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Multi-lump solutions to the KPI equation with a zero degree of derivation

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Abstract

We construct solutions to the Kadomtsev-Petviashvili equation (KPI)by using an extended Darboux transform. From elementary functions we give a method that provides different types of solutions in terms of wronskians of order N. For a given order, these solutions depend on the degree of summation and the degree of derivation of the generating functions. In this study, we restrict ourselves to the case where the degree of derivation is equal to 0. In this case, we obtain multi-lump solutions and we study the patterns of their modulus in the plane (x,y) and their evolution according time and parameters.

Keywords

Kadomtsev-Petviashvili equation; Wronskians; Lump; Extended Darboux transform.

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1. Introduction

We consider the Kadomtsev-Petviashvili equation (KPI) in the following form

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

where subscripts x, y and t denote partial derivatives.
Kadomtsev and Petviashvili [1] proposed first this equation in 1970. This equation is a universal model describing weakly nonlinear waves in media with dispersion of velocity. For example, it is used for surface and internal water waves [2,3], and in nonlinear optics [4]. Zakharov extended the inverse scattering transform (IST) to this KPI equation, and obtained several exact solutions.
A lump solution of the KPI equation was first constructed by Petviashvili [5] in 1976. The first rational solutions were found in 1977 by Manakov, Zakharov, Bordag and Matveev [6]. Krichever [7,8] showed that the dynamics of the lumps of the KPI equation is governed by the Calogero-Moser system. Other researches were led and more general solutions of the KPI equation were obtained. We can mention the following works of Satsuma and Ablowitz in 1979 [9], Matveev in 1979 [10], Freeman and Nimmo in 1983 [11,12], Ablowitz and Villarroel [13,14] in 1997-1999, Biondini and Kodama [15–17] in 2003-2007.
We construct solutions of the KPI equation in terms of a second derivative with respect to x of a logarithm of a wronskian of order N. For any order N the solutions depend in the degree of summation S and the degree of derivation D. So these N order solutions depend in general on $N(S(D+5)+1)$ real parameters. In the following, we restrict ourselves to the case where the order N is equal to 1 and the degree of derivation D equal to 0. In this case, we get lines of multi lump solutions to the KPI equation.
We construct some explicit solutions of order 1, depending on several real parameters, and give the representations of their modulus in the plane of the coordinates (x,y) according to parameters and time t.

2. Multiparametric solutions of order N to the KPI equation

We consider the Kadomtsev-Petiaviasvili (KPI) equation

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

We will use in the following the wronskian of order N of the functions $f_1,\dots,f_N$ which is the determinant denoted $W(f_1,\dots,f_N)$, defined by $\det(\partial_x^{i-1} f_j)_{1\le i\le N,1\le j\le N}$, ∂_x^i being the partial derivative of order i with respect to x and $\partial_x^0 f_j$ being the function f_j.
We consider a_j, b_{jsd}, d_{js}, e_{js},g_{js}, h_{js}, arbitrary real numbers with $c_{js} = g_{js} + ih_{js}$.
We consider the following elementary functions f_{js} defined by

$$f_{js}(x,y,t) = e^{ic_{js}x - ic_{js}^2 y + ic_{js}^3 t + d_{js} + ie_{js}}$$
$$= e^{i(g_{js}+ih_{js})x - i(g_{js}+ih_{js})^2 y + i(g_{js}+ih_{js})^3 t + d_{js} + ie_{js}}. \quad (2)$$

Then, we have the following statement;
Theorem 2.1: Let ψ_j, be the functions defined by

$$\psi_j(x,y,t) = a_j + \sum_{s=1}^{S}\sum_{d=0}^{D} b_{jsd}\partial_{c_{js}}^d e^{ic_{js}x - ic_{js}^2 y + ic_{js}^3 t + d_{js} + ie_{js}}$$

$$= a_j + \sum_{s=1}^{S} \sum_{d=0}^{D} b_{jsd} \partial_{c_{js}}^{d} f_{js}(x,y,t). \quad (3)$$

Then the function u defined by

$$u(x,y,t) = -2\partial_x^2 \ln(W(\psi_1(x,y,t),\ldots,\psi_N(x,y,t))) \quad (4)$$

is a solution to the (KPI) equation (1) depending on real $N(S(D+5)+1)$ parameters a_j, $b_{j,s,d}$, $g_{j,s}$, $h_{j,s}$, $d_{j,s}$, $e_{j,s}$, $1 \leq j \leq N$, $1 \leq s \leq S$, $0 \leq d \leq D$.

