Journal of Theoretical and Applied Physics
a SpringerOpen Journal

RESEARCH Open Access

Wigner distribution function of superposed quantum states for a time-dependent oscillator-like Hamiltonian system

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Abstract

A phase space distribution function of quantum mechanics, so-called the Wigner distribution function (WDF), for superposed states of the general time-dependent oscillator-like Hamiltonian system is investigated. Superposition of not only two different coherent states but also two different squeezed states are considered respectively. Analytical representation of WDF for the superposition states is derived rigorously on the basis of fundamental relations. We confirmed the existence of nonclassical properties in the system from the appearance of interference term in the WDF.

Keywords: Wigner distribution function, Coherent state, Squeezed state, Time-dependent Hamiltonian

PACS: 03.65.-w; 03.65.Ca; 42.50.St

Background

The research of time-dependent oscillator-like Hamiltonian systems (TDOHSs) have received great concern from both quantum and classical points of view for several decades, thanks to their usefulness in describing the dynamics of various physical phenomena. When studying classical dynamics of a TDOHS, one usually consider the system which has time-dependent parameters in the Hamiltonian and assume that its time variation is sufficiently slow. The analytical solution of Schrödinger equation for a TDOHS can be obtained using Lewis-Riesenfeld invariants. According to invariant operator theory of Lewis-Riesenfeld [1], the quantum solutions of TDOHSs are described in terms of their classical solutions. Hence, it is possible to derive analytical solutions of Schrödinger equation for TDOHS so far as their classical solution is known. Among numerous reports relevant to exact or approximate analyses of classical motion of individual TDOHSs, see in particular references [2-5].

When investigating the properties of quantum states, a number of probability distributions associated with quantum mechanics are necessary. In 1932, Wigner discovered a useful distribution function, now known as the Wigner distribution function (WDF), in the context of quantum statistical mechanics [6]. The WDF is a powerful tool when describing quantum behaviors of wide range of dynamical systems with various states such as coherent and squeezed states, and a rather complex state superposed them [7-24]. In particular, the WDF has come to play an ever-increasing role in the description of coherent laser beams and of electromagnetic wave propagation in time-varying magnetoplasma. Although WDF takes negative values in some regions in phase space, it yields exact marginal distributions that cannot be attained in terms of the Husimi distribution function [25]. Once the WDF is known for a quantum system, most of the quantum mechanical properties of the system can be deduced from a straightforward evaluation with its use. Zurek used WDF in order to describe the decoherence problem and the transition from quantum to classical regime [9]. There are other broad range of applications of WDF and these include tomography of a trapped ion [10], electrodynamics in normal/complex media [11-14], atomic physics [15], signal processing in optics [16], and quantum information science [17].

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The WDF for superposed state is investigated in this work, basing on the quantum solutions derived from invariant operator theory. The general type of Hamiltonian for TDOHS is established, and its analytic quantum solution is addressed in section 'Quantum description of time-dependent oscillator-like Hamiltonian system'. In section 'Wigner distribution function: superposition of coherent states', we investigate the characteristics of superposition of two coherent states for the general TDOHS, via the corresponding WDF. Among several types of coherent states, we consider the Glauber coherent state which is basic and the prototype for most classes of coherent states [26]. The WDF for the superposition of two squeezed states is investigated in section 'Wigner distribution function: superposition of squeezed states'. Our developments for WDF are applied in a particular system in section 'Application to a particular system'. The summary and conclusion of this research are given in the last section.

Quantum description of time-dependent oscillator-like Hamiltonian system

Let us consider a general time-dependent Hamiltonian of the form

$$\hat{H}(\hat{q},\hat{p},t) = \frac{\hat{p}^2}{2f(t)} + g(t)(\hat{q}\hat{p}+\hat{p}\hat{q}) + \frac{1}{2}f(t)\omega^2(t)\hat{q}^2 + h_1(t)\hat{q} + h_2(t)\hat{p} + k(t), \tag{1}$$

where $f(t)$, $g(t)$, $h_1(t)$, $h_2(t)$, and $k(t)$ are arbitrary time functions, and $\omega(t)$ is a time-variable frequency. Notice that $f(t) \neq 0$. From appropriate choice of time functions, we can obtain a particular type of TDOHS. For example, if we take $f(t) = me^{\gamma t}$, $\omega(t) = \omega_0$, and all other time functions are zero, where m, γ, and ω_0 are real positive constants, the system becomes a harmonic oscillator with exponentially increasing mass (HOEIM). The classical equation of motion of the HOEIM is the same as that of the damped harmonic oscillator characterized by a damping constant γ. This implies that the Hamiltonians that give their classical equation of motion are identical to each other. Therefore, they have the same quantum structure yielding common Schrödinger solutions (wave functions) for the two systems. However, the mathematical representations of their energy operator are different from each other [27], leading to different time evolutions of quantum energy for each. This outcome stems from the fact that a Hamiltonian of nonconservative TDOHS such as the damped harmonic oscillator is different from the energy operator of the system [28]. Any type of mass-accreting oscillators is basically understood by a pail-rain model presented in refs. [29,30]. Although the preparation of mechanical oscillator with exponentially increasing mass is rather difficult than that with linearly increasing mass, it is often used as a model to explain some physical phenomena. For example, Kim used it to describe massive scalar fields in quantum cosmology [31]. Another choice, which is $f(t) = m$, $\omega(t) = \omega_0$, $h_1 \neq 0$, and all other time functions are zero, gives simple harmonic oscillator of natural frequency ω_0 driven by arbitrary time-dependent forces.

Besides, the outcome of superposition states for such oscillator-like systems and their relevant WDFs are very interesting in physics due to their novel appearance indicating nonclassical features of the system. These states, in fact, gained much importance in quantum information theory as resources of quantum cryptography [32] and teleportation [33]. The nonclassicality of superposition states can also be used in several ways to refine resolutions of quantum measurements beyond the standard limits [34].

