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Nonstandard Lagrangian cosmology

Rami Ahmad El-Nabulsi

Abstract

We show that many independent scenarios as the 'accelerated expansion of the universe', 'eternal inflation', 'eternally oscillating universe', 'nonsingular oscillating universe', and 'collapse of an oscillating universe' may occur without modifying the gravity theory or introducing scalar fields of any type. This is achieved by replacing the standard Lagrangian in the Friedmann-Robertson-Walker spacetime model by an exponentially nonstandard Lagrangian which modify the Euler-Lagrange equation although the standard variational approach is used.

Keywords: Nonstandard Lagrangians; FRW cosmology; Spacetime model

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Introduction

The cosmic microwave background radiation, the large structure of the universe, the data obtained from the Supernova Legacy Survey of high redshift type SNeIa, the Gaussian primordial density perturbations and so on confirm that the universe is spatially flat and is presently undergoing a phase of accelerated expansion with a density parameter $\Omega_k = -0.015^{+0.020}_{-0.016}$ (within a 2% margin of error) [1-3]. Dark energy was proposed to explain the accelerated expansion of the universe with an equation of state (EoS) parameter $\gamma = p/\rho < -1/3$. Here, p and ρ are respectively the pressure and the density of the perfect fluid matter. The nature of dark energy is still an important challenge. At the moment, there exist different approaches to address this problem, e.g., quintessence [4], K-essence [5], Chaplygin gas [6], among others. Some recent and old attempts to resolve the dark energy problem are found in [7]. In fact, some of these models suffer from fine-tuning problems; besides, a dynamical time-varying dark energy is expected.

The main aim of this communication is to add a new theoretical approach to those introduced in the literature which is based on the notion of nonstandard Lagrangians (NSL) termed nonnatural by Arnold [8]. In fact, in the progress of years, it was observed that NSL play an important role in different branches of applied and theoretical physics, i.e., nonlinear differential equations

[9-11], Yang-Mills color confinement issue [12], dissipative systems [13-20], quantum field theories [21-23], Dirac-Born-Infeld field Lagrangian, p-adic string for tachyon field and cosmology [24-27]. It is striking that NSL do not have at the moment any physical foundation. This is an open problem, and much work is required. For a given dynamical problem, NSL are fundamentally generating functions of different equations of motions. It is noteworthy that NSL in general are based on standard calculus of variations, and for the sake of clarity it should be stressed that the term NSL in our framework basically refers to any 'Lagrangian functional that modifies the corresponding Euler-Lagrange equation and consequently Hamilton's equations of motion'.

In this communication, we will try to apply NSL formalism to cosmology. More precisely, we will deal with exponential nonstandard Lagrangian types introduced recently in [16]. Nevertheless in this work, we will generalize the arguments of [16] by considering time-dependent exponentially NSL. We will show that these particular types of NSL may offer some new cosmological features. The paper is organized as follows: in the 'Time-dependent exponential NSL: action, modified Euler-Lagrange equation, and modified FRW cosmology' section, we will introduce the basic machinery for the case of time-dependent exponential NSL and discus two independent cosmological scenarios based on the Friedmann-Robertson-Walker (FRW) spacetime. In the 'Generalized coordinate-dependent exponential NSL: action, modified Euler-Lagrange equation, and modified FRW cosmology' section, we discuss new types of

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exponential NSL which depend on the generalized co-ordinates and discuss their cosmological implications. In the 'Statefinder diagnostic' section, we analyze the statefinder diagnostic for the models obtained in this paper, and finally the 'Conclusions' section is given.

Time-dependent exponential NSL: action, modified Euler-Lagrange equation, and modified FRW cosmology

We start by introducing the basic properties of the time-dependent exponential NSL: let $L(\dot{q}, q, t) \in C^2([a, b] \times \mathbb{R} n \times \mathbb{R}^n; \mathbb{R})$ be an admissible smooth Lagrangian function with $(\dot{q}, q, t) \to L(\dot{q}, q, t)$ assumed to be a C^2 function with respect to all its arguments with $q \in C^1([a, b]; \mathbb{R}^n)$. NSL with a time-dependent exponential Lagrangian $\xi(t)$ $e^{L(\dot{q}(t), q(t), t)}$ are defined by

$$S = \int_a^b \xi(t) e^{L(\dot{q}(t), q(t), t)} dt \tag{1}$$

Here, $\xi(t)$ is an arbitrary time-dependent function. If $q(t)$ is a solution of the functional (1) and is subject to given boundary conditions $q(a) = q_a$ and $q(b) = q_b$, then it satisfies the following modified Euler-Lagrange equation:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{1}{\xi}\frac{\partial L}{\partial \dot{q}}\frac{d\xi}{dt} + \xi \frac{\partial L}{\partial \dot{q}}\left(\frac{\partial L}{\partial t} + \dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}}\right). \tag{2}$$

For the proof, the author is refereed to [16].

Before exploring our approach, the following two main remarks are constructive:

Remark 1. For a given dynamical system with N degrees of freedom $q = (q_1, ..., q_N)$ characterized by the Hamiltonian function $H = \Sigma_{k=1}^N \dot{q}_k(\partial L/\partial \dot{q}_k) - L$, it is effortless to check that the Hamiltonian is not a constant of motion, i.e., the conservation of energy is violated. The associated Hamilton's equations are derived easily using the standard procedure. To check this, we define the $F = (q, p, \dot{q}, \dot{p}) \triangleq \Sigma_{k=1}^N p_k \dot{q}_k - H$ where $p_k = \partial L/\partial \dot{q}_k$ is the momentum and $\{q, p, \dot{q}, \dot{p}\}$ are four sets of independent variables. Requiring that $\delta\left\{\int_a^b e^{\xi F(q, p, \dot{q}, \dot{p})} dt\right\} = 0$ and varying the variables q_k and p_k separately, we get our modified Euler-Lagrange equations $\delta F/\delta q_k = 0$ and $\delta F/\delta p_k = 0$. When written explicitly, these are

$$\frac{\partial F}{\partial p_k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{p}_k}\right) = \frac{1}{\xi}\frac{\partial F}{\partial \dot{p}_k}\frac{d\xi}{dt} + \xi \frac{\partial F}{\partial \dot{p}_k}\left(\dot{p}_k\frac{\partial F}{\partial p_k} + \ddot{p}_k\frac{\partial F}{\partial \dot{p}_k}\right)$$

$$\frac{\partial F}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_k}\right) = \frac{1}{\xi}\frac{\partial F}{\partial \dot{q}_k}\frac{d\xi}{dt} + \xi \frac{\partial F}{\partial \dot{q}_k}\left(\dot{q}_k\frac{\partial F}{\partial q_k} + \ddot{q}_k\frac{\partial F}{\partial \dot{q}_k}\right)$$

