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Determination of the amplitude-frequency for strongly nonlinear oscillator by two approximate analytical techniques

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Abstract

In this paper, we investigate two of the analytical approximate techniques, energy balance method and amplitude-frequency formulation, and these approximate techniques are applied to solve the strongly nonlinear differential equation of a mass attached to the center of a stretched elastic wire. We present a comparative study between the energy balance method and amplitude-frequency formulation with exact solution. The approximate results reveal that these methods are very effective and convenient for determining the frequencies of nonlinear dynamical systems.

Keywords: Nonlinear oscillator; Nonlinear differential equation; Energy balance method; Amplitude-frequency formulation; Exact solution

Introduction

Nonlinear phenomena play important roles in applied mathematics, physics and also in engineering problems in which each parameter varies depending on different factors. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts which are not simply understood through common observations. Moreover, obtaining exact solutions for these problems is a great purpose which has been quite untouched.

With the rapid development of nonlinear science, many different methods were proposed to solve various nonlinear problems, such as perturbation method, homotopy perturbation method, energy balance method, amplitude-frequency formulation, variational iteration method, variational approach method, etc. [1-16].

In this paper, we investigate two of the analytical approximate techniques, energy balance method and amplitude-frequency formulation, and these approximate techniques are applied to solve the strongly nonlinear differential equation of a mass attached to the center of a stretched elastic wire (Figure 1).

The differential equation of this dynamical system is in the following form [1]:

$$\frac{d^2}{dt^2}u(t)+u(t)-\frac{\lambda\cdot u(t)}{\sqrt{1+u(t)^2}}=0 \quad , \quad 0<\lambda\leq 1, \tag{1}$$

with initial conditions [1]:

$$u(0)=A, \frac{d}{dt}u(0). \tag{2}$$

This system oscillates between symmetric bounds $[-A, A]$, and its angular frequency and corresponding periodic solution are dependent on the amplitude A.

In this paper, our main purpose is to present a comparative study between the energy balance method and amplitude-frequency formulation with exact solution.

The description of energy balance method

In this section, we consider a general nonlinear oscillator in the following form [2]:

$$\frac{d^2}{dt^2}u(t)+f(u(t))=0, \tag{3}$$

in which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained:

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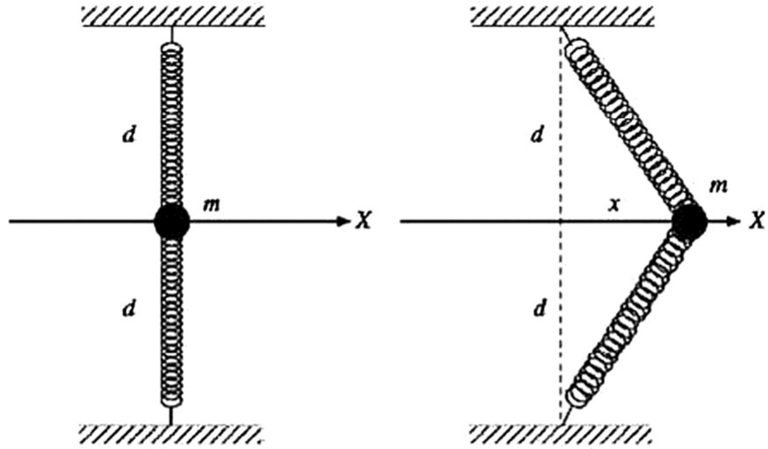

Figure 1 The oscillation of a mass attached to the center of a stretched elastic wire [1].

$$J(u(t)) = \int_0^t \left[-\frac{1}{2} \cdot \frac{d}{dt} u(t)^2 + F(u(t)) \right] dt, \tag{4}$$

where $T = 2\pi/\omega$ is the period of the nonlinear oscillator, $F(u(t)) = \int f(u(t))du$. Its Hamiltonian can be written in the following form:

$$\Delta H = \frac{1}{2} \cdot \frac{d}{dt} u(t)^2 + F(u(t)) = F(A) \tag{5}$$

or

$$R(t) = \frac{1}{2} \cdot \frac{d}{dt} u(t)^2 + F(u(t)) - F(A) = 0. \tag{6}$$

Oscillatory systems contain two important physical parameters, the frequency ω and the amplitude of oscillation, A. So let us consider such initial conditions:

$$u(0) = A, \frac{d}{dt} u(0) = 0. \tag{7}$$

We use the following trial function to determine the angular frequency ω:

$$u(t) = A\cos(\omega\, t). \tag{8}$$

Substituting Equation 8 into Equation 6, we obtain the following residual equation:

$$R(t) = \frac{1}{2} A^2 \omega^2 \sin^2(\omega\, t) + F(A\cos(\omega\, t)) - F(A) = 0. \tag{9}$$