Classical coordinate satisfies the differential equation of the form

$$\ddot{q} + \frac{\dot{f}}{f}\dot{q} + \left(-2\frac{\dot{f}g}{f} - 4g^2 + \omega^2 - 2\dot{g}\right)q = \frac{\dot{f}h_2}{f} + 2gh_2 - h_1/f + \dot{h}_2, \tag{2}$$

and the conjugate canonical momentum is obtained from $p = f[\dot{q} - 2gq - h_2]$. The classical solution of coordinate and momentum is composed of complementary functions, [$q_c(t)$ and $p_c(t)$], and particular ones [$q_p(t)$ and $p_p(t)$]:

$$q_{cl}(t) = q_c(t) + q_p(t), \tag{3}$$

$$p_{cl}(t) = p_c(t) + p_p(t). \tag{4}$$

Since Equation (2) is a second order differential equation, there are two linearly independent complementary functions for coordinate. If we denote them $q_{c,1}(t)$ and $q_{c,2}(t)$, the general complementary function is given by

$$q_c(t) = c_1 q_{c,1}(t) + c_2 q_{c,2}(t), \tag{5}$$

where c_1 and c_2 are arbitrary real constants. Once $q_c(t)$ is explicitly known, we can easily find the complementary function for momentum from

$$p_c(t) = f(t)[\dot{q}_c(t) - 2g(t)q_c(t)]. \tag{6}$$

Meanwhile, in terms of particular solutions, we can construct the annihilation and the creation operators on the basis of Lewis-Riesenfeld invariant operator theory [35]:

$$\hat{a} = \hat{X} + i\hat{Y}, \qquad \hat{a}^\dagger = \hat{X} - i\hat{Y}. \tag{7}$$

Here, operators $\hat{X}$ and $\hat{Y}$ are given by

$$\hat{X} = \frac{1}{2\rho}\sqrt{\frac{\Omega}{\hbar}}[\hat{q} - q_{\rm p}(t)], \quad (8)$$

$$\hat{Y} = \sqrt{\frac{1}{\hbar\Omega}}\{f(2g\rho - \dot{\rho})[\hat{q} - q_{\rm p}(t)] + \rho[\hat{p} - p_{\rm p}(t)]\}, \quad (9)$$

where Ω is an arbitrary real constant, and $\rho(t)$ is a time function that obeys the following differential equation

$$\ddot{\rho}(t)+\frac{\dot{f}}{f}\dot{\rho}(t)+\left(\omega^2-2\frac{\dot{f}g}{f}-4g^2-2\dot{g}\right)\rho(t)-\frac{\Omega^2}{4f^2\rho^3(t)} = 0. \quad (10)$$

The invariant operator of the system is given by

$$\hat{I} = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \quad (11)$$

This satisfies the basic relation, $d\hat{I}/dt = 0$ [1], and plays an important role when developing quantum theory of the system.

One can easily check that the boson commutation relation between $\hat{a}$ and $\hat{a}^\dagger$ holds: $[\hat{a}, \hat{a}^\dagger] = 1$. This grantees that we can manage the system in the usual way associated with quantum mechanics, and the corresponding quantum solutions can be derived from the conventional method. Hence, it is possible to derive the wave function of the system in Fock state from Schrödinger equation and, as a result, it is written in the form [35]

$$\psi_n(q,t) = \phi_n(q,t)\exp[i\epsilon_n(t)], \quad (12)$$

where $\phi_n(q,t)$ is the eigenstate of Equation (11), which is given by

$$\phi_n(q,t) = \sqrt[4]{\frac{\Omega}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left[\sqrt{\frac{\Omega}{2\rho^2\hbar}}(\hat{q} - q_{\rm p})\right] \times \exp\left\{\frac{i}{\hbar}p_{\rm p}\hat{q} - \frac{1}{2\rho\hbar}\left[\frac{\Omega}{2}\frac{1}{\rho} + if(2g\rho - \dot{\rho})\right](\hat{q} - q_{\rm p})^2\right\}, \quad (13)$$

and $\epsilon_n(t)$ is a time-dependent phase of the form

$$\epsilon_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{\Omega}{2f(t')\rho^2(t')}dt' - \frac{1}{\hbar}\int_0^t\left[\mathcal{L}_p(t') - \frac{h_2^2(t')f(t')}{2} + k(t')\right]dt', \quad (14)$$

with

$$\mathcal{L}_{\rm p}(t) = \frac{f(t)}{2}\dot{q}_{\rm p}^2(t) - 2g(t)f(t)q_{\rm p}(t)\dot{q}_{\rm p}(t) - \left(\frac{1}{2}f(t)\omega^2(t) - 2g^2(t)f(t)\right)q_{\rm p}^2(t). \quad (15)$$

The wave function Equation (12) is necessary when deriving the WDF of a particular superposition state.

Wigner distribution function: superposition of coherent states

In this section, we study the WDF for a superposition of two coherent states. Among diverse distribution functions in phase spaces, the WDF is the most significant mathematical tool in the realm of not only (quantum) optics but also other dynamical systems. For this reason, the WDF has been a major topic in quantum statistical physics as well as in the context of quantum optics. It is well known that the WDF can be negative in phase space. This is the reason why the WDF can not be regarded as a real probability density but a 'quasiprobability density'. If the quantum wave packet is a Gaussian type, the associated WDF is positive in every region [18].

Let us first start from the definition of coherent state $|\alpha\rangle$, which is the eigenstate of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \quad (16)$$

By solving this equation, we have the eigenvalue such that

$$\alpha = \alpha_0 e^{i\varphi}, \quad (17)$$

where, α_0 and φ are given by

$$\alpha_0 = [X^2(t) + Y^2(t)]^{1/2}, \qquad \varphi = \tan^{-1}\frac{Y(t)}{X(t)}, \quad (18)$$

with

$$X(t) = \frac{1}{2\rho(t)}\sqrt{\frac{\Omega}{\hbar}}q_{\rm c}(t), \quad (19)$$

$$Y(t) = \sqrt{\frac{1}{\hbar\Omega}}\{f(t)[2g(t)\rho(t) - \dot{\rho}(t)]q_{\rm c}(t) + \rho(t)p_{\rm c}(t)\}. \quad (20)$$

On the other hand, the eigenstate is represented in terms of Fock state wave function such that

$$|\alpha\rangle = \exp\left(-\frac{1}{2}\alpha_0^2\right)\sum_n \frac{\alpha^n}{\sqrt{n!}}|\phi_n(t)\rangle. \quad (21)$$

