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Some fixed points preserver

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Abstract

Let $\mathrm{B}(\mathrm{X})$ and $\mathrm{M}_n(\mathrm{F})$ be the algebra of all bounded linear operators on a complex Banach space X with $\dim \mathrm{X} \geq 3$ and the algebra of all $n \times n$ matrices over a field F with $\mathrm{char}\mathrm{F} \neq 2$, respectively. Also let $F(A)$ be the space of all fixed points of an operator $A \in \mathrm{B}(\mathrm{X})$. In this paper, we characterize the forms of linear maps $\phi : \mathrm{B}(\mathrm{X}) \to \mathrm{B}(\mathrm{X})$ which satisfy $F(A) = 0 \Leftrightarrow F(\phi(A)) = 0$ and linear maps $\phi : \mathrm{M}_n(\mathrm{F}) \to \mathrm{M}_n(\mathrm{F})$ which preserve the fixed points of matrices.

Keywords: Preserver problem, Operator algebra, Fixed point.

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1. Introduction

Let $\mathsf{B}(\mathsf{X})$ denote the algebra of all bounded linear operators on a Banach space X. Recall that $x \in \mathsf{X}$ is a fixed point of an operator $A \in \mathsf{B}(\mathsf{X})$, whenever we have $Ax = x$. For $A \in \mathsf{B}(\mathsf{X})$, denote by $\mathrm{Lat}A$ and $F(A)$ the lattice of A, that is, the set of all invariant subspaces of A and the set of all fixed points of A, respectively. Jafarian and Sourour [2] characterized the linear maps on $\mathsf{B}(\mathsf{X})$ preserving the lattice of operators. Later on, authors in [1] characterized the maps on $\mathsf{B}(\mathsf{X})$ preserving the lattice of sum and several products of operators.

It is clear that for an linear operator A, $F(A)$ is a vector space and also is one of the elements of $\mathrm{Lat}A$. Denote by $\dim F(A)$, the dimension of $F(A)$. Authors in [5] and [6] characterized the maps on $\mathsf{B}(\mathsf{X})$ preserving the dimension of fixed points of product and the dimension of fixed points of sum of operators, respectively.

Let $\mathsf{M}_n(\mathsf{F})$ be the algebra of all $n \times n$ matrices over a field F with $\mathrm{char}\mathsf{F} \neq 2$ and $n \geq 3$. In this paper, we characterize the forms of linear maps $\phi : \mathsf{B}(\mathsf{X}) \to \mathsf{B}(\mathsf{X})$ which satisfy

$$F(A) = 0 \Leftrightarrow F(\phi(A)) = 0 \qquad (A \in \mathsf{B}(\mathsf{X}))$$

and linear maps $\phi : \mathsf{M}_n(\mathsf{F}) \to \mathsf{M}_n(\mathsf{F})$ which satisfy

$$F(A) \subseteq F(\phi(A)) \qquad (A \in \mathsf{M}_n(\mathsf{F})).$$

2. Zero fixed points preservers

Proposition 2.1. Let $A \in \mathsf{B}(\mathsf{X})$. If for every $T \in \mathsf{B}(\mathsf{X})$, we have

$$F(T) = 0 \Longrightarrow F(A+T) = 0,$$

then $A = 0$.

Proof. If $A \neq 0$, then there exists a nonzero vector $x \in \mathsf{X}$ such that $Ax \neq 0$. Thus there exists a linear functional f such that $f(x) = 1$ and $f(Ax) \neq 0$. Set $T = (x - Ax) \otimes f$. We have $F(T) = 0$ but $F(A+T) \neq 0$, because $f(x - Ax) \neq 1$ and $(A+T)x = x$. The proof is complete.

Lemma 2.2. Let X be an infinite dimensional complex Banach space. Suppose $\phi : \mathsf{B}(\mathsf{X}) \to \mathsf{B}(\mathsf{X})$ is a surjective linear map which satisfies the following condition:

$$F(A) = 0 \Leftrightarrow F(\phi(A)) = 0 \qquad (A \in \mathsf{B}(\mathsf{X})).$$

Then ϕ is unital.