If by chance the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Equation 8 is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives

$$\omega = \sqrt{\frac{2[F(A) - F(A\cos(\omega\, t))]}{A^2 \sin^2(\omega\, t)}}. \tag{10}$$

Its period can be written in the following form:

$$T = \frac{2\pi}{\sqrt{\frac{2[F(A) - F(A\cos(\omega\, t))]}{A^2 \sin^2(\omega\, t)}}}. \tag{11}$$

Therefore, we can obtain the following approximate solution:

$$u(t) = A\cos\left(\sqrt{\frac{2[F(A) - F(A\cos(\omega\, t))]}{A^2 \sin^2(\omega\, t)}} . t \right). \tag{12}$$

The description of amplitude-frequency formulation

In this section, we consider a generalized nonlinear oscillator in the following form [3]:

$$\frac{d^2}{dt^2} u(t) + f(u(t)) = 0, \tag{13}$$

with initial conditions:

$$u(0) = A, \frac{d}{dt} u(0) = 0. \tag{14}$$

For solving nonlinear differential equation by means of amplitude-frequency formulations, we use two trial functions in the following form:

$$u_1(t) = A\cos(\omega_1\, t) \tag{15}$$

and

Table 1 Comparison of the approximate frequencies with the exact frequencies when λ = 0.1

A	Energy balance method	Amplitude-frequency formulation	Exact solution
0.1	0.948880	0.948880	0.948881
1	0.961360	0.961106	0.961098
10	0.994166	0.993715	0.993713
100	0.999414	0.999363	0.999364

$$u_2(t) = A\cos(\omega_2\, t) \tag{16}$$

The residuals are

$$R_1(t) = -A\cos(\omega_1\, t) + f(\cos(\omega_1\, t)) \tag{17}$$

and

$$R_2(t) = -A\omega_2{}^2\cos(\omega_2\, t) + f(\cos(\omega_2\, t)). \tag{18}$$

The original frequency-amplitude formulation reads

$$\omega^2 = \frac{\omega_1{}^2 R_2(t) - \omega_2{}^2 R_1(t)}{R_2 - R_1}. \tag{19}$$

He used the following formulation [3]; Geng and Cai improved the formulation by choosing another location point [4]. In other words, He solved the problem at the point zero, and they discussed at $\pi/3$, and by this work, they improved the method.

$$\omega^2 = \frac{\omega_1{}^2 R_2(\omega_2\, t = 0) - \omega_2{}^2 R_1(\omega_1\, t = 0)}{R_2 - R_1} \tag{20}$$

This is the improved form by Geng and Cai:

$$\omega^2 = \frac{\omega_1{}^2 R_2\left(\omega_2\, t = \frac{\pi}{3}\right) - \omega_2{}^2 R_1\left(\omega_1\, t = \frac{\pi}{3}\right)}{R_2 - R_1}. \tag{21}$$

By considering $\cos(\omega_1 t) = \cos(\omega_2 t) = k$ and substituting the obtained ω into $u(t) = \cos(\omega t)$, we can obtain the constant k in ω^2 equation in order to have the frequency without irrelevant parameter. To improve its accuracy, we can use the following trial function when they are required:

$$u_1(t) = \sum_{i=1}^{m} A_i \cos(\omega_i\, \mathrm{t}), \quad \mathrm{u}_2(t) = \sum_{i=1}^{m} A_i \cos(\Omega_i\, \mathrm{t}), \tag{22}$$

or we can use

$$u_1(t) = \frac{\sum_{i=1}^{m} A_i \cos(\omega_i\, \mathrm{t})}{\sum_{j=1}^{n} B_j \cos(\omega_j\, \mathrm{t})}, \quad u_2(t) = \frac{\sum_{i=1}^{m} A_i \cos(\Omega_i\, \mathrm{t})}{\sum_{j=1}^{n} B_j \cos(\Omega_j\, \mathrm{t})} \tag{23}$$