Proof: The corresponding Lax pair to the (KPI) equation (1)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0,$$

is given by

$$\begin{cases} \phi_t = -\phi_{3x} + \frac{3}{2}u\phi_x + v\phi, \\ \phi_y = i\phi_{xx} - iu\phi. \end{cases} \quad (5)$$

Each solution ϕ to this system gives a solution u to the KPI equation. This system is known to be covariant by the Darboux transformation. If $\phi_1,\ldots,\phi_N,\phi$ are solutions of the system (5) respectively associated to u and v, then $\phi[N]$ defined by $\phi[N] = \frac{W(\phi_1,\ldots,\phi_N,\phi)}{W(\phi_1,\ldots,\phi_N)}$ is another solution of this system (5) where in particular u replaced by $u[N] = u - 2\ln W(\phi_1,\ldots,\phi_N)_{xx}$ giving a new solution to the KPI equation.

If we choose $u = 0$ and $v = 0$ then the functions ψ_j defined in Ref. [3] verify the following system

$$\begin{cases} \psi_t = -\psi_{3x}, \\ \psi_y = i\psi_{xx}. \end{cases} \quad (6)$$

Then the solution of the system (6) associated can be written as $\varphi(x,y,t) = \frac{W(\psi_1,\ldots,\psi_N,\psi)}{W(\psi_1,\ldots,\psi_N)}$ and u_N can be expressed as

$$u[N] = -2(\ln W(\psi_1,\ldots,\psi_N))_{xx},$$

which proves the result.

We will call the order N of the wronskian, the order of the solution. The number S of terms of the summation will be call the degree of the summation of the solution and D the degree of derivation of the solution.

For each order N of the solution there is a lot of choices concerning the degrees of the summation and derivation.

In the following, we restrict ourselves to the study where $D = 0$.

3. Solutions of order 1 with a degree of summation equal to 1 ($S = 1$) depending on 6 real parameters

In this case, we observe lines of lumps whose intensities and directions depend on 6 real parameters. These structures are more sensitive to g_1 and h_1 parameters than to others. If a_1 or b_1 is equal to 0, we get the trivial solution equal to 0.

Solutions to the KPI equation can be written as

$$v(x,y,t) = -2\frac{n(x,y,t)}{d(x,y,t)^2} \quad (7)$$

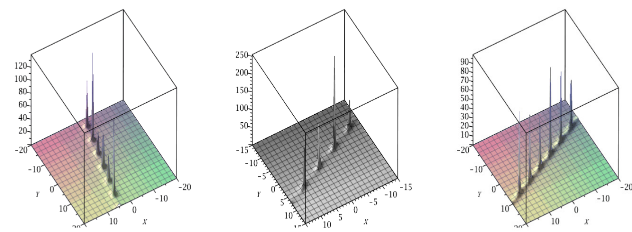

Figure 1. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 0.1$, $h_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$; in the center for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 0.9$, $d_{1,1} = 1$, $e_{1,1} = 1$; on the right for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 0,1$.

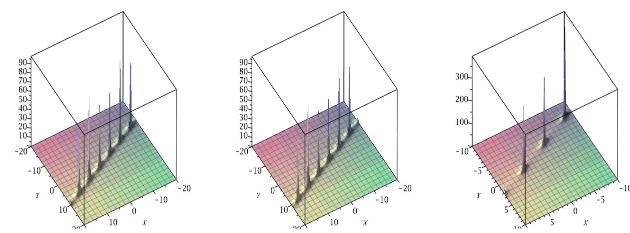

Figure 2. Solution of order 1 to KPI, on the left for $t = 0$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 0.9$, $d_{1,1} = 1$, $e_{1,1} = 1$; on the right for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 1$, $d_{1,1} = 10$, $e_{1,1} = 10$.