Using $F = (q, p, \dot{q}, \dot{p}) \triangleq \Sigma_{k=1}^N p_k \dot{q}_k - H$, we find correspondingly the modified equations of motion:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} - \left(\frac{1}{\xi}\frac{d\xi}{dt} + \xi\left(-\frac{\partial H}{\partial q_k}\dot{q}_k + \ddot{q}_k p_k\right)\right)p_k$$

One may naturally ask how these modified equations of motion will modify the dynamics accordingly and result in a nonconserved energy. To inspect rapidly about this, we consider the standard time-independent Lagrangian $L = (\dot{q}^2 - q^2)/2$. The resulting equation of motion as derived from Equation 2 is $\ddot{q}(1 + \xi\dot{q}^2) + \dot{q}\dot{\xi}/\xi + q(1 - \xi\dot{q}^2) = 0$. This is a second-order nonlinear ordinary differential equation. Solving this equation numerically for $\xi = 1/t$ and the initial conditions $q(1) = 1$ and $q'(1) = 0$ results in the following oscillatory dynamics as illustrated in Figure 1.

If for instance, we choose $\xi = e^{-1/t}$ and the initial conditions $q(1) = 0$ and $q'(1) = 1$, then the dynamics is illustrated in Figure 2.

One more illustration corresponds to $L = \dot{q} + q^2/2$ and $\xi = e^{-1/t}$. The resulting equation of motion is $\ddot{q} + \dot{q} q + tq = 1$, and the numerical solution is graphically illustrated in Figure 3 for $q(1) = 0$ and $\dot{q}(1) = 1$.

One can still portray the oscillatory dynamics of a given system by extending the concept of Hamiltonian which violates the energy conservation.

Remark 2. We already know that Einstein's general theory of relativity (EGR) is considered a successful theory as it explains many astrophysical and cosmological fundamental problems. In fact, the main aim of this work is not to replace it with a new theory that violates its basic principles. In contrast, we would like to simply extend the Lagrangian formalism in order to obtain general exact solutions of the standard cosmological model without violating the basic principle of EGR. The models that will be constructed in the next paragraphs and sections will simply demonstrate the existence of new types of dynamical evolution of the universe which do not exist in the standard cosmological model and which are based on EGR. In addition, most of the phenomenological approaches discussed in the literature invoke higher gauge symmetries beyond those observed at very low energy limit where all well-known symmetries are not broken at all energy levels. However, there are some arguments which conjecture emergent gauge theories in nature from a more fundamental theory with relatively different degrees of freedom and conceivably without an intrinsic gauge invariance, i.e., violation of diffeomorphism invariance. According to the Witten-Weinberg theorem, if gauge theories are explained from the emergence principle,
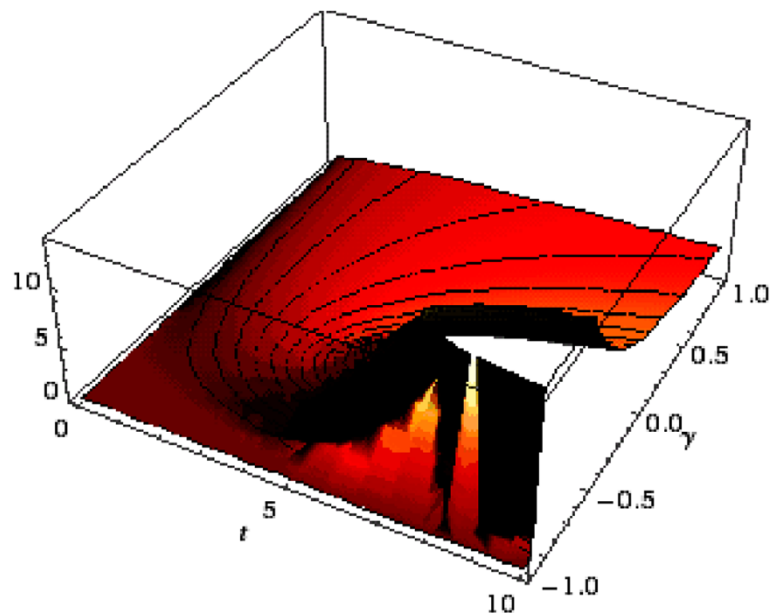
Figure 1 Numerical solution of $\ddot{q}(1+\xi\dot{q}^2)+\dot{q}\dot{\xi}/\xi+q(1-\xi\dot{q}^2)=0$ for $\xi = 1/t$, $\dot{q}(1)=1$, and $\dot{q}(1)=0$.

then spacetime and Lorentz invariance are emergent as well [28,29].

We consider the flat barotropic and isotropic FRW spacetime usually described by the metric $ds^2 = -N^2dt^2 + a(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$ where N is the lapse function and $a(t)$ is the scale factor of the universe. Assuming that the universe is filled with barotropic fluid with EoS $p = \gamma\rho$, $\gamma \in \mathbb{R}$ and ρ is the energy density of the barotropic fluid, the ADM Lagrangian formulation of the problem gives after simple mathematical algebra the following Lagrangian density [30]:

$$L = \frac{6a\dot{a}^2}{N} + 16\pi GNa^3\rho \qquad (3)$$

where G is the gravitational coupling constant. It is noteworthy that Equation 3 is obtained after writing the Einstein-Hilbert action which is a four-dimensional
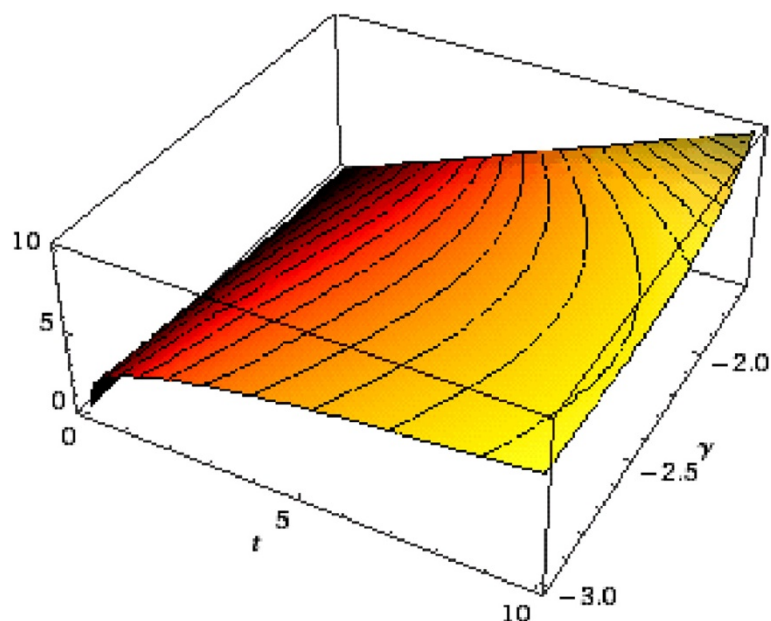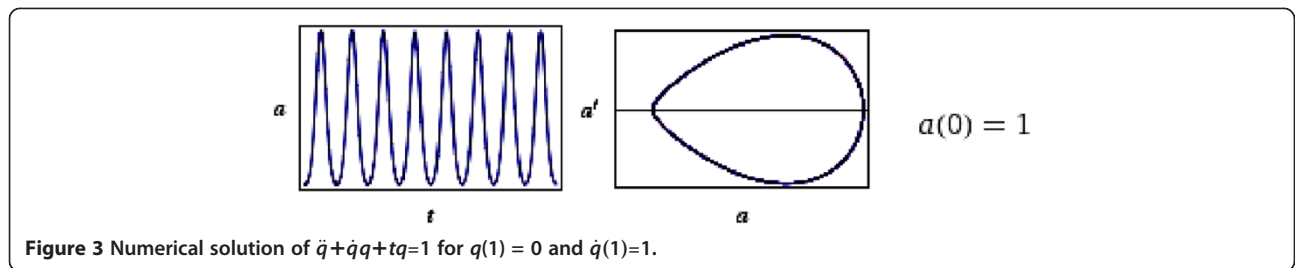

Figure 2 Numerical solution of $\ddot{q}(1+\xi\dot{q}^2)+\dot{q}\dot{\xi}/\xi+q(1-\xi\dot{q}^2)=0$ for $\xi = e^{-1/t}$, $q(1) = 0$, and $\dot{q}(1)=1$.

Figure 3 Numerical solution of $\ddot{q}+\dot{q}q+tq=1$ for $q(1)=0$ and $\dot{q}(1)=1$.

integration, then computing the Ricci scalar and the determinant of the metric, and afterward performing the integration by parts with respect to the time assuming that the geometry of the universe is the same everywhere. We refer the reader to [31] for the details of the calculation.

Remark 3. The FRW metric is a solution of EGR which is invariant under general coordinate transformations. One may ask naturally if it is meaningful to take a special solution of EGR corresponding to the Lagrangian (3) and to plug it into the nonrelativistic Equation 2. To make it clear, our approach is based on its main part on the intermediate step approach simply because the standard cosmological equations can be derived from the variational principle for the Einstein-Hilbert action. More precisely, the variation is made with respect to the scale factor and the lapse function which after that is replaced by the time-scale invariant condition [32,33]. There is no need in that case to extend the NSL formalism to EGR and relativistic quantum field theories and undoubtedly run into major difficulties like causality and superluminal motion [34].

Usually, the energy density is not constant and depends on the scale factor, i.e., $\rho = \rho(a)$. We can derive the corresponding Friedmann equations after applying Equation 2 respectively for $q = a$ and $q = N$ with the ensuing choice of the gauge $N = 1$. We obtain

$$8\pi G\rho + \frac{8\pi Ga}{3}\frac{d\rho}{da} - \frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} = \frac{2}{\xi}\frac{\dot{a}}{a}\frac{d\xi}{dt} + 12\xi\frac{\dot{a}^4}{a} + 96\xi\pi G\rho a\dot{a}^2$$

$$+ 32\xi\pi G\dot{a}^2 a^2\frac{d\rho}{da} + 24\xi\dot{a}^2\ddot{a}$$

$$(4)$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} \tag{5}$$

The matter content is assumed to be a perfect fluid, and accordingly, the covariant derivative of the stress-energy tensor $T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu}$ gives $\rho = \rho(a) = \rho_0 a^{-3(\gamma+1)}$;

ρ_0 is an integration constant which is set equal to $\rho_0 = 3/8\pi G$ for mathematical convenience. Equation 5 gives

$$a(t) = \left(\frac{3(1+\gamma)}{2}t\right)^{\frac{2}{3(1+\gamma)}} \tag{6}$$

where we have assumed that $a(t = 0) = 0$. This solution describes a power-law expansion of the universe. We plot in Figure 4 the evolution of the scale factor with time:

In order to find the value of the EoS parameter, we choose the ansatz $\xi = t^x, x \in \mathbb{R}$. The following constraint equation is then obtained from Equation 4:

$$\frac{1}{3(1+\gamma)}\left(\frac{2}{3(1+\gamma)} + x\right)t^{-2}$$

$$= 12\left(\left(\frac{3(1+\gamma)}{2}\right)^{-\frac{2(1+2\gamma)}{(1+\gamma)}} - \left(\frac{3(1+\gamma)}{2}\right)^{-\frac{2\gamma}{(1+\gamma)}}\left(\frac{2+3\gamma}{3(1+\gamma)}\right)\right)t^{\frac{2}{1+\gamma}-4+x}$$

$$(7)$$

This equation gives $x = \frac{2\gamma}{1+\gamma}$ for a consistent solution, and after replacing into Equation 7, we find $x \approx -4.6$ and $\gamma \approx -0.7$. This value of the EoS parameter is within the observational limits. The universe is then dominated by a vacuum energy and is accelerated in time. The energy density of the universe decays as $\rho \propto a^{-0.9}$. The results obtained here is interesting as it shows that the cosmic speedup may be obtained without implementing into Einstein's field equation exotic scalar fields. Note that in our approach, three different epochs in the history of the universe exist as well: matter-dominated epoch ($\gamma = 0$) with $a(t) = t^{2/3}$, radiation-dominated epoch ($\gamma = \frac{1}{3}$) with $a(t) = t^{1/2}$, and dark energy-dominated epoch with $a(t) = t^{2.2}$. For $\gamma = -1$ which corresponds to a vacuum-dominated epoch, it is easy to check from Equation 5 that the universe is inflating exponentially with time. We conclude that the universe is superaccelerating exponentially, and then its expansion is decelerating during the
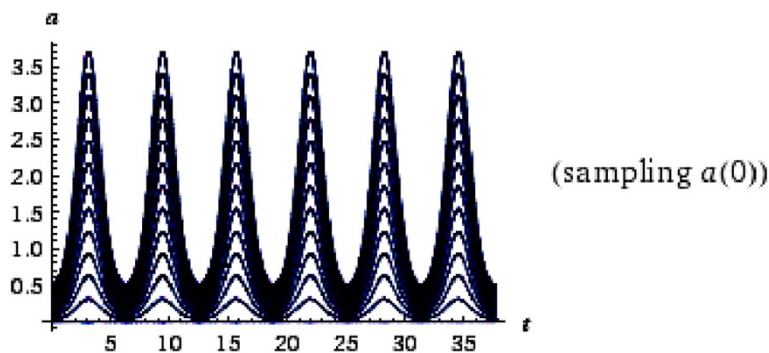
Figure 4 Evolution of *a(t)* for -1 < γ < 1 (Equation 6).

radiation- and matter-dominated epochs and afterward accelerating during the dark energy-dominated epoch.

