

# Solution of Inverse Kinematic Problem of a 2DOF Robot Using Decomposition Method

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Received Spring 2015, Accepted Summer 2015

## Abstract

The inverse kinematics problem of a two degree of freedom (2DOF) planar robot arms is solved using Adomian's decomposition method (ADM), after converting to a system of two nonlinear algebraic equations. The advantage of the method is that it gives the solutions as functions of the desired position of the end effector and the length of the arms. The accuracy of the solutions can be increased up to desired order. The solutions haven't any singularity. The problem must be solved once for any structure and the results can be used for any path and finally, the method is fast and simple to understand.

**Keywords:** *Inverse Kinematics, ADM, Robots arm, 2DOF.*

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## 1. Introduction

A robot is a mechanical intelligent motion agent which can perform tasks on its own, or with external guidance. In practice a robot is usually an electro-mechanical machine which guides by computer and electronic programming. A manipulator is an arm like mechanism on a robot that consists of a series of segments, usually sliding or rotating which grasp and move objects with a number of degrees of freedom, under automatic or manual control by an end-effector connected to it. The kinematics is a fundamental part of the multidisciplinary research area of robotics. The position kinematic model relates the joints positions and the end-effector posture. The direct kinematics analysis is the process of calculating the end-effector posture from the joint positions given, while the inverse kinematics analysis is the process of obtaining the joint positions necessary to establish a desired end-effector posture [11]. The method commonly used for direct kinematic modeling, is based on the Denavit–Hartenberg convention that is a consistent and concise description of the kinematic relations between the links of a kinematic chain introduced at 1955 [6]. The inverse kinematics is an ill posed problem because of the problems such as

multiple or even infinite solutions, nonlinearity, singularities and uncertainties. Scientists and engineers use artificial neural network (ANN) to solve the inverse kinematics problems [7, 8, 9]. In this approach, a network will be trained to learn a desired set of joint angles positions from a given set of end-effector positions.

The ADM was first introduced by George Adomian at the beginning of the 1980's for solving a wide range of problems whose mathematical models yield equation or system of equations involving algebraic, differential, integral and integro-differential terms [2, 3, 5]. This method is simple and efficient for the solution of algebraic or system of algebraic equations that gives accurate approximate solutions. Abbaui and Cherruault [1] applied ADM to solve nonlinear algebraic equation  $f(x)=0$  and proved the convergence of the series solution. E. Babolian et al. applied the ADM to solve a system of nonlinear equations [4]. A. R. Vahidi, et.al. used restarted ADM to improve the solutions of systems of nonlinear algebraic equations [12].

In this paper the inverse kinematic problem of a 2DOF robot arm is solved using the well-known ADM after converting to a system of nonlinear algebraic equations. The problem is

introduced in section 2. Section 3 belongs to describing the method and its application to the problem. Section 4 gives the application of results in controlling the end effector to move on three different lines in a planar surface.

## 2. The problem

The structure of a planar 2DOF manipulator is given in Figure 1.

In the structure given in figure 1,  $l_1$  and  $l_2$  are the constant lengths of the arms and the variables  $\theta_1$  and  $\theta_2$  are joint angles measured from positive horizontal axes. The arm lengths considered to be equal to reduce the structural limitations. Direct kinematics formulation of this structure is the position of end effector,  $M(p, q)$ , as functions of  $\theta_1$  and  $\theta_2$ . This obtains easily by the simple geometrical relations

$$p = l\cos\theta_1 + l\cos\theta_2, \quad (1)$$

$$q = l\sin\theta_1 + l\sin\theta_2. \quad (2)$$

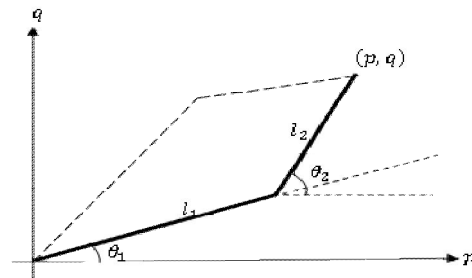


Figure 1. The structure of the planar 2DOF manipulator.