In position space, we insert Equation (13) into the above equation and, after some algebra, we have

$$\langle q|\alpha\rangle = \sqrt[4]{\frac{\Omega}{2\rho^2\hbar\pi}} \times \exp\left\{\alpha\sqrt{\frac{\Omega}{\rho^2\hbar}}(q - q_{\rm p}) + \eta(q) - \frac{1}{2}\alpha_0^2 - \frac{1}{2}\alpha^2\right\}, \quad (22)$$

where

$$\eta(q) = -\frac{1}{4\rho\hbar}\left[\frac{\Omega}{\rho} + 2if(2g\rho - \dot{\rho})\right](q - q_{\rm p})^2 + \frac{i}{\hbar}p_{\rm p}q. \tag{23}$$

This is known as the(?) Glauber coherent state which exhibits no conspicuous nonclassical effects.

However, quantum superposition of coherent states exhibit diverse nonclassical properties such as normal and higher-order squeezing, sub-Poissonian statistics and quantum interference [36]. Various schemes have been proposed to generate and observe such states extensively [37]. As a familiar example of such state, let us consider a superposition of two coherent states which are $\pi/2$ out of phase with respect to each other [19]:

$$|\psi(t)\rangle = \frac{1}{\sqrt{N}}(|\alpha\rangle + e^{i\phi}|i\alpha\rangle), \tag{24}$$

where ϕ is a relative phase, and N is the normalization factor of the form

$$N = 2[\,1 + e^{-\alpha_0^2}\cos(\alpha_0^2 + \phi)]\,. \tag{25}$$

The researches for other types of superposition states are also found in other reports [20,21]. On experimental side, the ability to create and maintain superposition states is a key element in attempts for observing nonclassical behaviors of larger and more complex quantum systems.

In particular, for $\phi = 0$, $\phi = \pi$, and $\phi = \pi/2$, Equation (24) becomes

$$|\psi^{+}(t)\rangle = \frac{1}{\sqrt{N_+}}(|\alpha\rangle + |i\alpha\rangle), \tag{26}$$

$$|\psi^{-}(t)\rangle = \frac{1}{\sqrt{N_-}}(|\alpha\rangle - |i\alpha\rangle), \tag{27}$$

$$|\psi^{0}(t)\rangle = \frac{1}{\sqrt{N_0}}(|\alpha\rangle + e^{i\pi/2}|i\alpha\rangle), \tag{28}$$

respectively, where

$$N_+ = 2(1 + e^{-\alpha_0^2}\cos\alpha_0^2), \tag{29}$$

$$N_- = 2(1 - e^{-\alpha_0^2}\cos\alpha_0^2), \tag{30}$$

$$N_0 = 2(1 - e^{-\alpha_0^2}\sin\alpha_0^2). \tag{31}$$

If we use Equation (22), the superposition state in configuration space is obtained to be

$$\begin{aligned}\langle q|\psi(t)\rangle \;=\; & \sqrt[4]{\frac{\Omega}{2\rho^2\hbar\pi}}\frac{2}{\sqrt{N}} \\ & \times\exp\left\{\eta(q) + (1+i)\frac{\alpha}{2}\sqrt{\frac{\Omega}{\rho^2\hbar}}(q - q_{\rm p}) - \frac{1}{2}\alpha_0^2 + \frac{i\phi}{2}\right\} \\ & \times\cosh\left[(1-i)\frac{\alpha}{2}\sqrt{\frac{\Omega}{\rho^2\hbar}}(q - q_{\rm p}) - \frac{1}{2}\alpha^2 - \frac{i\phi}{2}\right].\end{aligned} \tag{32}$$

As is well known, the fundamental mode of Gaussian wave packet such as coherent states and their superposition can be studied by utilizing the WDF. The definition of WDF for our superposed coherent state is

$$W(q,p,t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle\psi(t)|q+y\rangle\langle q-y|\psi(t)\rangle e^{2ipy/\hbar}dy. \tag{33}$$

The substitution of Equation (32) into the above equation leads to

$$\begin{aligned}W(q,p,t) \;=\; & \frac{1}{\pi\hbar N}e^{-\alpha_0^2}\exp\left(-\frac{2}{\hbar\Omega}I(q,p,t)\right) \\ & \times\left[e^{-\alpha_0^2}(e^{\Pi_1} + e^{-\Pi_2}) \right.\\ & \left. + 2e^{\Pi_1/2 - \Pi_2/2}\cos\left(\frac{1}{2}\Pi_1 - \frac{1}{2}\Pi_2 - \alpha_0^2 + \phi\right)\right],\end{aligned} \tag{34}$$

where

$$\begin{aligned}I(q,p,t) = & \frac{\Omega^2}{4\rho^2(t)}(q - q_{\rm p}(t))^2 \\ & + \left[\rho(p - p_{\rm p}(t)) + f(2g\rho(t) - \dot{\rho}(t))(q - q_{\rm p}(t))\right]^2,\end{aligned} \tag{35}$$

$$\begin{aligned}\Pi_1 = & 2\sqrt{\frac{\Omega}{\hbar\rho^2}}(q - q_{\rm p})\alpha_0\cos\varphi \\ & + \frac{4\rho\alpha_0\sin\varphi}{\sqrt{\Omega\hbar}}\left[(p - p_{\rm p}) + f\frac{2g\rho - \dot{\rho}}{\rho}(q - q_{\rm p})\right],\end{aligned} \tag{36}$$

$$\begin{aligned}\Pi_2 = & 2\sqrt{\frac{\Omega}{\hbar\rho^2}}(q - q_{\rm p})\alpha_0\sin\varphi \\ & - \frac{4\rho\alpha_0\cos\varphi}{\sqrt{\Omega\hbar}}\left[(p - p_{\rm p}) + f\frac{2g\rho - \dot{\rho}}{\rho}(q - q_{\rm p})\right].\end{aligned} \tag{37}$$

Thus, the full expression of WDF is obtained. The cosine function in the last term of Equation (34) reflects the quantum interference between the two bells associated with $\langle q|\alpha\rangle$ and $\langle q|i\alpha\rangle$, respectively. This is a strong evidence for the appearance of nonclassical features of superposition state, which cannot be explained using the knowledge of classical mechanics.