However, in most cases because of the sufficient accuracy, trial functions are as follows and just the first term:

$$u_1(t) = A\cos t, \quad u_2(t) = a\cos(\omega\, t) + (A - a)\cos(3\omega\, t) \tag{24}$$

and

$$u_1(t) = A\cos t, \quad \mathrm{u}_2(t) = \frac{A(1+c)\cos(\omega\, \mathrm{t})}{1 + \mathrm{c}\cos(2\omega\, \mathrm{t})}, \tag{25}$$

where a and c are unknown constants. In addition, we can set $\cos(t) = k$ in $u_1(t)$ and $\cos(\omega t) = k$ in $u_2(t)$.

The application of energy balance method for the nonlinear oscillator

In this section, we consider the nonlinear equation in the following form [1]:

$$\frac{d^2}{dt^2}u(t) + u(t) - \frac{\lambda \cdot u(t)}{\sqrt{1 + u(t)^2}} = 0 \quad , \quad 0 < \lambda \le 1, \tag{26}$$

with initial conditions [1]:

$$u(0) = A, \frac{d}{dt}u(0) = 0. \tag{27}$$

Table 2 Comparison of the approximate frequencies with the exact frequencies when λ = 0.5

A	Energy balance method	Amplitude-frequency formulation	Exact solution
0.1	0.708424	0.708423	0.708423
1	0.788075	0.786524	0.786171
10	0.970480	0.968168	0.968102
100	0.9997067	0.996813	0.996812

Table 3 Comparison of the approximate frequencies with the exact frequencies when $\lambda = 0.75$

A	Energy balance method	Amplitude-frequency formulation	Exact solution
0.1	0.502788	0.502787	0.502786
1	0.656958	0.654164	0.652771
10	0.955378	0.951854	0.951696
100	0.995597	0.995215	0.995214

For this problem,

$$f(u(t)) = u(t) - \frac{\lambda \cdot u(t)}{\sqrt{1+u(t)^2}} \tag{28}$$

and

$$F(u(t)) = \frac{1}{2}u(t)^2 - \lambda\sqrt{1+u(t)^2}. \tag{29}$$

Its variational principle can be easily obtained:

$$J(u(t)) = \int_0^t \left(-\frac{1}{2}\cdot\frac{d}{dt}u(t)^2 + \frac{1}{2}u(t)^2 - \lambda\sqrt{1+u(t)^2}\right)dt. \tag{30}$$

Its Hamiltonian, therefore, can be written in the following form:

$$\begin{aligned}\Delta H &= \frac{1}{2}\cdot\frac{d}{dt}u(t)^2 + \frac{1}{2}u(t)^2 - \lambda\sqrt{1+u(t)^2} \\ &= \frac{1}{2}A^2 - \lambda\sqrt{1+A^2}\end{aligned} \tag{31}$$

or

$$\begin{aligned}R(t) = &\frac{1}{2}\cdot\frac{d}{dt}u(t)^2 \\ &+\frac{1}{2}u(t)^2 - \lambda\sqrt{1+u(t)^2} - \frac{1}{2}A^2 \\ &+\lambda\sqrt{1+A^2} \\ =& 0.\end{aligned} \tag{32}$$

Substituting Equation 8 into Equation 32, we obtain

$$\begin{aligned}R(t) &= \frac{1}{2}A^2\omega^2\sin^2(\omega\, t) + \frac{1}{2}A^2\cos^2(\omega\, t) - . \\ \lambda&\sqrt{1+A^2\cos^2(\omega\, t)} - \frac{1}{2}A^2 + \lambda\sqrt{1+A^2} = 0\end{aligned} \tag{33}$$

If we collocate at $\omega t = \pi/4$, we obtain the following result:

$$\omega_{EBM} = \sqrt{1 + \frac{2\lambda\sqrt{4+2A^2}}{A^2} - \frac{4\lambda\sqrt{1+A^2}}{A^2}}. \tag{34}$$

Its period can be written in the following form:

$$T_{EBM} = \frac{2\pi}{\sqrt{1 + \frac{2\lambda\sqrt{4+2A^2}}{A^2} - \frac{4\lambda\sqrt{1+A^2}}{A^2}}}. \tag{35}$$

The exact period is [1]

$$T_{Exact} = 4\int_0^{\frac{\pi}{2}} \left(1/\sqrt{1 - \frac{2\lambda}{\sqrt{1+A^2\sin^2 t} + \sqrt{1+A^2}}}\right)dt. \tag{36}$$