The inverse kinematic problem is to find the values of  $\theta_1$  and  $\theta_2$  for a known position  $M(p, q)$  of end effector. Defining  $x = \cos\theta_1$  and  $y = \sin\theta_2$ , (1) and (2) converts to the algebraic equations

$$p = lx + l\sqrt{1 - y^2} \quad (3)$$

$$q = l\sqrt{1 - x^2} + ly, \quad (4)$$

or

$$\frac{p}{l} - x = \sqrt{1 - y^2} \quad (5)$$

$$\frac{q}{l} - y = \sqrt{1 - x^2} \quad (6)$$

After squaring and some simple calculations (5) and (6) can be rewritten in the form

$$x = \frac{p^2 - l^2}{2pl} + \frac{l}{2p}(x^2 + y^2), \quad (7)$$

$$y = \frac{q^2 - l^2}{2ql} + \frac{l}{2q}(x^2 + y^2). \quad (8)$$

In the section 3, the system of equations (7) and (8) will be solved using the ADM.

### 3. The method and solution

In this section, at first the ADM will be described for the general form of a system of nonlinear algebraic equations then it will be used to solve the underlying inverse kinematic problem.

#### 3.1 The ADM for a system of nonlinear equations

Consider the following system of nonlinear equations

$$f_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, n, \quad (9)$$

where  $f_i: R^n \rightarrow R$ . Equation (9) can be written in the canonical form

$$x_i = c_i(x_1, \dots, x_n) + N_i(x_1, \dots, x_n), \quad i = 1, 2, \dots, n, \quad (10)$$

where  $c_i$ 's are constants and  $N_i$ 's are nonlinear functions of their arguments generally. The standard ADM [9] uses the solution  $x_i$  in terms of the series

$$x_i = \sum_{j=0}^{\infty} x_{i,j} \quad i = 1, 2, \dots, n. \quad (11)$$

The nonlinear functions,  $N_i$ 's in (10) are expressed in terms of an infinite series called Adomian polynomials<sup>12</sup> as

$$N_i(x_1, \dots, x_n) = \sum_{j=0}^{\infty} A_{i,j} \quad i = 1, 2, \dots, n \quad (12)$$

$A_{i,j}$ 's given in (12) depend upon

$$x_{1,0}, x_{1,1}, \dots, x_{1,j}, x_{2,1}, \dots, x_{2,j}, \dots, x_{n,1}, \dots, x_{n,j}.$$

In view of the equation (11) and (12),

$$N_i(\sum_{j=0}^{\infty} x_{1,j} \lambda^j, \dots, \sum_{j=0}^{\infty} x_{n,j} \lambda^j) = \sum_{j=0}^{\infty} A_{i,j} \lambda^j, \quad i = 1, 2, \dots, n, \quad (13)$$

from which we obtain

$$A_{i,j} = \frac{1}{j!} \left[ \frac{d^j}{d\lambda^j} N_i \left( \sum_{j=0}^{\infty} x_{1,j} \lambda^j, \dots, \sum_{j=0}^{\infty} x_{n,j} \lambda^j \right) \right]_{\lambda=0}, \quad i = 1, 2, \dots, n. \quad (14)$$

Where  $\lambda$  is the parameter introduced for convenience. Hence the equation (10) can be written as

$$\sum_{j=0}^{\infty} x_{i,j} \lambda^j = c_i + \sum_{j=0}^{\infty} A_{i,j} \lambda^j, \quad i = 1, 2, \dots, n. \quad (15)$$

The ADM defines the components  $x_{i,j}$ ,  $j \geq 0$ , by the following recursive relations

$$x_{i,0} = c_i, \quad (16)$$

$$x_{i,j+1} = A_{i,j}, \quad i = 1, 2, \dots, n, \quad j = 0, 1, 2, \dots \quad (17)$$

Finally the approximate solution  $x_i$  can be approximated by the truncated series

$$\varphi_{i,k} = \sum_{j=0}^k x_{i,j}, \quad (18)$$

that

$$\lim_{k \rightarrow \infty} \varphi_{i,k} = x_i, \quad i = 1, 2, \dots, n, \quad (19)$$

gives a converge solution which the convergence is proved<sup>13</sup>.