For $\phi = 0$, $\phi = \pi$, and $\phi = \pi/2$, we respectively have

$$W^{+}(q,p,t) = \frac{1}{\pi\hbar N_{+}} e^{-\alpha_0^2} \exp\left(-\frac{2}{\hbar\Omega} I(q,p,t)\right) \times \left[e^{-\alpha_0^2}(e^{\Pi_1} + e^{-\Pi_2}) + 2e^{\Pi_1/2-\Pi_2/2} \cos\left(\frac{1}{2}\Pi_1 - \frac{1}{2}\Pi_2 - \alpha_0^2\right)\right], \quad (38)$$

$$W^{-}(q,p,t) = \frac{1}{\pi\hbar N_{-}} e^{-\alpha_0^2} \exp\left(-\frac{2}{\hbar\Omega} I(q,p,t)\right) \times \left[e^{-\alpha_0^2}(e^{\Pi_1} + e^{-\Pi_2}) - 2e^{\Pi_1/2-\Pi_2/2} \cos\left(\frac{1}{2}\Pi_1 - \frac{1}{2}\Pi_2 - \alpha_0^2\right)\right], \quad (39)$$

$$W^{0}(q,p,t) = \frac{1}{\pi\hbar N_{0}} e^{-\alpha_0^2} \exp\left(-\frac{2}{\hbar\Omega} I(q,p,t)\right) \times \left[e^{-\alpha_0^2}(e^{\Pi_1} + e^{-\Pi_2}) - 2e^{\Pi_1/2-\Pi_2/2} \sin\left(\frac{1}{2}\Pi_1 - \frac{1}{2}\Pi_2 - \alpha_0^2\right)\right]. \quad (40)$$

The phase of interference term for these three cases are different from each other. Thus, the shape of interference pattern varies depending on the value of ϕ.

Wigner distribution function: superposition of squeezed states

As well as coherent state, the squeezed state also plays an important role, since it enables us to reduce quantum noise in one quadrature at the expense of increasing the noise in its counterpart quadrature. Squeezed states are described via an operator of the form

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^{\dagger}, \quad (41)$$

where μ and ν satisfy

$$|\mu|^2 - |\nu|^2 = 1. \quad (42)$$

As that of $\hat{a}$ and $\hat{a}^{\dagger}$ in the previous section, the usual boson commutation relation for $\hat{b}$ and $\hat{b}^{\dagger}$ also holds: $[\hat{b}, \hat{b}^{\dagger}] = 1$. If we represent the corresponding eigenvalue equation in the form

$$\hat{b}|\beta\rangle = \beta|\beta\rangle, \quad (43)$$

$|\beta\rangle$ is the squeezed state. By solving Equation (43) in configuration space, we have the squeezed state such that

$$\langle q|\beta\rangle = \mathcal{N}_q \exp\left\{-\frac{1}{\rho\hbar}\left[\frac{\Lambda}{\mu-\nu}\left(\frac{1}{2}q^2 - q_{\rm p}q\right) - i\rho p_{\rm p}q\right] + \frac{\mu\alpha + \nu\alpha^*}{\rho(\mu-\nu)}\sqrt{\frac{\Omega}{\hbar}}q\right\}, \quad (44)$$

where $\mathcal{N}_q$ is the normalization factor of the form

$$\mathcal{N}_q = \left(\frac{\Omega}{2\rho^2\hbar\pi}\frac{1}{(\mu-\nu)(\mu^*-\nu^*)}\right)^{1/4} \times \exp\left[-\frac{\Omega}{4\rho^2\hbar}\frac{1}{(\mu-\nu)(\mu^*-\nu^*)} \times \left(q_{\rm p} + 2\rho\sqrt{\frac{\hbar}{\Omega}}\alpha_0\cos\varphi\right)^2 + i\delta_{{\rm s},q}(\alpha,\alpha^*)\right], \quad (45)$$

with some phase $\delta_{{\rm s},q}(\alpha,\alpha^*)$, and

$$\Lambda = \frac{\Omega}{2\rho}(\mu+\nu) + if(2g\rho - \dot{\rho})(\mu-\nu). \quad (46)$$

If we choose the phase $\delta_{{\rm s},q}(\alpha,\alpha^*)$ in the form

$$\delta_{{\rm s},q}(\alpha,\alpha^*) = -\frac{q_{\rm p}^2}{4\hbar\rho}\left(2(2g\rho - \dot{\rho})f + \frac{i\Omega(\mu\nu^* - \mu^*\nu)}{\rho(\mu-\nu)(\mu^*-\nu^*)}\right) - \alpha_0^2\sin\varphi\cos\varphi + iq_{\rm p}\sqrt{\frac{\Omega}{\hbar\rho^2}} \times \frac{[(|\mu|^2 - \mu\nu^* - 1/2)\alpha - (|\mu|^2 - \mu^*\nu - 1/2)\alpha^*]}{(\mu-\nu)(\mu^*-\nu^*)}, \quad (47)$$

Equation (44) reduces to

$$\langle q|\beta\rangle = \left(\frac{\Omega}{2\pi\hbar\rho^2(\mu-\nu)(\mu^*-\nu^*)}\right)^{1/4} \times \exp\left\{-\frac{1}{2\rho\hbar}\frac{\Lambda}{(\mu-\nu)}(q-q_{\rm p})^2 + \frac{i}{\hbar}p_{\rm p}q + \frac{\mu\alpha+\nu\alpha^*}{\rho(\mu-\nu)}\sqrt{\frac{\Omega}{\hbar}}(q-q_{\rm p}) - \frac{1}{4}(\alpha^2 - \alpha^{*2}) - \frac{\alpha_0^2[1+\cos(2\varphi)]}{2(\mu-\nu)(\mu^*-\nu^*)}\right\}, \quad (48)$$

and hence, another type of squeezed state $\langle q|i\beta\rangle$ can be represented as

$$\langle q|i\beta\rangle = \left(\frac{\Omega}{2\pi\hbar\rho^2(\mu-\nu)(\mu^*-\nu^*)}\right)^{1/4} \times \exp\left\{-\frac{1}{2\rho\hbar}\frac{\Lambda}{(\mu-\nu)}(q-q_{\rm p})^2+\frac{i}{\hbar}p_{\rm p}q + \frac{i(\mu\alpha+\nu\alpha^*)}{\rho(\mu-\nu)}\sqrt{\frac{\Omega}{\hbar}}(q-q_{\rm p})+\frac{1}{4}(\alpha^2-\alpha^{*2}) - \frac{\alpha_0^2[1-\cos(2\varphi)]}{2(\mu-\nu)(\mu^*-\nu^*)}\right\}. \tag{49}$$