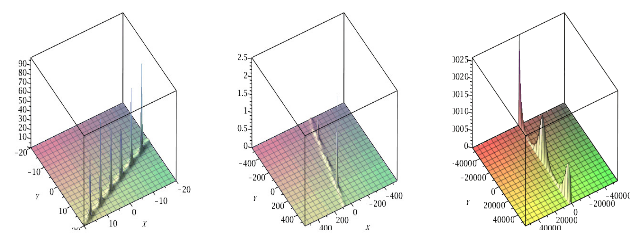

Figure 3. Solution of order 1 to KPI, on the left for $t = 10$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 0.1$, $h_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 1$; in the center for $t = 10^2$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 0.9$, $d_{1,1} = 1$, $e_{1,1} = 1$; on the right for $t = 10^3$, $a_1 = 1$, $b_{1,1,0} = 1$, $g_{1,1} = 1$, $h_{1,1} = 1$, $d_{1,1} = 1$, $e_{1,1} = 0,1$.

with

$$n(x,y,t) = b_{1,1,0}(ig_{1,1} - h_{1,1})^2$$

$$\times \exp(ig_{1,1}x - h_{1,1}x - iyg_{1,1}{}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}{}^2$$

$$+itg_{1,1}{}^3 - 3tg_{1,1}{}^2h_{1,1} - 3itg_{1,1}h_{1,1}{}^2 + th_{1,1}{}^3 + d_{1,1} + ie_{1,1})a_1.$$

and

$$d(x,y,t) = b_{1,1,0}$$

$$\times \exp(-h_{1,1}x+ig_{1,1}x-3tg_{1,1}{}^2h_{1,1}-3itg_{1,1}h_{1,1}{}^2+itg_{1,1}{}^3$$

$$+2yg_{1,1}h_{1,1}+iyh_{1,1}{}^2+th_{1,1}{}^3-iyg_{1,1}{}^2+d_{1,1}+ie_{1,1})+a_1.$$

We give figures 1-3 in the (x,y) plane of coordinates. If $a_1=0$ or $b_{110}=0$, we get the trivial solution $u=0$. Otherwise, we remark that parameters a_1 or b_{110} are negligible and plays very little role in the structure of solutions. So if these parameters are not equal to 0 we fix them equal to 1.

4. Solutions of order 1 with a degree of summation equal to 2 ($S=2$) depending on 11 real parameters

In this case, we observe lines of lumps but contrary to the order 1, with a bifurcation in two branches whose intensities and directions depend on 11 real parameters. These structures are more sensitive to g_i and h_i parameters than to others. A generic solution can be considered as two lumps chains meeting in a double point; we get also a whole set of degenerated configurations. In this case, we take $N=1$, $S=2$ and the solutions to the KPI equation can be written as

$$v(x,y,t)=-2\frac{n(x,y,t)}{d(x,y,t)^2} \quad (8)$$

Details of $n(x,y,t)$ and $d(x,y,t)$ are presented in appendix 1. We give some figures 4 to 7 in the (x,y) plane of coordinates. There is a formation of multi-lines of lumps meeting in a same point which varies according to the values of the parameters. A maximum of three lines has been observed.

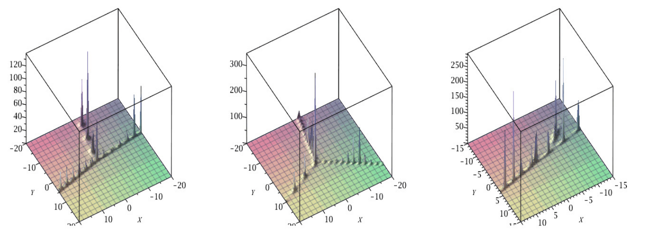

Figure 4. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=0.1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$; in the center for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=0.1$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$; on the right for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=0.5$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$.

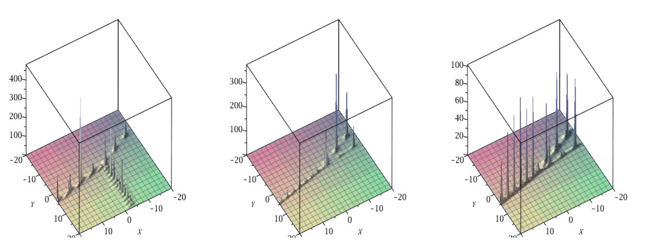

Figure 5. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=0.5$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$; in the center for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=0.1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$; on the right for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=0.5$, $e_{1,1}=1$, $e_{1,2}=2$.