#### 3.2 Application of ADM to inverse kinematics problem

In order to solve the system of nonlinear algebraic equations(7) and (8) using ADM,

let  $x = \sum_{i=0}^{\infty} x_i$ ,  $y = \sum_{i=0}^{\infty} y_i$ ,  $x^2 + y^2 = \sum_{i=0}^{\infty} A_i$  to obtain

$$\sum_{i=0}^{\infty} x_i = \frac{p^2 - l^2}{2pl} - \sum_{i=0}^{\infty} A_i, \quad (20)$$

$$\sum_{i=0}^{\infty} y_i = \frac{q^2 - l^2}{2ql} - \sum_{i=0}^{\infty} A_i. \quad (21)$$

The equations (20) and (21) give the recursive relations

$$x_0 = \frac{p^2 - l^2}{2pl}, \quad (22)$$

$$y_0 = \frac{q^2 - l^2}{2ql}, \quad (23)$$

$$x_{i+1} = A_i, \quad (24)$$

$$y_{i+1} = A_i. \quad (25)$$

Here  $x, y$  are used rather than  $x_1, x_2$  adopted in section 2. Choosing  $k = 1$  in (18) the approximate solution  $x \approx \varphi_{1,1}$  and  $y \approx \varphi_{2,1}$  for  $l = 1$  will be obtained as

$$x \approx \varphi_{1,1} = \frac{-q^4 + p^6 q^2 (-1 + 12q^2) - p^2 q^2 (2 - 8q^2 + q^4) + p^4 (-1 + 8q^2 - 26q^4 + 4q^6)}{16p^5 q^4} \quad (26)$$

$$y \approx \varphi_{2,1} = \frac{-q^4 + p^6 q^2 (-1 + 4q^2) - p^2 q^2 (2 - 8q^2 + q^4) + p^4 (-1 + 8q^2 - 26q^4 + 12q^6)}{16p^4 q^5} \quad (27)$$

Using (26) and (27) in (3) and (4), the approximate solutions of the direct problem  $(p_c, q_c)$  obtains as follows.

$$p_c(p, q) \approx l\varphi_{1,1} + l\sqrt{1 - \varphi_{2,1}^2} \quad (28)$$

$$q_c(p, q) \approx l\sqrt{1 - \varphi_{1,1}^2} + l\varphi_{2,1} \quad (29)$$

Obviously, the point  $(p_c, q_c)$  includes the approximate horizontal and vertical coordinates of the end effector as functions of desired coordinates  $(p, q)$ . The procedure can be repeated to obtain more accurate approximations up to desired accuracy.

#### 4. Results and discussions

The end effector coordinates obtained as the functions of desired coordinates in the section 3 using the ADM. The results for two terms of the approximate solution (18) are illustrated as (28) and (29). Here we examine these functions to obtain some desired paths of the robot end effector given in the figure 1. Three paths are considered. At first we choose the horizontal path  $q = 1$  while  $p$  is in  $[0, 1.8]$ . Some points of this line, by the increment 0.1, are calculated using (28) and (29) and plotted in figure 2.

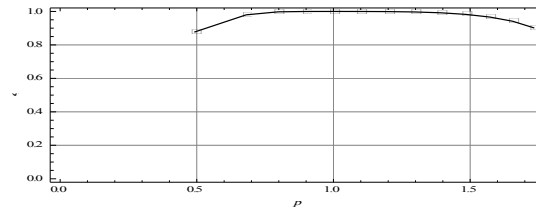


Figure 2. The calculated path of robot end effector for desired horizontal line by two terms of (18).

There are some important points in the figure 1 that must be considered.

- Due to the structure of robot arms, as shown in the figure 1, for the line  $q = 1$  there is an upper limit for the value of horizontal coordinate, i.e.  $p_{max} = \sqrt{3}$ .
- Due to the square root in the (28) and (29), there aren't real solutions for all points in the surface. We just show the points with real  $p_c$  and  $q_c$ .

As the second path, we choose the vertical line  $p = 1$ . The figure 3 shows some points of this line.

It can be seen in the figure 3 that as in horizontal line given in the figure 2, there is a lower limit due to the approximate calculations and an upper limit arising from the structure of the robot arm given in the figure 1.

As the last example we calculate the path of an inclined line with the slope of one, i.e.  $p = q$ . Some points of this line are given in figure 4.

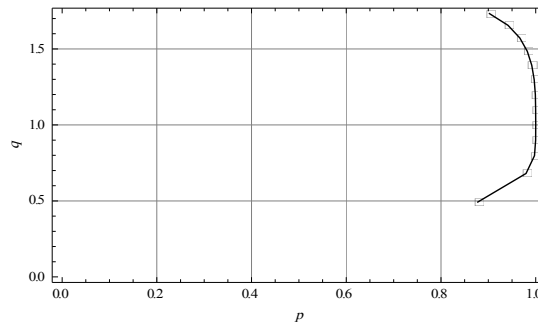


Figure 3. The calculated path of robot end effector for desired vertical line by two terms of (18).