In the presence of the cosmological constant Λ (in units $\hbar = c = 1$), the Lagrangian of the theory is

$$L = \frac{6a\dot{a}^2}{N} + 16\pi G N a^3 \rho + 2N a^3 \Lambda \qquad (8)$$

The equations of motion as deduced from the modified Euler-Lagrange Equation 2 are

$$8\pi G\rho + \frac{8\pi Ga}{3}\frac{d\rho}{da} - \frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} + \Lambda = \frac{2}{\xi}\frac{\dot{a}}{a}\frac{d\xi}{dt} + 12\xi\frac{\dot{a}^4}{a}$$
$$+ 96\xi\pi G\rho a\dot{a}^2 + 32\xi\pi G\dot{a}^2 a^2 \frac{d\rho}{da} + 24\xi\dot{a}^2\ddot{a} + 12\xi\Lambda a\dot{a}^2 \qquad (9)$$

and

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \qquad (10)$$

Now, with the help of $\rho = \rho(a) = \rho_0 a^{-3(\gamma+1)}$, Equation 10, and the chain rule $\frac{d\xi}{dt} = \dot{a}\frac{d\xi}{da}$, we can write Equation 9 as

$$-(1+3\gamma)a^{-3(\gamma+1)} + \frac{4\Lambda}{3} = \frac{2}{\xi}\left(a^{-2-3\gamma} - \frac{\Lambda}{3}a\right)\frac{d\xi}{da}$$
$$+ 12\xi\left(2\ddot{a} - 3\gamma a^{1-3(\gamma+1)} + \Lambda a\right)\left(a^{-1-3\gamma} - \frac{\Lambda}{3}a^2\right)$$
$$+ 12\xi\left(a^{-3-6\gamma} + \frac{\Lambda^2}{9}a^3 - \frac{\Lambda}{3}a^{-3\gamma}\right) + 2\frac{\ddot{a}}{a} \qquad (11)$$

We conjecture that $\xi(a) = ya^m, (y, m) \in \mathbb{R}$ and accordingly from Equation 11, we find

$$-(1+3\gamma)a^{-3(\gamma+1)} + \frac{4\Lambda}{3} = 2ma^{-3(\gamma+1)} - \frac{2m\Lambda}{3}$$
$$+ \left(24ya^{m-1-3\gamma} + 8y\Lambda a^{m+2} + 2a^{-1}\right)\ddot{a}$$
$$+ (-12\gamma + 12\gamma\Lambda + 8\Lambda)ya^{m-3\gamma}$$
$$+ 12(1-3\gamma)ya^{m-6\gamma-3} - \frac{8y\Lambda^2}{3}a^{m+3} \qquad (12)$$

In order to obtain a plausible solution, we need to get rid of the $\ddot{a}$ factor in Equation 12. We then find $\gamma = -1$, $m = -3$, and $y = -\frac{1}{4(3-\Lambda)}$. After replacing these parameters into Equation 12, we get $\Lambda \approx 3.5$. This scenario corresponds to the case of an eternal inflation where $a(t) = e^{(1+\Lambda/3)t} \approx e^{2t}$. The value of the EoS parameter $\gamma = -1$ corresponds to a vacuum energy density or a cosmological constant, and this implies an eternal inflation which generates diverging space-time volume [35,36].

These two illustrations show the significance of the time-dependent exponentially NSL in cosmology and which are not predicted in the standard model. The main differences between the first scenario and the second one are the following:

1. In the absence of the cosmological constant, the universe starts inflation, and after the end of inflation, the universe decelerates than accelerates, a situation which offers a solution to the singularity and trans-Planckian problems [37-39]. From the cosmological point of view, this can be problematic

since areas characterized by nonzero measure are required in order that ours will be an inflationary universe [40,41].

2. In the presence of the cosmological constant, the universe is eternally inflating and is in agreement with string theory landscape which refers to the large number of possible false vacua which is around 10^{500} [42,43]. Therefore, eternal inflation is not forbidden, but in contrast, it is regarded as an attractive feature since it may be formed in a merely single area of the universe.

Generalized coordinate-dependent exponential NSL: action, modified Euler-Lagrange equation, and modified FRW cosmology

For this second approach characterized by a generalized coordinate-dependent exponential NSL, we initiate by introducing the basic properties. We reconsider the admissible smooth Lagrangian function $L(\dot{q}, q, t) \in C^2$ $([a, b] \times R^n \times R^n; R)$. For the case of a generalized coordinate-dependent NSL $e^{\omega(q) L^{\varepsilon(t)}(\dot{q}(t), q(t), t)}$, we define the action functional of the theory by

$$S = \int_a^b e^{\omega(q) L^{\varepsilon(t)}(\dot{q}(t), q(t), t)} dt \qquad (13)$$

where $\omega(q)$ is an arbitrary q-dependent function and $\varepsilon(t)$ is a time-dependent function. The associated modified Euler-Lagrange equation is

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \dot{q}\frac{1}{\omega}\frac{d\omega}{dq}\frac{\partial L}{\partial \dot{q}} + \frac{1}{\varepsilon}\frac{d\varepsilon}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{1}{\omega\varepsilon}L\frac{d\omega}{dq}$$

$$- \frac{1}{L}\left(\frac{\partial L}{\partial t} + \dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}}\right)\frac{\partial L}{\partial \dot{q}}$$

$$+ \frac{\partial L}{\partial \dot{q}}\left(\frac{d\varepsilon}{dt}\ln L + \varepsilon\frac{1}{L}\left(\frac{\partial L}{\partial t} + \dot{q}\frac{\partial L}{\partial q} + \ddot{q}\frac{\partial L}{\partial \dot{q}}\right)\right)$$

$$+ \frac{\partial L}{\partial \dot{q}}\left(\dot{q}\frac{d\omega}{dq}L^{\varepsilon(t)} + \left(\frac{d\varepsilon}{dt}\ln L + \varepsilon\frac{1}{L}\left(\frac{\partial L}{\partial t} + \dot{q}\frac{\partial L}{\partial q}\right.\right.\right.$$

$$\left.\left.\left. + \ddot{q}\frac{\partial L}{\partial \dot{q}}\right)\right)L^{\varepsilon(t)}\right) \qquad (14)$$

Remark 4. The arguments discussed in Remark 1 concerning the violation of the energy conservation are applicable in that case as well.