The application of amplitude-frequency formulation for the nonlinear oscillator

In this section, we consider the nonlinear equation in the following form [1]:

$$\frac{d^2}{dt^2}u(t) + u(t) - \frac{\lambda\cdot u(t)}{\sqrt{1+u(t)^2}} = 0 \quad , \quad 0 < \lambda \le 1, \tag{37}$$

with initial conditions [1]:

$$u(0) = A, \frac{d}{dt}u(0) = 0. \tag{38}$$

For small values of A, we can write

$$\frac{1}{\sqrt{1+u(t)^2}} = 1 - \frac{1}{2}u(t)^2. \tag{39}$$

We can write nonlinear equation in the following form:

Table 4 Comparison of the approximate frequencies with the exact frequencies when $\lambda = 0.95$

A	Energy balance method	Amplitude-frequency formulation	Exact solution
0.1	0.231391	0.231388	0.231367
1	0.529168	0.524765	0.520335
10	0.943122	0.938597	0.938333
100	0.994420	0.993936	0.993934

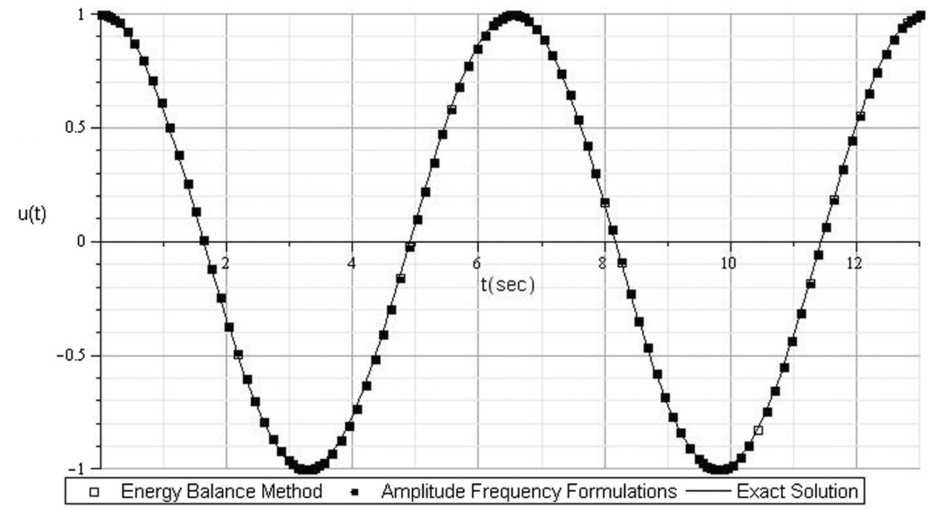

Figure 2 Comparison of energy balance method and amplitude-frequency formulation with exact solution when *A* = 1, *λ* = 0.1.

$$\frac{d^2}{dt^2}u(t)+(1-\lambda)u(t)+\frac{1}{2}\lambda u(t)^3=0 \tag{40}$$

by using two trial functions in the following form:

$$u_1(t)=A\cos(t) \tag{41}$$

and

$$u_2(t)=A\cos(\omega\, t). \tag{42}$$

For Equation 40, we obtain the following residuals:

$$R_1(t)=-\lambda Ak+\frac{1}{2}\lambda A^3k^3. \tag{43}$$

By simple calculation, we obtain

$$R_2(t)=-3Ak-\lambda Ak+\frac{1}{2}\lambda A^3k^3. \tag{44}$$

With considering $\cos(\omega_1 t) = \cos(\omega_2 t) = k$, we have

$$\omega^2=\frac{{\omega_1}^2R_2-{\omega_2}^2R_1}{R_2-R_1} \tag{45}$$

$$\omega^2=\frac{-3Ak-\lambda Ak+\frac{1}{2}\lambda A^3k^3-\omega^2\left(-\lambda Ak+\frac{1}{2}\lambda A^3k^3\right)}{\left(-3Ak-\lambda Ak+\frac{1}{2}\lambda A^3k^3\right)-\left(-\lambda Ak+\frac{1}{2}\lambda A^3k^3\right)}. \tag{46}$$