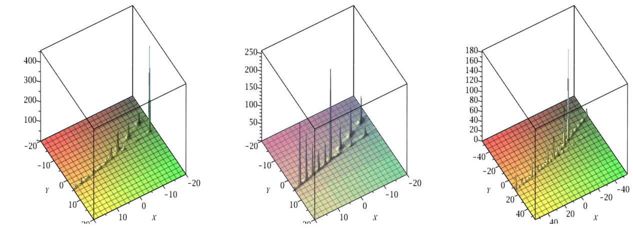

Figure 6. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=0.1$, $e_{1,2}=2$; in the center for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=0.1$; on the right for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$.

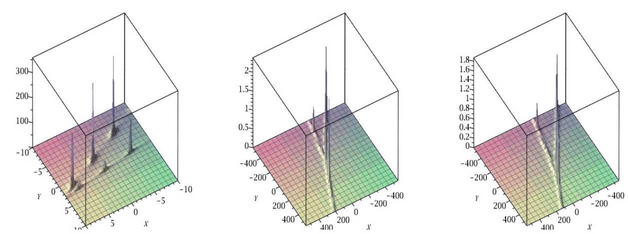

Figure 7. Solution of order 1 to KPI, on the left for $t=1$, $a_1=10^2$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $g_{1,1}=1$, $g_{1,2}=2$, $h_{1,1}=1$, $h_{1,2}=2$, $d_{1,1}=10$, $d_{1,2}=20$, $e_{1,1}=10$, $e_{1,2}=20$; in the center for $t=10^2$, $a_1=10^2$, $b_{1,1,0}=10$, $b_{1,2,0}=20$, $g_{1,1}=0.1$, $g_{1,2}=0.2$, $h_{1,1}=0.1$, $h_{1,2}=0.2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$; on the right for $t=10^3$, $a_1=10^3$, $b_{1,1,0}=10$, $b_{1,2,0}=20$, $g_{1,1}=0.1$, $g_{1,2}=0.2$, $h_{1,1}=0.1$, $h_{1,2}=0.2$, $d_{1,1}=1$, $d_{1,2}=2$, $e_{1,1}=1$, $e_{1,2}=2$.

Remark

Sub-particular case of solutions of order 1 without derivative ($D=0$) with a degree of summation equal to 2 (S=2)

We can choose for the functions f_j, the functions defined by $f_j=e^{ic_jx-ic_j^2y+ic_j^3t+ie_j}$ with c_j, e_j real numbers for $j=1$ and $j=2$, $a_1=0$ and $b_{110}=b_{120}=b_1$.
then the function $\psi_1(x,y,t)$ can be written as

$$\psi_1(x,y,t)=b_1(e^{ic_1x-ic_1^2y+ic_1^3t+ie_1}+e^{ic_2x-ic_2^2y+ic_2^3t+ie_2})$$

$$=2b_1e^{i(c_2+c_1)x-i(c_2^2+c_1^2)y+(c_2^3+ic_1^3)t+i(e_2+e_1)}$$

$$\times\cosh(i(c_2-c_1)x-i(c_2^2-c_1^2)y+i(c_2^3-c_1^3)t+i(e_2-e_1))$$

So the solution to the KPI equation can be written as

$$a(x,y,t) = -2(\ln \psi_1(x,y,t))$$

$$= \frac{-2((c_2-c_1)^2)}{\cosh(i(c_2-c_1)x - i(c_2^2-c_1^2)y + i(c_2^3-c_1^3)t + i(e_2-e_1))}$$

5. Solutions of order 1 with a degree of summation equal to 3 ($S=3$) depending on 16 real parameters

In this case, we observe lines of lumps but contrary to order 1 and 2, with a bifurcation in three branches whose intensities and directions depend on the 16 real parameters. These structures are more sensitive to g_i and h_i parameters than to others. A generic solution can be considered as three lumps chains meeting in a triple point; we get also a whole set of degenerated configurations. In this section, we take N equal to 1 and $S=3$. The general expression with all the parameters being too complex is given in appendix. We give here only this with some parameters values to simplify. For example we choose :
$a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=2$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$.
So, the solutions to the KPI equation can be written as

$$v(x,y,t) = -2\frac{n(x,y,t)}{d(x,y,t)^2}. \quad (9)$$

with

$$n(x,y,t) = -2i(12\exp(4ix-4x+20y-56it+4+4i-56t)$$
$$+27\exp(3ix-3x+18y-54it-54t+3+3i)$$
$$+\exp(ix-x+2y-2it-2t+1+i)$$
$$+8\exp(2ix-2x+8y-16it-16t+2+2i)$$
$$+6\exp(5ix-5x+26y-70it+5+5i-70t)$$
$$+2\exp(3ix-3x+10y-18it+3+3i-18t))$$

and

$$d(x,y,t) = \exp(-x+ix-2t-2it+2y+1+i)$$
$$+2\exp\exp(-2x+2ix-16t-16it+8y+2+2i)$$
$$+3\exp(-3x+3ix-54t-54it+18y+3+3i)+1.$$

We give figures 8-11 in the (x,y) plane of coordinates.