In this work, we choose $\omega(q) = \frac{1}{2}q^2$; nevertheless, other choices are as well promising. For the case of the Lagrangian given by Equation 3, we can derive as well the resultant Friedmann equations after applying Equation 14 respectively for $q = a$ and $q = N$ with the ensuing choice of the gauge $N = 1$. We get

$$48\pi Ga^2\rho + 16\pi Ga^3\frac{d\rho}{da} - 12a\ddot{a} - 30\dot{a}^2 = 12a\dot{a}\frac{1}{\varepsilon}\frac{d\varepsilon}{dt}$$

$$- \frac{2}{\varepsilon}\left(6\dot{a}^2 + 16\pi Ga^2\rho\right) - \frac{12\dot{a}^2}{6\dot{a}^2 + 16\pi Ga^2\rho}\left(6\dot{a}^2 + 48\pi Ga^2\rho\right.$$

$$\left. + 16\pi Ga^3\frac{d\rho}{da} + 12a\ddot{a}\right) + 12a\dot{a}\left(\frac{d\varepsilon}{dt}\ln\left(6a\dot{a}^2 + 16\pi Ga^3\rho\right)\right.$$

$$\left. + \varepsilon\dot{a}\frac{6\dot{a}^2 + 48\pi Ga^2\rho + 16\pi Ga^3\frac{d\rho}{da} + 12a\ddot{a}}{6a\dot{a}^2 + 16\pi Ga^3\rho}\right)$$

$$+ 12a\dot{a}\left(6a\dot{a}^2 + 16\pi Ga^3\rho\right)^{\varepsilon(t)}\left(\dot{a}a + \frac{d\varepsilon}{dt}\ln\left(6a\dot{a}^2 + 16\pi Ga^3\rho\right)\right.$$

$$\left. + \varepsilon\dot{a}\frac{6\dot{a}^2 + 48\pi Ga^2\rho + 16\pi Ga^3\frac{d\rho}{da} + 12a\ddot{a}}{6a\dot{a}^2 + 16\pi Ga^3\rho}\right) \qquad (15)$$

and

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3}\frac{\varepsilon(t) + 2}{\varepsilon(t) - 2} \qquad (16)$$

With the help of $\rho = \rho(a) = \rho_0 a^{-3(\gamma + 1)}$ with $\rho_0 = 3/8\pi G$, we can simplify Equation 16 to

$$\dot{a}^2 = a^{-1-3\gamma}\frac{\varepsilon(t) + 2}{\varepsilon(t) - 2} \qquad (17)$$

At this stage, the number of solutions is large depending on the mathematical forms of $\varepsilon(t)$. We will pick some special forms and discuss their cosmological impacts. Let us choose first:

$$\varepsilon(t) = 2\frac{t^{2m} + 1}{t^{2m} - 1}, \quad m \in \mathbb{R} \qquad (18)$$

Accordingly, Equation 17 gives

$$a(t) = \frac{3(1 + \gamma)}{2(m + 1)}t^{\frac{2(m+1)}{3(1+\gamma)}} \qquad (19)$$

where we assumed $a(t = 0) = 0$. Cosmic acceleration occurs if $\gamma < \frac{2m-1}{3}$. For $m = -2$, we find $\varepsilon(t) \to -2$ for very large time, and from Equation 15, we get after a long but simple numerical algebra $\gamma \approx -1.5$. The universe is then dominated by phantom energy and is accelerating with time. The phantom divide line is crossed, and the energy density increases with time without the Big Rip singularity. This value is in agreement with recent astronomical observations [1-3] which set the value of the equation of the state parameter at $\gamma = -1.06^{+0.13}_{-0.08}$. We plot in Figure 5 the variations of the scale factor with cosmic time.

One more amazing illustration corresponds to
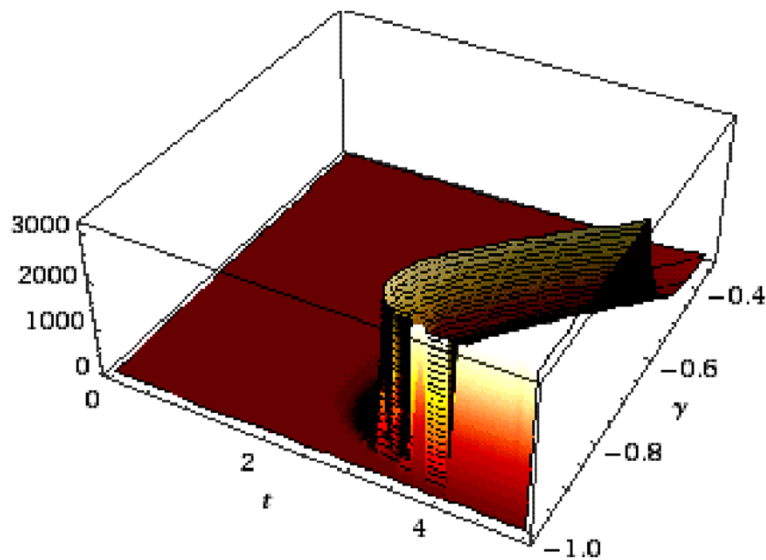
$$\varepsilon(t) = 2\frac{\cos^2 t + 1}{\cos^2 t - 1} \qquad (20)$$

Figure 5 Evolution of *a(t)* for -3 < γ < 5/3 (Equation 19).

From Equation (17), we find $a^{(1+3\gamma)/2}\dot{a} = \sin t$ which, after replacement into Equation 15, gives $\gamma \approx -1$ after simple algebra, i.e., $H = \dot{a}/a \approx \sin t$ and $a(t) \approx e^{\sin t}$. This particular case corresponds to an eternally oscillating universe characterized by a periodic Hubble parameter and dominated by vacuum energy. Similar cosmological evolutions but through totally different approaches were discussed in [44-46]. We plot in Figures 6, 7, and 8 the variation of Equation 20, the evolution of the scale factor in time, and a sample family solution, respectively.

A third interesting illustration corresponds to the case

$$\varepsilon(t) = 2\frac{e^{2t} + 1}{e^{2t} - 1} \qquad (21)$$

From Equation 17, we find

$$a(t) = \left(\frac{3(1+\gamma)}{2}(e^t - 1)\right)^{\frac{2}{3(1+\gamma)}}, \qquad (22)$$

and after substitution into Equation 15 gives $\gamma \approx -0.9$ after simple numerical algebra. This case corresponds to

an eternal inflation universe dominated by vacuum energy. We plot in Figures 9 and 10 the variations of Equation 21 in time and the evolution of the scale factor with time.