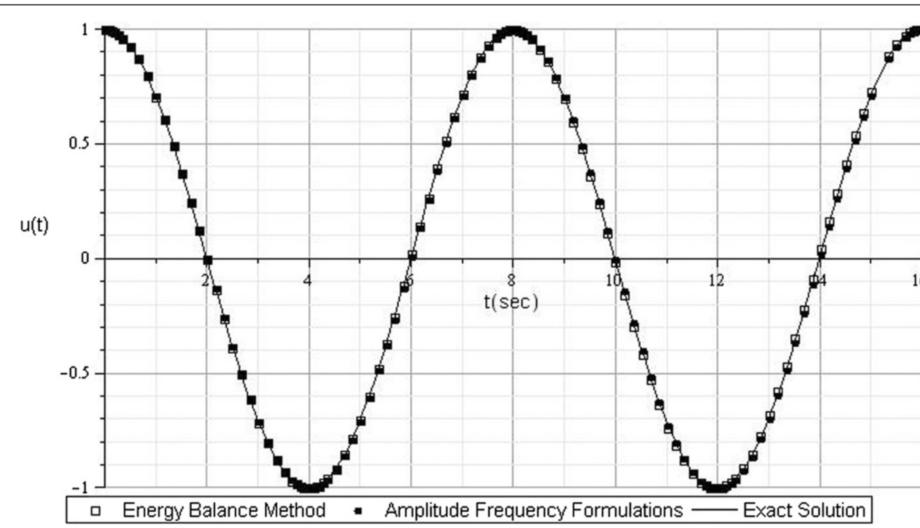

Figure 3 Comparison of energy balance method and amplitude-frequency formulation with exact solution when *A* = 1, *λ* = 0.5.

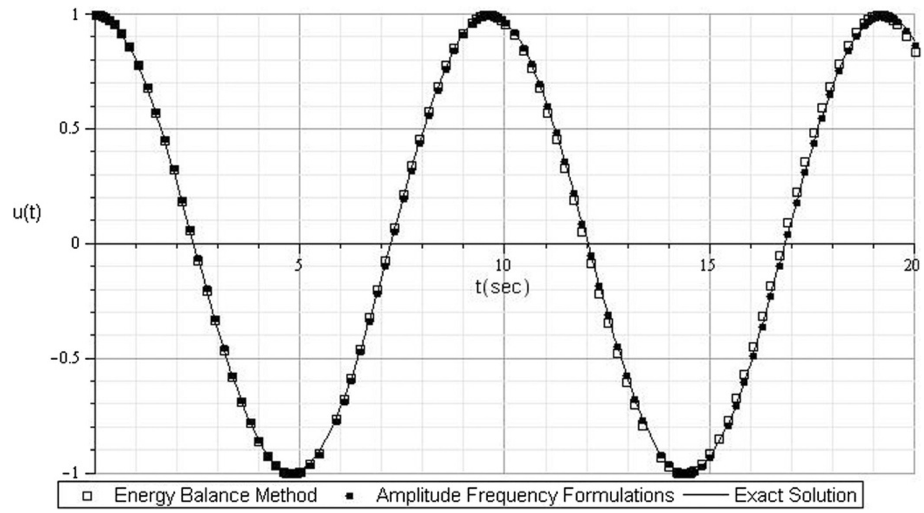

Figure 4 Comparison of energy balance method and amplitude-frequency formulation with exact solution when $A = 1$, $\lambda = 0.75$.

Therefore, we have

$$\omega_{AFF} = \sqrt{1-\lambda + \frac{1}{2}\lambda A^2 k^2}. \quad (47)$$

We can rewrite $u(t) = A\cos(\omega t)$ in the following form:

$$u(t) = A\cos\left(\sqrt{1-\lambda + \frac{1}{2}\lambda A^2 k^2}\cdot \mathrm{t}\right). \quad (48)$$

We can rewrite the main equation in the following form:

$$\frac{d^2}{dt^2}u(t) + \left((1-\lambda) + \frac{1}{2}\lambda A^2 k^2\right)u(t) = \frac{1}{2}\lambda A^2 k^2 u(t) - \frac{1}{2}\lambda u(t)^3. \quad (49)$$