We can easily confirm from a little relevant evaluation that $\langle q|\beta\rangle$ is properly normalized, leading to

$$\int_{-\infty}^{\infty}\langle\beta|q\rangle\langle q|\beta\rangle dq = 1. \tag{50}$$

However, the same evaluation for $\langle q|i\beta\rangle$ gives

$$\int_{-\infty}^{\infty}\langle i\beta|q\rangle\langle q|i\beta\rangle dq = \exp\left[-\frac{\alpha_0^2[1-\cos(2\varphi)]}{(\mu-\nu)(\mu^*-\nu^*)} - \frac{[\alpha A-\alpha^*A^*]^2}{2(\mu-\nu)(\mu^*-\nu^*)}\right], \tag{51}$$

where

$$A = |\mu|^2 - 2\mu\nu^* + |\nu|^2. \tag{52}$$

We thus see that $\langle q|i\beta\rangle$ cannot be properly normalized with the factor $\mathcal{N}_q$. To fix this defect, we need to redefine the normalization constant of $\langle q|i\beta\rangle$, such that

$$\mathcal{N}_{q,(\rm new)} = \mathcal{N}_q \exp\left[\frac{\alpha_0^2[1-\cos(2\varphi)]}{2(\mu-\nu)(\mu^*-\nu^*)} + \frac{[\alpha A-\alpha^*A^*]^2}{4(\mu-\nu)(\mu^*-\nu^*)}\right]. \tag{53}$$

This yields exact normalization for $\langle q|i\beta\rangle$ and further evaluation of $\langle q|i\beta\rangle$ using this can be fulfilled, resulting in

$$\langle q|i\beta\rangle = \left(\frac{\Omega}{2\pi\hbar\rho^2(\mu-\nu)(\mu^*-\nu^*)}\right)^{1/4} \times \exp\left\{-\frac{1}{2\rho\hbar}\frac{\Lambda}{(\mu-\nu)}(q-q_{\rm p})^2+\frac{i}{\hbar}p_{\rm p}q + \frac{i(\mu\alpha+\nu\alpha^*)}{\rho(\mu-\nu)}\sqrt{\frac{\Omega}{\hbar}}(q-q_{\rm p})+\frac{1}{4}(\alpha^2-\alpha^{*2}) + \frac{[\alpha A-\alpha^*A^*]^2}{4(\mu-\nu)(\mu^*-\nu^*)}\right\}. \tag{54}$$

Now, let us express the superposition state in the form

$$\langle q|\Psi(t)\rangle = \frac{1}{\sqrt{\mathcal{N}}}(\langle q|\beta\rangle + e^{i\phi}\langle q|i\beta\rangle), \tag{55}$$

where Equation (54) is used for $\langle q|i\beta\rangle$. The normalization factor of Equation (55) is given by

$$\mathcal{N} = 2 + \exp\left[-\frac{1}{4(\mu-\nu)(\mu^*-\nu^*)}(2\alpha_0^2B_1+\alpha^2B_2 + \alpha^{*2}B_3 + 4\alpha_0^2\cos^2\varphi) - i\phi\right] + \exp\left[-\frac{1}{4(\mu-\nu)(\mu^*-\nu^*)}(2\alpha_0^2B_1^*+\alpha^{*2}B_2^* + \alpha^2B_3^* + 4\alpha_0^2\cos^2\varphi) + i\phi\right], \tag{56}$$

with

$$B_1 = (1+2i)(|\mu|^4+|\nu|^4) - 2[\,i(\mu\nu^*+\mu^*\nu)+2\mu^*\nu]\,|\mu|^2 - 2[\,i(\mu\nu^*+\mu^*\nu)+2\mu\nu^*]\,|\nu|^2 + 2(3+2i)|\mu|^2|\nu|^2, \tag{57}$$

$$B_2 = -3|\mu|^4+|\nu|^4-2(1+2i)|\mu|^2|\nu|^2+2[\,1+2(2+i)\mu\nu^*]\,|\mu|^2 + 2(1+2i\mu\nu^*)|\nu|^2 - 4(1+i)\mu^2\nu^{*2} - 2(\mu\nu^*+\mu^*\nu), \tag{58}$$

$$B_3 = -3|\nu|^4+|\mu|^4-2(1+2i)|\mu|^2|\nu|^2-2[\,1-2(2+i)\mu^*\nu]\,|\nu|^2 - 2(1-2i\mu^*\nu)|\mu|^2 - 4(1+i)\mu^{*2}\nu^2 + 2(\mu\nu^*+\mu^*\nu). \tag{59}$$

We can express the WDF of superposition of the two squeezed states in the form

$$\mathcal{W}(q,p,t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle\Psi(t)|q+y\rangle\langle q-y|\Psi(t)\rangle e^{2ipy/\hbar}dy. \tag{60}$$