Oscillatory behavior of the universe may be obtained if, for instance, we choose

$$\varepsilon(t) = 2\frac{t^2 J_0^2(t) + 1}{t^2 J_0^2(t) - 1} \qquad (23)$$

where $J_n(z)$ is the Bessel function of the first kind. Its variation is plotted in Figure 11.

From Equation 17, we find after simple integration

$$a(t) = \left(\left(-\frac{3(1+\gamma)}{2}\right)\left(c - \frac{t^2}{2} {}_0\tilde{F}_1\left(;2,-\frac{t^2}{4}\right)\right)\right)^{\frac{2}{3(1+\gamma)}} \qquad (24)$$

where ${}_0\tilde{F}_1(;a,t)$ is the regularized hypergeometric function and c is a constant of integration. Assuming the initial condition $a(0) = 1$, we obtain $\gamma \approx -1$; therefore, Equation 24 is reduced to
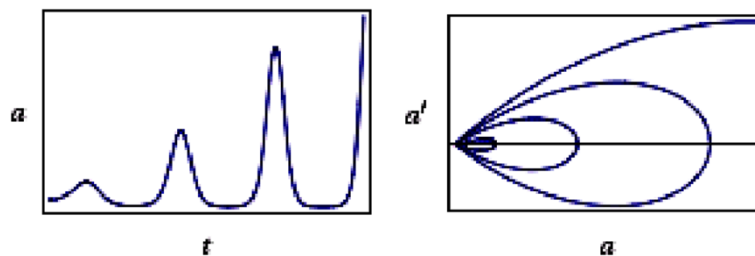

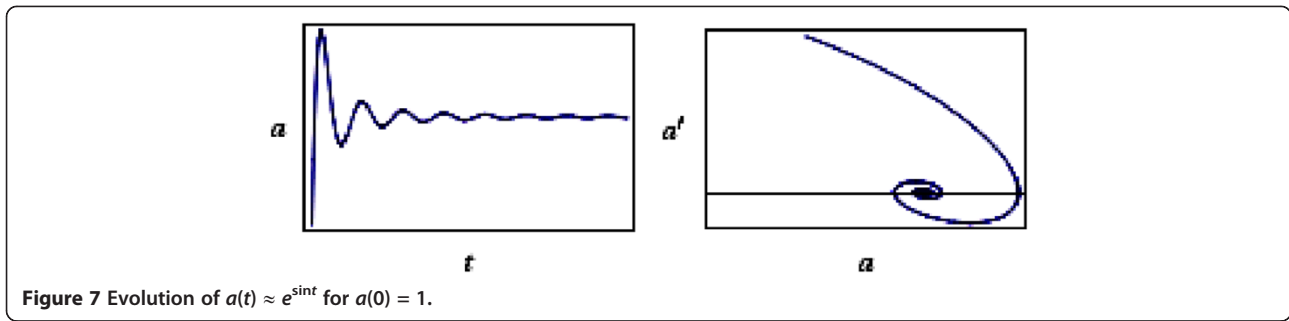**Figure 6 Variation of Equation 20 with cosmic time.**

Figure 7 Evolution of $a(t) \approx e^{\sin t}$ for $a(0) = 1$.

$$a(t) = e^{\frac{t^2}{2} \, _0\tilde{F}_1 \left(;2,-\frac{t^2}{4} \right)}$$

$$\underset{t \to \infty}{\approx} e^{\sin(t+\frac{\pi}{4}) \left(\frac{3\sqrt{\frac{1}{t}}}{4\sqrt{2\pi}} - \frac{105\left(\frac{1}{t}\right)^{\frac{5}{2}}}{512\sqrt{2\pi}} + \frac{72765\left(\frac{1}{t}\right)^{\frac{9}{2}}}{65536\sqrt{2\pi}} + O\left(\frac{1}{t}\right)^{\frac{13}{2}} \right) + \cos(t+\frac{\pi}{4}) \left(-\sqrt{\frac{2t}{\pi}} - \frac{15\left(\frac{1}{t}\right)^{\frac{3}{2}}}{64\sqrt{2\pi}} + + \frac{4725\left(\frac{1}{t}\right)^{\frac{7}{2}}}{8192\sqrt{2\pi}} - \frac{8513505\left(\frac{1}{t}\right)^{\frac{11}{2}}}{1048576\sqrt{2\pi}} + O\left(\frac{1}{t}\right)^{\frac{13}{2}} \right)}$$

(25)

This corresponds to an oscillating nonsingular scenario as shown in Figure 12.

One final model corresponds to

$$\varepsilon(t) = 2\frac{J_0^2(t) + t^2}{J_0^2(t) - t^2}$$

(26)

Its variation is plotted in Figure 13.

After performing the integration, we find

$$a(t) = \left(\left(-\frac{3(1+\gamma)}{2} \right) \left(\frac{1}{2} G_{1,3}^{2,0} \left(\left(\frac{t}{2} \right)^2 \Big| \begin{matrix} 1 \\ 0,0,0 \end{matrix} \right) + k \right) \right)^{\frac{2}{3(1+\gamma)}},$$

(27)

where k is an integration constant and $G_{p,q}^{m,n} \left(z \Big| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right)$

is the Meijer G-function defined by

$$G_{p,q}^{m,n} \left(z \Big| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=n+1}^{p} \Gamma(a_j - s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + s)} x^s ds$$

(28)

A numerical analysis shows that $\gamma \approx -1$. With the initial condition $a(0) = 1$, Equation 27 is then reduced to

$$a(t) = e^{\frac{1}{2} \left(G_{1,3}^{2,0} \left(\frac{1}{4} \Big| \begin{matrix} 1 \\ 0,0,0 \end{matrix} \right) - G_{1,3}^{2,0} \left(\left(\frac{t}{2} \right)^2 \Big| \begin{matrix} 1 \\ 0,0,0 \end{matrix} \right) \right)}$$

(29)

and its variations in time are plotted in Figure 14.