The right side of Equation 49 vanishes completely:

$$B = \int_0^{\frac{T}{4}}\left(\frac{1}{2}\lambda A^2 k^2 u(t) - \frac{1}{2}\lambda u(t)^3\right)(\cos(\omega\ \mathrm{t}))\mathrm{dt} = 0 \quad , \quad \mathrm{T} = \frac{2\pi}{\omega} \quad (50)$$

$$B_1 = \int_0^{\frac{T}{4}}\left(\frac{1}{2}\lambda A^2 k^2 A\cos(\omega\ \mathrm{t})\right)(\cos(\omega\ \mathrm{t}))\mathrm{dt} = \frac{1}{2}\lambda A^3 k^2 \frac{\pi}{4} \quad (51)$$

$$B_2 = \int_0^{\frac{T}{4}}\left(-\frac{1}{2}\lambda(A^3\cos^3(\omega\ \mathrm{t})\right)(\cos(\omega\ \mathrm{t}))\mathrm{dt} = -\frac{1}{2}\lambda A^3 \frac{3\pi}{16} \quad (52)$$

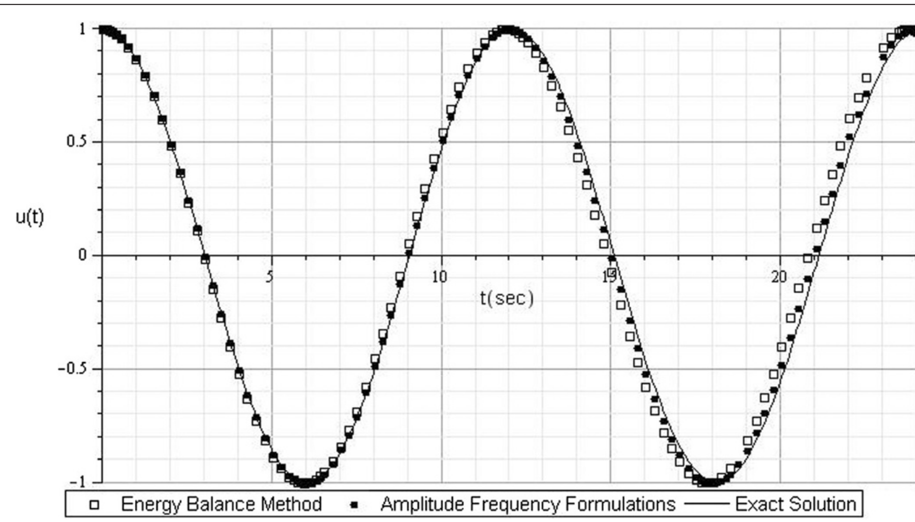

Figure 5 Comparison of energy balance method and amplitude-frequency formulation with exact solution when $A = 1$, $\lambda = 0.95$.

So, we have

$$B = B_1 + B_2 \;\rightarrow\; \frac{1}{2}\lambda A^3 k^2 \frac{\pi}{4} - \frac{1}{2}\lambda A^3 \frac{3\pi}{16} = 0. \tag{53}$$

With solving and simplifying Equation 53, we have

$$k^2 = \frac{3}{4}. \tag{54}$$

With substituting Equation 54 into Equation 47, we have approximate frequency in the following form:

$$\omega_{AFF} = \sqrt{1-\lambda + \frac{3}{8}\lambda A^2}. \tag{55}$$

Its period can be written in the following form:

$$T_{AFF} = \frac{2\pi}{\sqrt{1-\lambda + \frac{3}{8}\lambda A^2}}. \tag{56}$$

The comparison of the approximate frequencies with exact frequencies

To illustrate the accuracy of the energy balance method and amplitude-frequency formulation, we present the comparison results of analytical approximate techniques with exact solution in Tables 1, 2, 3 and 4 and Figures 2, 3, 4 and 5 for different values of λ.

Conclusions

In this paper, we investigated and applied two of the analytical approximate techniques, energy balance method and amplitude-frequency formulation, for solving the strongly nonlinear differential equation of a mass attached to the center of a stretched elastic wire.

To illustrate the accuracy of the energy balance method and amplitude-frequency formulation, we presented a comparative study between the analytical approximate techniques with exact solution. The approximate results reveal that these methods are very effective and convenient for determining the frequencies of nonlinear dynamical systems.

Competing interests

Both authors declare that they have no competing interests.

Authors' contributions

AA conceived of the study and participated in drafted the manuscript. HEK carried out the software works and the solution of equations by analytical approximate techniques and participated in drafting the manuscript. Both authors read and approved the final manuscript.

Authors' information

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