiency of $\begin{pmatrix} \hat{\theta}\theta^* x_k \\ \theta^* z_k \\ y_k \end{pmatrix}$, with models 5 and 7 respectively. So the overall efficiency for this network will be $\bar{\theta}^{overall} = \bar{\theta}_1 \bar{\theta}_2$.

Now we should proof that, $\bar{\theta}^{overall} = 1$ it means that $\bar{\theta}_1 = \bar{\theta}_2 = 1$ there are 2 cases to consider:

Case1: let $\bar{\theta}_1 = 1$. In these case if $\bar{\theta}_2 = 1$ the proof is completed. Otherwise, $\bar{\theta}_2 \prec 1$. Let $(\bar{\theta}_2, \bar{\lambda}^2, \bar{\delta}^2)$ be the optimal solution for model 7 for evaluating $D\widehat{M}U$. by replace this solution in model 6 restrictions we will have (let $\bar{\theta}_2 \hat{\theta} = \tilde{\theta}$)

$$\sum_{j=1}^{n} \bar{\lambda}_j^2 x_{ij} \le \bar{\theta}_2 \, \bar{\theta}_1 (\hat{\theta}\theta * x_k) = \bar{\theta}_2 (\hat{\theta}\theta * x_k) = \tilde{\theta}\theta * x_k \quad (a)$$

So

$$\sum_{j=1}^{n} \bar{\lambda}_j^2 x_{ij} \le \tilde{\theta}\theta * x_k \quad (a)$$

And

$$\sum_{j=1}^{n} (\bar{\lambda}_j^2 - \bar{\delta}_j^2) z_j \ge 0 \qquad (b)$$

$$\sum_{j=1}^{n} \bar{\delta}_j^2 y_j \ge y_k \qquad (c)$$

Restrictions a, b and c show that $(\tilde{\theta}_2, \bar{\lambda}^2, \bar{\delta}^2)$ is a feasible solution for model 7. Another hand, $\hat{\theta}$ is an optimal value for model 7. According to assumption of $\bar{\theta}_2 \prec 1$, we can conclude that $\tilde{\theta} = \hat{\theta} \, \bar{\theta}_2 \prec \hat{\theta}$ that it's in contrast with the optimality of $\hat{\theta}$.

Second case: in this case suppose that $\bar{\theta}_1 \prec 1$ and $\bar{\theta}_2 \le 1$. (It means $\bar{\theta}_1 \prec 1$ and $\bar{\theta}_2 \prec 1$ or $\bar{\theta}_2 = 1$)

Let $(\bar{\theta}_1, \bar{\lambda}^1, \bar{\delta}^1)$ be the optimal solution of model 2 for evaluating $D\bar{M}U$, by replace this solution in model 2 restriction we will have: (let $\tilde{\theta} = \bar{\theta}_1 \hat{\theta}$)

$$\sum_{j=1}^{n} \bar{\lambda}_j^1 x_{ij} \le \bar{\theta}(\hat{\theta}\theta * x_k) = \tilde{\theta}\theta * x_k \qquad (d)$$

$$\sum_{j=1}^{n} (\bar{\lambda}_j^1 - \bar{\delta}_j^1) z_j \ge (1 - \bar{\theta}_1)(\theta * z_{dk}) = \theta * z_{dk} - \bar{\theta}_1 \, \theta * z_{dk} \qquad (e)$$

According to assumption of $\bar{\theta}_1 \prec 1$, we can conclude that $\bar{\theta}_1 \theta * z_k \le \theta * z_k$ .so constraint d result:

$$\sum_{j=1}^{n} (\bar{\lambda}_j^1 - \bar{\delta}_j^1) z_j \ge \theta * z_d - \theta * z_k = 0 \qquad (f)$$

And

$$\sum_{j=1}^{n} \bar{\delta}_j^1 y_j \ge y_k \qquad (g)$$

Restrictions e, f and g indicating that $(\tilde{\theta}_1, \bar{\lambda}^1, \bar{\delta}^1)$ satisfy all the constraints in model 6 so it's a feasible solution for model 7. In another hand $\hat{\theta}$ is an optimal value for model 7. According to assumption of $\bar{\theta}_1 \prec 1$, we can conclude that $\tilde{\theta} = \hat{\theta} \, \bar{\theta}_1 \prec \hat{\theta}$, that it's in contrast with the optimality of $\hat{\theta}$.

Therefore $\bar{\theta}_1 = \bar{\theta}_2 = 1$ must be true.

5. Conclusion

Traditional studies in DEA view each system as a black-box and ignoring the performance of components in calculating the relative efficiency corresponding to each DMU.

In order to taking in to account the relationship between the process efficiency and overall efficiency,

Kao and Hwang [8] defined the overall efficiency of a two-stage DMUs as a geometric mean of stages, under constant returns to scale technology. Chen et al. [9] defined the overall efficiency as a convex combination of two stage (arithmetic mean), this definition is able to calculate the overall efficiency of decision making units under both technologies (VRS and CRS).

After solving each DEA model, in order to improving the DMUs efficiency, introduce one efficient market to evaluated DMU is so important. Conventional DEA network models cannot ensure that the market of each evaluated DMU is efficient. This paper introduces a method to calculate bench mark corresponded to each DMU.

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