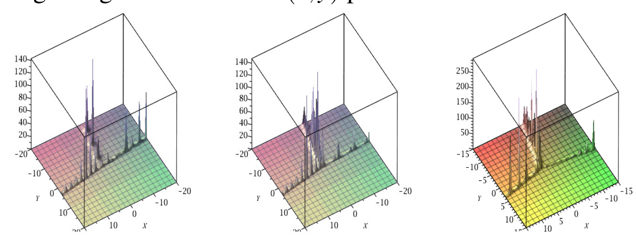

Figure 8. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=0.1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=2$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$; in the center for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=0.1$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=2$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$; on the right for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=0.1$, $h_{1,1}=1$, $h_{1,2}=2$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$.

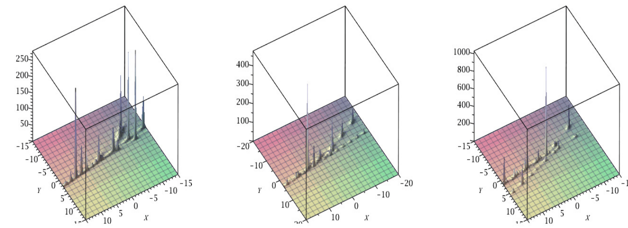

Figure 9. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=0.5$, $h_{1,2}=2$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$; in the center for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=0.5$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$; on the right for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=2$, $h_{1,3}=0.7$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$.

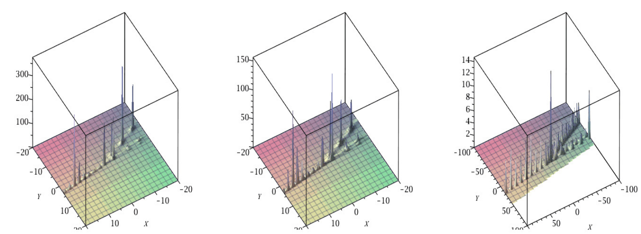

Figure 10. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=0.5$, $h_{1,2}=2$, $h_{1,3}=3$, $d_{1,1}=0.1$, $d_{1,2}=0.2$, $d_{1,3}=0.3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$; in the center for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=0.5$, $h_{1,3}=3$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=0.1$, $e_{1,2}=0.2$, $e_{1,3}=0.3$; on the right for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $g_{1,1}=1$, $g_{1,2}=2$, $g_{1,3}=3$, $h_{1,1}=1$, $h_{1,2}=2$, $h_{1,3}=0.7$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$.

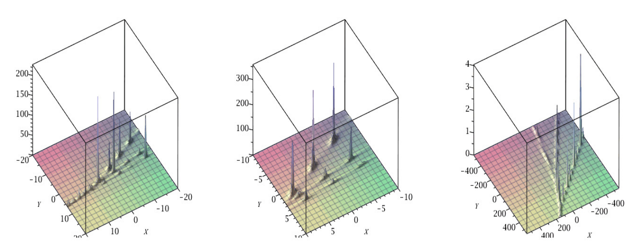

Figure 11. Solution of order 1 to KPI, on the left for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $g_{1,1} = 1$, $g_{1,2} = 2$, $g_{1,3} = 3$, $h_{1,1} = 0.5$, $h_{1,2} = 2$, $h_{1,3} = 3$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$; in the center for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $g_{1,1} = 1$, $g_{1,2} = 2$, $g_{1,3} = 3$, $h_{1,1} = 1$, $h_{1,2} = 0.5$, $h_{1,3} = 3$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$; on the right for $t = 10^3$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $g_{1,1} = 1$, $g_{1,2} = 2$, $g_{1,3} = 3$, $h_{1,1} = 1$, $h_{1,2} = 2$, $h_{1,3} = 0.7$, $d_{1,1} = 10$, $d_{1,2} = 20$, $d_{1,3} = 30$, $e_{1,1} = 10$, $e_{1,2} = 20$, $e_{1,3} = 30$.