The integration, after inserting Equation (55) with Equations (56) to (59) into the above equation, yields

$$\begin{aligned}\mathcal{W}(q,p,t) = {} & \frac{1}{\pi\hbar}\frac{1}{\mathcal{N}}\exp\left(-\frac{2}{\hbar\Omega}I_s(q,p,t)\right)\left\{\exp\left\{\frac{1}{2(\mu-\nu)(\mu^*-\nu^*)}\right.\right.\\ & \times\left[(\alpha A-\alpha^*A^*)\left(\alpha A-\alpha^*A^*+2i\sqrt{\frac{\Omega}{\hbar\rho^2}}(q-q_{\rm p})\right)\right.\\ & \left.\left.-(\alpha+\alpha^*)^2\right]+\frac{2}{\sqrt{\hbar\Omega}}(\alpha+\alpha^*)C\right\}\\ & +\exp\left\{\frac{1}{2(\mu-\nu)(\mu^*-\nu^*)}\left[(\alpha+\alpha^*)\left(-(\alpha+\alpha^*)\right.\right.\right.\\ & \left.\left.+2\sqrt{\frac{\Omega}{\hbar\rho^2}}(q-q_{\rm p})\right)+(\alpha A-\alpha^*A^*)^2\right]-\frac{2i}{\sqrt{\hbar\Omega}}(\alpha A-\alpha^*A^*)C\right\}\\ & +\exp\left\{\frac{1}{2}(\mu-\nu)(\mu^*-\nu^*)D_+^2-\frac{2i}{\sqrt{\hbar\Omega}}(\mu-\nu)(\mu^*-\nu^*)D_+C\right.\\ & -\frac{1}{2}(\alpha^2-\alpha^{*2})+\frac{[\alpha A-\alpha^*A^*]^2}{4(\mu-\nu)(\mu^*-\nu^*)}-i\phi+D_-\sqrt{\frac{\Omega}{\hbar\rho^2}}(q-q_{\rm p})\\ & \left.-\frac{\alpha_0^2\cos^2\varphi}{(\mu-\nu)(\mu^*-\nu^*)}\right\}\\ & +\exp\left\{-\frac{1}{2}(\mu-\nu)(\mu^*-\nu^*)D_+^2+\frac{2}{\sqrt{\hbar\Omega}}(\mu-\nu)(\mu^*-\nu^*)D_+C\right.\\ & +\frac{1}{2}(\alpha^2-\alpha^{*2})+\frac{[\alpha A-\alpha^*A^*]^2}{4(\mu-\nu)(\mu^*-\nu^*)}+i\phi+iD_-\sqrt{\frac{\Omega}{\hbar\rho^2}}(q-q_{\rm p})\\ & \left.\left.-\frac{\alpha_0^2\cos^2\varphi}{(\mu-\nu)(\mu^*-\nu^*)}\right\}\right\},\end{aligned} \tag{61}$$

where

$$I_s(q,p,t) = \frac{\Omega^2}{4\rho^2}\frac{(q-q_{\rm p}(t))^2}{(\mu-\nu)(\mu^*-\nu^*)}+(\mu-\nu)(\mu^*-\nu^*)C^2, \tag{62}$$

$$\begin{aligned}C = {} & \rho(p-p_{\rm p}(t))\\ & +\left((2g\rho-\dot{\rho})f-\frac{i\Omega}{2\rho}\frac{\mu^*\nu-\mu\nu^*}{(\mu-\nu)(\mu^*-\nu^*)}\right)\\ & \times(q-q_{\rm p}(t)),\end{aligned} \tag{63}$$

$$D_+ = \frac{\alpha\mu+\alpha^*\nu}{\mu-\nu}+\frac{i(\alpha^*\mu^*+\alpha\nu^*)}{\mu^*-\nu^*}, \tag{64}$$

$$D_- = \frac{\alpha\mu+\alpha^*\nu}{\mu-\nu}-\frac{i(\alpha^*\mu^*+\alpha\nu^*)}{\mu^*-\nu^*}. \tag{65}$$

This is the complete expression of WDF for the superposition of the two squeezed states. The WDF represented here is useful when investigating quantum corrections of a classical distribution function of TDOHS in squeezed state. For $\mu = 1$ and $\nu = 0$, Equation (61) reduces to that of the superposition of coherent states given in Equation (34). The integration of the WDF over either of the coordinate or momentum variables yields the probability distribution for the other, such that

$$\int_{-\infty}^{\infty}\mathcal{W}(q,p,t)dq = |\langle p|\Psi(t)\rangle|^2, \tag{66}$$

$$\int_{-\infty}^{\infty}\mathcal{W}(q,p,t)dp = |\langle q|\Psi(t)\rangle|^2. \tag{67}$$

These relations hold in general on every situation for a quantum state and guarantee the WDF to be a quantal distribution function in spite of its peculiar properties such as its allowed appearance of negativity. It is