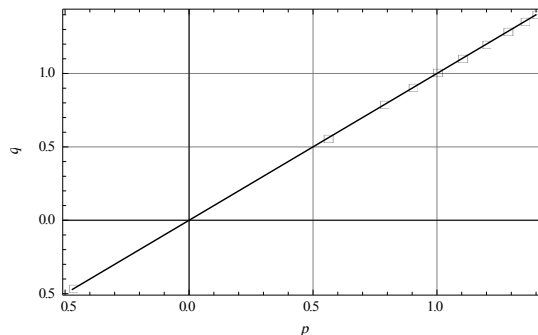


Figure 4. The calculated path of robot end effector for inclined line by two terms of (18).

The structure of the robot arm for the given line,  $p = q$ , imposes the upper limit of  $\sqrt{2}$  on both  $p$  and  $q$ . This limit can be seen in the figure 4.

More accurate solutions can be obtained by calculating more terms in ADM. The

figures 5-7 are plotted with the approximate solutions  $p_c$  and  $q_c$  considering  $k = 6$  in the series solution (18) for three predefined paths.

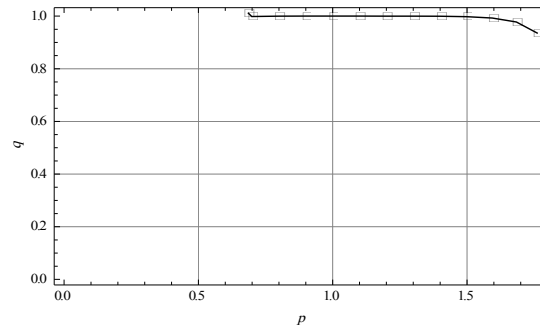


Figure 5. The calculated path of robot end effector for horizontal line by seven terms of (18).

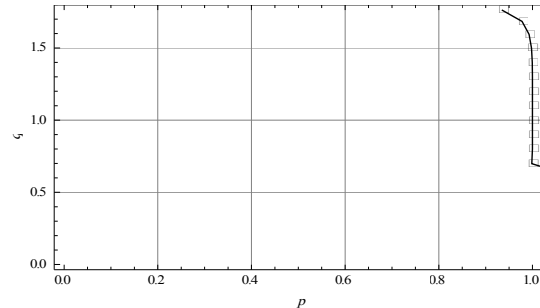


Figure 6. The calculated path of robot end effector for vertical line by seven terms of (18).

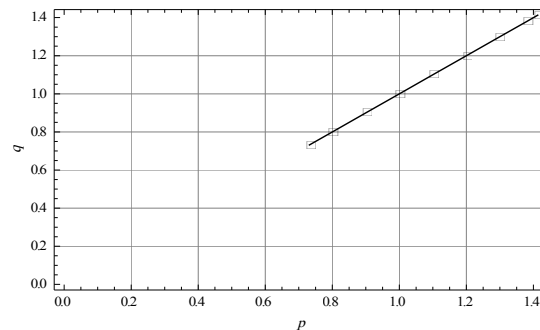


Figure 7. The calculated path of robot end effector for inclined line by seven terms of (18).

As expected, the lines ( $q = 1, p = 1$  &  $p = q$ ) will be obtained approximately. Comparing to the figures 2-4, it can be seen that increasing the number of calculated terms of (18), increases the accuracy of the results.

## 5. Conclusions

The studies show that the ADM can be used to solve the inverse kinematics problem of a planar 2DOF robot manipulator which can be used to control the robot end effector on desired line. Increasing the number of calculated terms of the approximate solutions obtained by the ADM, gives more accurate paths. The solutions exactly show the farthest possible position of the end effector limited by the lengths of the arms. The solutions have and lower limit for any desired line, due to the existence of square root in the direct kinematic equation. The study revealed some advantages of the ADM to the inverse kinematic problems. (a) The solutions can be obtained as functions of the desired position of the end effector and the length of the arms. (b) The accuracy of the solutions can be increased up to desired order. (c) The solutions haven't any singularity. (d) The problem must be solved once for any structure and the

results can be used for any path at any time. (e) The method is fast and simple to understand. We propose more studies in this field using the ADM, especially for more complicated structures with higher degrees of freedom.

## Acknowledgments

Authors would like to thank the Islamic Azad University, Yadegar-e Imam Khomeini (RAH) Shahr-e-Rey Branch for funding this work.



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