6. Solutions of order 1 with a degree of summation equal to 4 ($S = 4$) depending on 21 real parameters

In this case, we observe generically lines of lumps, with a bifurcation in four branches whose intensities and directions depend on the 21 real parameters. These structures are more sensitive to g_i and h_i parameters than to others. We get also a whole set of degenerated configurations.

In the case $S = 4$, the expression of the solutions with all the parameters being too long, we present only one of them with particular values of parameters . For example we choose : $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $b_{1,4,0} = 4$, $g_{1,1} = 1$, $g_{1,2} = 2$, $g_{1,3} = 3$, $g_{1,4} = 4$, $h_{1,1} = 1$, $h_{1,2} = 2$, $h_{1,3} = 3$, $h_{1,4} = 4$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $d_{1,4} = 4$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$, $e_{1,4} = 4$.

So, the solution to the KPI equation, with these choices of parameters, can be written as

$$v(x,y,t) = -2\frac{n(x,y,t)}{d(x,y,t)^2} \tag{10}$$

with

$$\begin{aligned}
n(x,y,t) = {} & 2i((27\exp(3ix-3x+18y-54it-54t+3+3i) \\
& +12\exp\exp(-7x+7ix+7+7i+50y-182it-182t) \\
& +8\exp(2ix-2x+8y-16it-16t+2+2i) \\
& +2\exp(-3x+3ix+3+3i+10y-18it-18t) \\
& +125\exp(5ix-5x+50y-250it-250t+5+5i) \\
& +60\exp(-8x+8ix+8+8i+68y-304it-304t) \\
& +90\exp(-7x+7ix+7+7i+58y-266it-266t) \\
& +6\exp(-5x+5ix+5+5i+26y-70it-70t) \\
& +32\exp(-6x+6ix+6+6i+40y-144it-144t) \\
& +\exp(ix-x+2y-2it-2t+1+i) \\
& +64\exp(4ix-4x+32y-128it-128t+4+4i) \\
& +36\exp(-5x+5ix+5+5i+34y-130it-130t) \\
& +12\exp(-4x+4ix+4+4i+20y-56it-56t) \\
& +20\exp(-9x+9ix+9+9i+82y-378it-378t) \\
& +80\exp(-6x+6ix+6+6i+52y-252it-252t))
\end{aligned}$$

and

$$\begin{aligned}
d(x,y,t) = {} & \exp(-x+ix-2t-2it+2y+1+i) \\
& +2\exp(-2x+2ix-16t-16it+8y+2+2i) \\
& +3\exp(-3x+3ix-54t-54it+18y+3+3i) \\
& +4\exp(-4x+4ix-128t-128it+32y+4+4i) \\
& +5\exp(-5x+5ix-250t-250it+50y+5+5i)+1
\end{aligned}$$

. Generically, we get five lines of lumps with degenerate situations. We present figure 12 in the (x,y) plane :

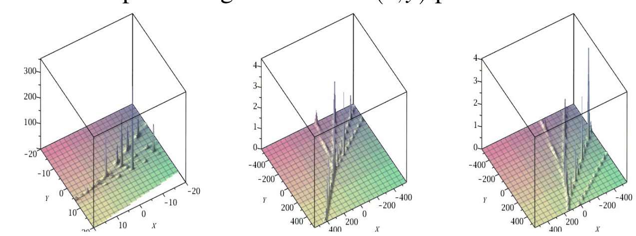

Figure 12. Solution of order 1 to KPI, on the left for $t = 1$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $b_{1,4,0} = 4$, $g_{1,1} = 1$, $g_{1,2} = 2$, $g_{1,3} = 3$, $g_{1,4} = 4$, $h_{1,1} = 1$, $h_{1,2} = 2$, $h_{1,3} = 3$, $h_{1,4} = 4$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $d_{1,4} = 4$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$, $e_{1,4} = 4$; in the center for $t = 10^2$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $b_{1,4,0} = 4$, $g_{1,1} = 0.1$, $g_{1,2} = 0.2$, $g_{1,3} = 0.3$, $g_{1,4} = 0.4$, $h_{1,1} = 0.1$, $h_{1,2} = 0.2$, $h_{1,3} = 0.3$, $h_{1,4} = 0.4$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $d_{1,4} = 4$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$, $e_{1,4} = 4$; on the right for $t = 10^3$, $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $b_{1,4,0} = 4$, $g_{1,1} = 0.1$, $g_{1,2} = 0.2$, $g_{1,3} = 0.3$, $g_{1,4} = 0.4$, $h_{1,1} = 0.1$, $h_{1,2} = 0.2$, $h_{1,3} = 0.3$, $h_{1,4} = 0.4$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $d_{1,4} = 4$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$, $e_{1,4} = 4$.