Proof. Let $\phi(A) = I$ and $T \in \mathsf{B}(\mathsf{X})$ such that $F(T) = 0$. It is easy to see that $F(-T + 2I) = 0$. So we have

$$F(\phi(-T+2I))=0 \Rightarrow F(-\phi(-T+2I)+2I)=0$$
$$\Rightarrow F(\phi(T-2I+2A))=0$$
$$\Rightarrow F(T-2I+2A)=0,$$

which by Proposition 2.1 implies that $-2I+2A=0$ and hence $A=I$.

Theorem 2.3. [4] Let X be an infinite dimensional complex Banach space and let $\phi:\mathsf{B}(\mathsf{X})\to\mathsf{B}(\mathsf{X})$ be a surjective linear mapping. Then ϕ is an automorphism of the algebra $\mathsf{B}(\mathsf{X})$ if and only if ϕ preserves injective operators in both directions and satisfies $\phi(I)=I$.

Theorem 2.4. Let X be an infinite dimensional complex Banach space. Suppose $\phi:\mathsf{B}(\mathsf{X})\to\mathsf{B}(\mathsf{X})$ is a surjective linear map which satisfies the following condition:

$$F(A)=0 \Leftrightarrow F(\phi(A))=0 \qquad (A\in\mathsf{B}(\mathsf{X})).$$

Then ϕ is an automorphism of the algebra $\mathsf{B}(\mathsf{X})$.

Proof. It is easy to check that $\ker A = F(I+A)$. This together with Lemma 2.2 and the preserving property of ϕ implies that ϕ preserves the injectivity of operators in both directions and so assertion follows from Theorem 2.3.

3. Fixed points preservers

Proposition 3.1. Let $A\in\mathsf{M}_n(\mathsf{F})$. A is an idempotent matrix if and only if

$$F(A)+F(I-A)=\mathsf{F}^n.$$

Proof. From

$$F(I-A)=\ker A,$$
$$A\in\mathsf{I}_n(\mathsf{F}) \Leftrightarrow F(A)=\operatorname{Im} A$$

and

$$A\in\mathsf{I}_n(\mathsf{F}) \Rightarrow \operatorname{Im} A+\ker A=\mathsf{F}^n, \qquad (3.1)$$

we can conclude the assertion.

Let $\mathsf{M}_n(\mathsf{R})$ be the algebra of all $n\times n$ matrices over a unital commutative ring R with 2 invertible.

Theorem 3.2. [7] Suppose that ϕ is an invertible linear transformation on $\mathsf{M}_n(\mathsf{R})$ fixing the identity. Then the following two announcements are equivalent.

(i) ϕ preserves idempotence.

(ii) ϕ is a Jordan automorphism.

Theorem 3.3. Suppose $\phi:\mathsf{M}_n(\mathsf{F})\to\mathsf{M}_n(\mathsf{F})$ is a linear map which satisfies the following condition:

$$F(A)\subseteq F(\phi(A)) \qquad (A\in\mathsf{M}_n(\mathsf{F})).$$

Then ϕ is a Jordan automorphism.

Proof. From $\mathsf{F}^n=F(I)\subseteq F(\phi(I))$ we obtain $\phi(I)=I$. For any arbitrary matrix

A we have

$$F(A) \subseteq \operatorname{Im} A. \qquad (3.2)$$

(3.2)

Let $A \in \mathsf{M}_n(\mathsf{F})$ be an idempotent matrix. By Proposition 3.1 we have $F(A) + F(I - A) = \mathsf{F}^n$. This together with assumption yields

$$\mathsf{F}^n \subseteq F(\phi(A)) + F(I - \phi(A))$$

$$\subseteq \operatorname{Im} \phi(A) + \ker \phi(A) = \mathsf{F}^n$$

and hence $F(\phi(A)) + F(I - \phi(A)) = \mathsf{F}^n$. Again using Proposition 3.1 yields that $\phi(A)$ is idempotent. Therefore ϕ is unital and preserves idempotent matrices and so assertion can be followed by Theorem 3.2.

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