In the second and third illustrations, the universe is spatially flat, nonsingular, oscillating, and dominated by the vacuum energy. However, in the fourth illustration, the amplitude of the oscillations increases with time, whereas in the last (fifth) illustration, the amplitude of the oscillations decreases with time and the universe
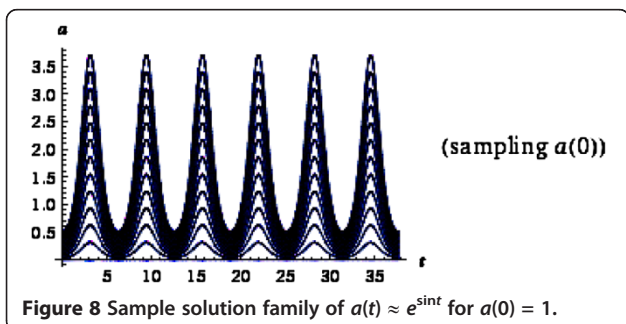

Figure 8 Sample solution family of $a(t) \approx e^{\sin t}$ for $a(0) = 1$.
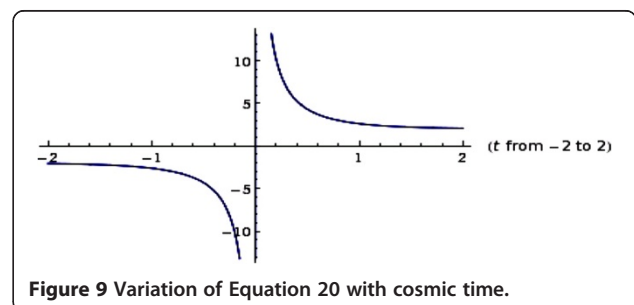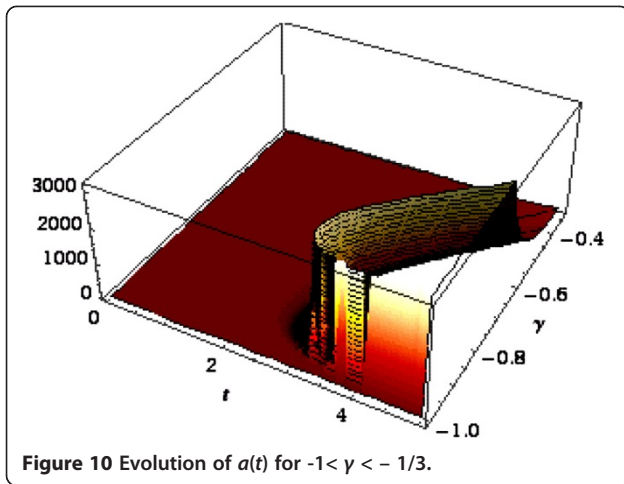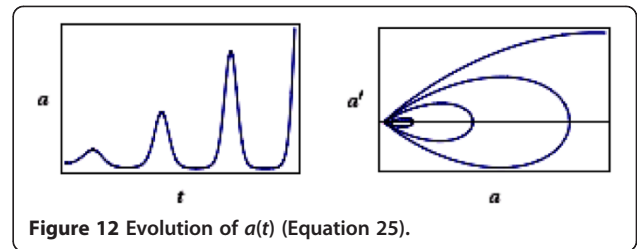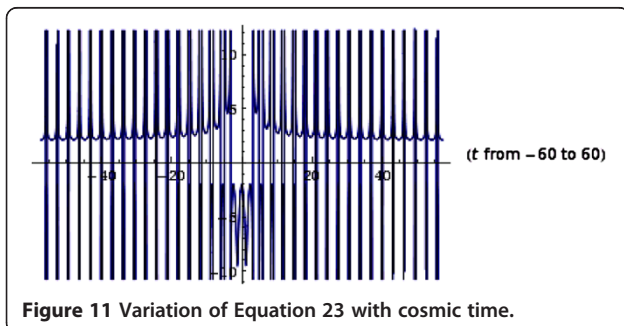

Figure 9 Variation of Equation 20 with cosmic time.

Figure 10 Evolution of *a*(t) for -1< γ < − 1/3.

tends toward a static spacetime. The universe in that case will stay stable eternally and will not end into a big crunch. This solution is amazing as it shows that the universe accelerates until some time, and then deceleration takes place until it reaches an asymptotic value which belongs to the class of Big Crunch or collapse of an oscillating universe [47-49]. This allows nonsingular emergent cosmological models to be constructed from nonstandard Lagrangians in which the universe starts oscillating with very large amplitude and tends toward a Big Crunch solution. The universe starts oscillating, but it may undergo a certain number of oscillations before it tunnels to the bounce point due to quantum mechanical effects and then expand forever.

Statefinder diagnostic

In order to have an improved thought of the major features of the independent models studied in this work, we can put them side by side with statefinder diagnostics (SFD) which is able to set apart between a wide variety of different models, including the Lambda cold dark matter (ΛCDM). In fact, the SFD introduces a pair of parameters $\{r, s\}$ as [50]

Figure 11 Variation of Equation 23 with cosmic time.

Figure 12 Evolution of *a*(t) (Equation 25).

$$\{r,s\} = \left\{ \frac{\ddot{a}}{aH^3}, \frac{r-1}{3(q-\frac{1}{2})} \right\} \qquad (30)$$

where $q = -\frac{\ddot{a}}{aH^2}$ is the deceleration parameter.

In fact, the statefinder parameters $\{r, s\}$ can be used to categorize between different models of dark energy [51-55]. It should be noted that for the quintessence-like model of dark energy, $s > 0$, whereas for the phantom-like model, $s < 0$. Besides, the evolution from phantom to quintessence or inverse is given by crossing of the fixed point $\{1, 0\}$ in the $\{r, s\}$ plane which corresponds to the ΛCDM cosmological model.

For the first model studied in this paper characterized by the scale factor (6), we obtain the following expressions for the statefinder parameters:

$$\{r,s\} = \left\{ \frac{(1+3\gamma)(2+3\gamma)}{2}, \frac{\gamma(\gamma+1)}{2+3\gamma} \right\} \qquad (31)$$

It is obvious that this model is able to obtain the ΛCDM phase of the universe, corresponding to the $r - s$ plane to the fixed point $\{1, 0\}$ for $\gamma = 0$ (pressureless matter) or $\gamma = -1$ (vacuum energy).

For the second model characterized by an eternal inflation, we have $\gamma = -1$ and $a(t) = e^{(1 + \Lambda/3)t} \approx e^{2t}$, and we get $\{r, s\} = \{1, 0\}$. For the third model, characterized by the scale factor (19), we find $\{r, s\} = \{3, -0.1\}$, i.e., $s < 0$ as the model corresponds to a phantom scenario, and for the fourth case characterized by $a(t) \approx e^{\sin t}$, we find
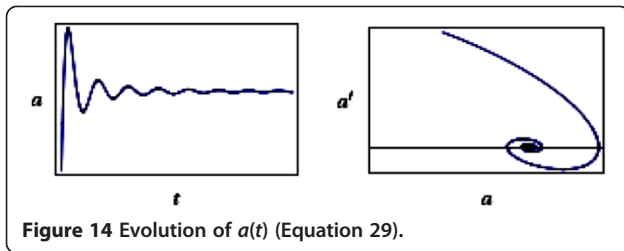