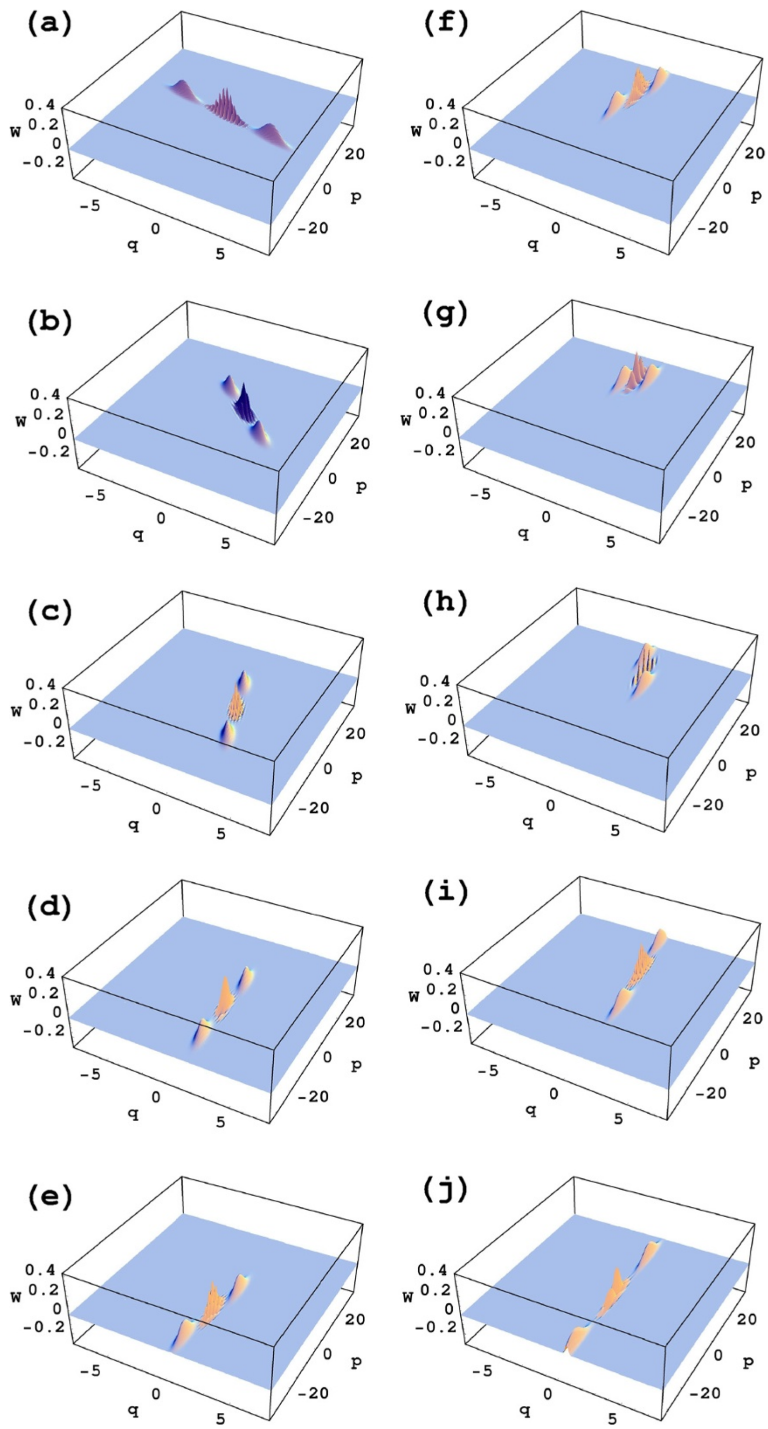

Figure 1 Time evolution of WDF of superposed coherent state for the system. This was presented in 'Application to a particular system' section. The value of t is 0.0 for **(a)**, 0.5 for **(b)**, 1.0 for **(c)**, 1.5 for **(d)**, 2.0 for **(e)**, 2.5 for **(f)**, 3.0 for **(g)**, 3.5 for **(h)**, 4.0 for **(i)**, and 4.5 for **(j)**. We have taken $c_1 = c_2 = 6$, $\hbar = 1$, $\omega_0 = 1$, $m_0 = 1$, $\gamma_1 = \gamma_2 = 1$, $c = 1$, $\Omega = 1$, and $\phi = \pi/2$ (these values will also be used in all of the subsequent figures).

well known that nonclassical states of a quantum system can be described in a best way via the WDF. The tomographic reconstruction of WDF in quantum optics is possible from the experimentally obtained data for a set of probability distributions of the light quadrature-phase amplitudes [38].

Application to a particular system

Since any type of time-functions is allowed in Equation (1), our development for WDF can be applied to diverse classes of TDOHS. Let us consider a particular case described in terms of time functions of the form

$$f(t) = m_0 e^{\gamma_1 t + c\sin(\gamma_2 t)}, \qquad (68)$$

$$\omega^2(t) = \omega_0^2 + \frac{1}{\sqrt{f(t)}}\frac{d^2\sqrt{f(t)}}{dt^2}, \qquad (69)$$

where m_0, c, γ_1, γ_2, and ω_0 are real constants, and all other time functions in Equation (1) are zero. Here, m_0 and ω_0 should be always positive for a physically acceptable system. This system is also treated by Lo [39] for somewhat different purpose. Notice that the quantum mechanical problem developed in this paper is described in terms of classical solutions. By solving Equation (10) for this system, we have

$$\rho(t) = \sqrt{\frac{\Omega}{2\omega_0 f(t)}}. \qquad (70)$$

Further, from Equation (2), we see that $q_c(t)$ is given by Equation (5) with

$$q_{c,1}(t) = \rho(t)\cos(\omega_0 t), \qquad (71)$$

$$q_{c,2}(t) = \rho(t)\sin(\omega_0 t), \qquad (72)$$

and the corresponding $p_c(t)$ is easily identified from Equation (6). However, there are no particular solutions for both coordinate and momentum $[q_p(t) = p_p(t) = 0]$ since $h_1 = h_2 = 0$ in Equation (1). Thus, we have obtained the complete classical solutions of the system.

The WDF of the coherent state is illustrated in Figure 1, and that of the squeezed state is in Figures 2 and 3. Figure 2 represents q-squeezing, while Figure 3 p-squeezing. The two bells in all figures correspond to macroscopically distinguishable quantum states $|\alpha\rangle$ and $|i\alpha\rangle$ (or $|\beta\rangle$ and $|i\beta\rangle$) which are separated in phase by $\pi/2$. These two bells rotate clockwise with time as a whole. The ripple appeared in the middle region between the two bells represents interference between $|\alpha\rangle$ and $|i\alpha\rangle$, and this interesting feature indicates the nonclassicality of the superposed quantum system. As you can see, negative values are allowed