Figure 13 Variation of Equation 26 with cosmic time.
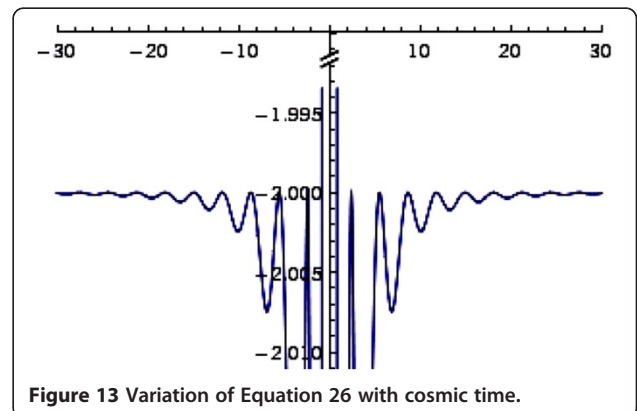
Figure 14 Evolution of *a(t)* (Equation 29).

$$\{r, s\} = \left\{ -\frac{\sin t (3 + \sin t)}{\cos^2 t}, \frac{1 + 3\sin t}{3\cos^2 t - 2\sin t} \right\} \quad (32)$$

Noticeably, the trajectories will never pass the ΛCDM point.

For the fifth model characterized by the scale factor (25), a long numerical analysis shows that at very large time $\{r, s\} \approx \{1, 0\}$, whereas for the sixth model, $\{r, s\} = \{\infty, -\infty\}$ which is in agreement with [56]. It is worth-mentioning that the accurate values of the statefinder parameters should be derived from the model-independent technique. In principle, these parameters can be extracted from some future astronomical observations, e.g., the SNAP-type experiment.

Conclusions

In summary, several solutions have been considered to address the problem of the cosmic speedup by modifying Einstein's gravitational theory or introducing new forms of exotic matter and interactions. In this communication, we have showed that time-dependent exponentially NSL are desirable approaches in cosmology. Two independent models were discussed accordingly.

The first cosmological model is a result of the modified Euler-Lagrange equation derived from an action function characterized by the time-dependent exponential Lagrangian $\xi(t)e^{L(\dot{q}(t),q(t),t)}$ with $\xi = t^x, x \in \mathbb{R}$. In the absence of the cosmological constant, it was observed that the universe is dominated by dark energy and its energy density decays as $\rho \propto a^{-0.9}$, whereas in the presence of Einstein's lambda, the universe is in the stage of eternal inflation which is in agreement with string theory landscape. In our approach, the first of the Friedmann Equations (5 and 10) are not altered, nevertheless the second Equations (4 and 9) are modified. These modified equations also modified the dynamics of the FRW cosmology with and without the presence of the cosmological constant.

The second cosmological model is a result of the modified Euler-Lagrange equation derived from an action function characterized by the coordinate-dependent exponential Lagrangian $e^{\omega(q)L^{\varepsilon(t)}(\dot{q}(t),q(t),t)}$ with $\omega(q) = \frac{1}{2}q^2$. We have discussed the cosmological dynamics for several forms of $\varepsilon(t)$. For the first form represented by Equation 18, it was observed that the universe is dominated by phantom energy and is accelerating with time. This is remarkable as the phantom divide line is crossed in that case and the energy density increases with time without Big Rip singularity. For the second form represented by Equation 20, it was realized that the universe is eternally oscillating and is dominated by vacuum energy. For the third form represented by Equation 21, the universe is eternally inflating and is dominated by vacuum energy. For the fourth form represented by Equation 23, we have obtained a nonsingular universe, oscillating in time, and dominated by vacuum energy. Finally, for the fifth form represented by Equation 26, the universe starts oscillating and is dominated by vacuum energy; however, the amplitude of its oscillations decreases with time and the universe undertakes a number of cosmic oscillations before it tunnels to the bounce point due to quantum mechanical effects and then expands eternally. In all these cases, the scale factor evolution, EoS, and statefinder parameters are calculated which facilitate to explore the accelerating universe. We have also displayed the scale factor and $\varepsilon(t)$ graphically using appropriate values of the EoS parameter to understand the behavior of the universe. In most of the models discussed in this paper, the EoS contains a negative sign which supports the acceleration of the universe.

In both NSL approaches introduced in this work, we have selected specific forms of $\xi(t)$ $\varepsilon(t)$ and $\omega(q)$ although other forms may be selected as well. The mathematical forms of these parameters show dissimilar phases of the accelerating universe depending upon them. It is natural to ask which forms describe the true universe. Our response is simple: we have tried in this paper to prove that NSL are not merely interesting to mathematicians but also must be taken seriously by physicists. Of course, a detailed analysis is required in order to select the best forms of the parameters $\xi(t)$ $\varepsilon(t)$ and $\omega(q)$. Nevertheless, the results obtained here show that NSL are motivating.

We are aware that energy violation of the classical mechanical approach characterized by a NSL resulted in a violation of the energy in general relativity. Nevertheless, the violation of energy conservation in general relativity theories is not innovative. Some works were done in [57-60], and many nice properties were raised accordingly. For example, in [58], it was observed that nonconservative gravitational field equations result in a cosmological model with a locally varying nonzero cosmological constant. In [60], many arguments were presented showing that the energy conservation is constantly violated in the standard FRW model of the universe even when it should not for asymptotically flat spacetime in a closed system and that such a violation can be associated analytically to the use of a radially focused FRW-based distance metric.

This is in contrast with what is observed in [61]. In our approach, the violation of energy came from NSL, not from quantum effects as it was revealed in [62] nor from negative energy photons [63].

The formalism introduced here may be applied to other cosmological spacetime models, e.g., anisotropic cosmological models. The physical meaning of any NSL necessitates an elucidation as we already know that any standard Lagrangian encodes most of the information of a given dynamics, i.e., classical, quantum, or cosmological. NSL are still in their early stages, and much work is required to be done in order to better understand their meanings and appreciate their roles. A different nonstandard approach to gravity exists already in the literature to model the dark sector of the universe; however, the approach constructed here is simpler and is based on classical formalism. It is notable that the number of exact cosmological solutions not based on scalar fields or exotic matters which are in agreement with astronomical observations is rather limited. The questions that arise are these: what is the physical meaning of NSL, and why can one obtain exact solutions starting from the modified Euler-Lagrange equations? We already know that the EGR belongs to the class of theories which are invariant with respect to reparametrization [64-67]. The Lagrangian (3) we choose is characterized by a lapse function which is set equal to 1 in our analysis. Restoring the gauge invariance condition or reparametrization by adding to the set of equations of motion of the Friedmann equation as a constraint related to this invariance may be done, but the whole issue still needs careful analysis. We stress in the end that our approach is characterized by breaking diffeomorphism invariance which represents an intricacy in the theory. However, this kind of violation which belongs to the class of emergent theory is still an open problem, and much work has already been done in this direction [68-74]; the issue is still under study.

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