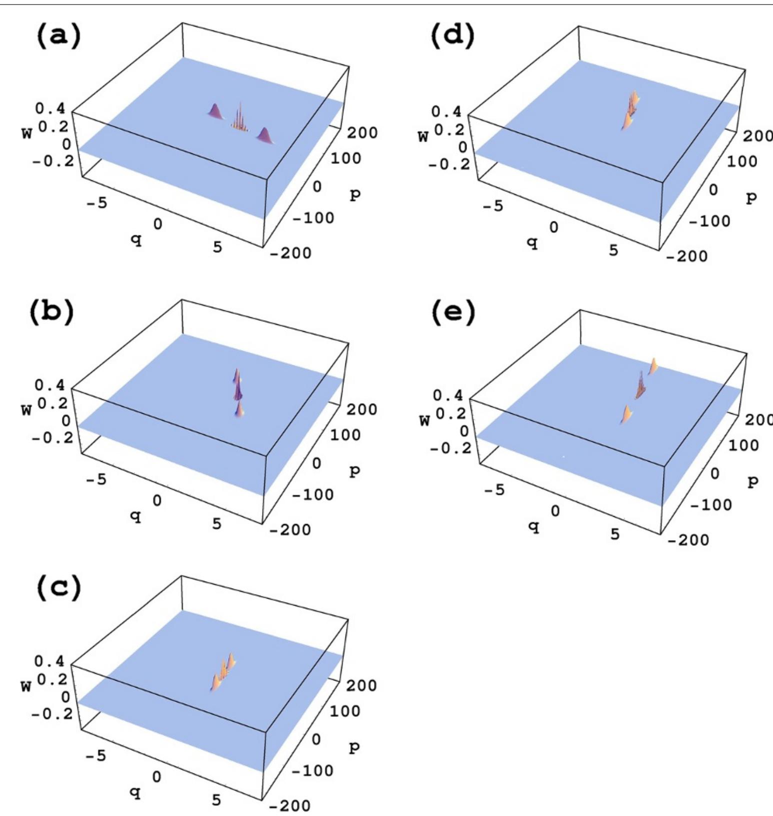

Figure 2 Time evolution of WDF of superposed squeezed state for the system. This was presented in 'Application to a particular system' section. We used $(\mu, \nu) = (\sqrt{2}, 1)$. The value of t is 0.0 for **(a)**, 1.0 for **(b)**, 2.0 for **(c)**, 3.0 for **(d)**, and 4.0 for **(e)**.

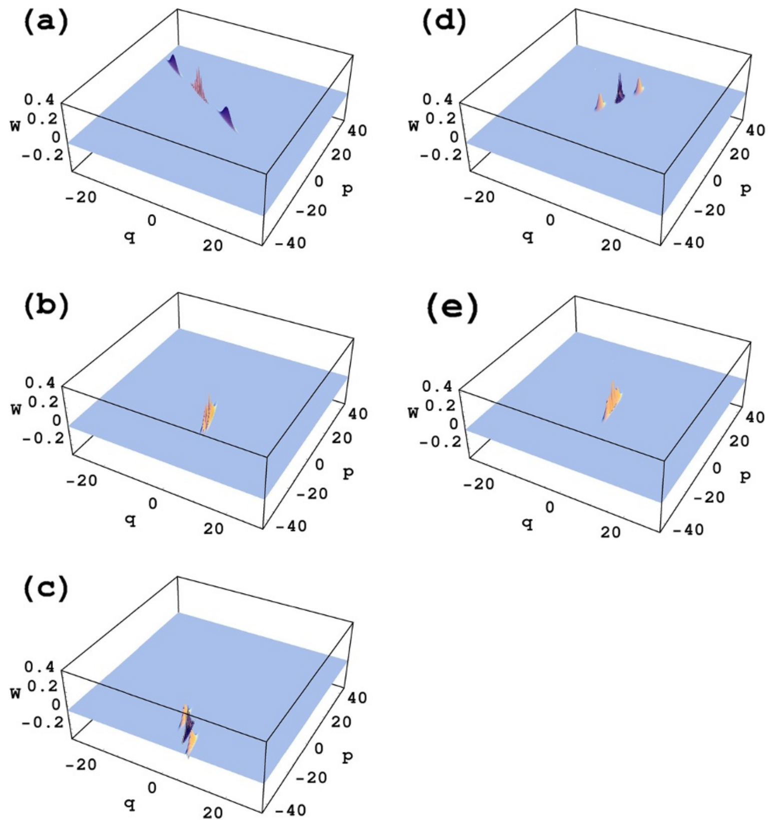

Figure 3 The same as Figure 2 but with different set of squeezing parameter which is $(\mu, \nu) = (\sqrt{2}, -1)$.

in the region of ripple, which is impossible in any classical system. A detailed phenomenological interpretation of such interference appeared in ref. [22].

Summary and conclusion

If we consider that we are free to choose any type of time functions given in Equation (1), the Hamiltonian we employed here is in a very general form. From particular choices of them, we can investigate quantum states and the corresponding WDF for various kinds of time-dependent harmonic oscillators. Coherent and squeezed states are constructed using the annihilation and the creation operators which are associated with the invariant operator theory. A distinguished feature of invariant operator theory is that its development concerning the establishment of quantum states can be attained through the introduction of classical solutions. Hence, we considered complementary functions, $q_c(t)$ and $p_c(t)$, and particular solutions, $q_p(t)$ and $p_p(t)$, when developing our theory for WDF.

The special type of superposition for the two different coherent states and for the two different squeezed states is regarded, respectively, and their corresponding WDFs are evaluated. Two elements of a superposition state is $\pi/2$ out of phase with respect to each other and have relative phase of ϕ as shown in Equations (24) and (55). The global approach described here in order to derive a general analytic form of WDF enables one to study the dynamics of the quantized TDOHS and their quantum properties in configuration space. The WDF has long been used as a very popular tool to study the quantum behaviors of, in principal, stationary systems. In recent years, it has also become a convenient implement when investigating rather complicated dynamical systems that are described by time-dependent Hamiltonian. Notably, there has been a very extensive research for the dynamics of photon statistics in quantum optics [21,40,41].

It is interesting that the value of WDF can be negative in some parts of the phase space for a certain quantum state. One can regard this as a reflection of the nonclassical effects, which is classically impossible [23]. A sophisticated approach for nonclassical characteristics of superposition state is in principle achieved by taking advantage of WDF [21,24]. We also confirmed the existence of nonclassical features in our system from the cosine term of the WDF, which represents the interference between the two elements of the superposed state. The pattern of stripes appeared in the interfered region of phase space for WDF varies depending on the value of ϕ. Indeed, the value of WDF in the large part of that region can be negative as a signature of nonclassicality.

Abbreviations

HOEIM: Harmonic oscillator with exponentially increasing mass; TDOHS: Time-dependent oscillator-like Hamiltonian system; WDF: Wigner distribution function.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally, read, and approved the final manuscript.

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Acknowledgements

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant no.: 2010-0016914).

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Received: 27 April 2012 Accepted: 11 October 2012
Published: 6 November 2012

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doi:10.1186/2251-7235-6-26
Cite this article as: Choi *et al.*: Wigner distribution function of superposed quantum states for a time-dependent oscillator-like Hamiltonian system. *Journal of Theoretical and Applied Physics* 2012 **6**:26.