7. Solutions of order 1 with a degree of summation equal to 5 (S=5) depending on 26 real parameters

In this case, we observe generically five lines of lumps and a whole set of degene-rated configurations. These structures are more sensitive to g_i and h_i parameters than to others.

In the case $S = 5$, the expression of the solutions with all the parameters being too long, we present only one of them with particular values of parameters . For example we choose : $a_1 = 1$, $b_{1,1,0} = 1$, $b_{1,2,0} = 2$, $b_{1,3,0} = 3$, $b_{1,4,0} = 4$, $b_{1,5,0} = 5$, $g_{1,1} = 1$, $g_{1,2} = 2$, $g_{1,3} = 3$, $g_{1,4} = 4$, $g_{1,5} = 5$, $h_{1,1} = 1$, $h_{1,2} = 2$, $h_{1,3} = 3$, $h_{1,4} = 4$, $h_{1,5} = 5$, $d_{1,1} = 1$, $d_{1,2} = 2$, $d_{1,3} = 3$, $d_{1,4} = 4$, $d_{1,5} = 4$, $e_{1,1} = 1$, $e_{1,2} = 2$, $e_{1,3} = 3$, $e_{1,4} = 4$, $e_{1,5} = 5$.

So, in this case, the solution to the KPI equation can be written as

$$v(x,y,t) = -2\frac{n(x,y,t)}{d(x,y,t)^2} \tag{11}$$

with

$$n(x,y,t) = -2i(27\exp(3ix-3x+18y-54it-54t+3+3i)$$
$$+8\exp(2ix-2x+8y-16it-16t+2+2i)$$
$$+6\exp(-5x+5ix+5+5i+26y-70it-70t)$$
$$+32\exp(-6x+6ix+6+6i+40y-144it-144t)$$
$$+64\exp(4ix-4x+32y-128it-128t+4+4i)$$
$$+12\exp(-4x+4ix+4+4i+20y-56it-56t)$$
$$+12\exp(-7x+7ix+7+7i+50y-182it-182t)$$
$$+\exp(ix-x+2y-2it-2t+1+i)$$
$$+2\exp(-3x+3ix+3+3i+10y-18it-18t)$$
$$+36\exp(-5x+5ix+5+5i+34y-130it-130t))$$

and

$$d(x,y,t) = \exp(-x+ix-2t-2it+2y+1+i)$$
$$2\exp(-2x+2ix-16t-16it+8y+2+2i)$$
$$+3\exp(-3x+3ix-54t-54it+18y+3+3i)$$
$$+4\exp(-4x+4ix-128t-128it+32y+4+4i)+1.$$

Generically, we get four lines of lumps with degenerate situations. We present figure 13 in the (x,y) plane :

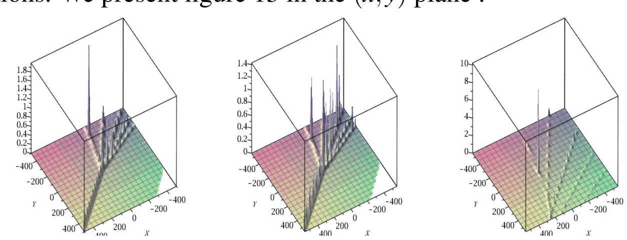

Figure 13. Solution of order 1 to KPI, on the left for $t=0$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $b_{1,4,0}=4$, $b_{1,5,0}=5$, $g_{1,1}=0.1$, $g_{1,2}=0.2$, $g_{1,3}=0.3$, $g_{1,4}=0.4$, $g_{1,5}=0.5$, $h_{1,1}=0.1$, $h_{1,2}=0.2$, $h_{1,3}=0.3$, $h_{1,4}=0.4$, $h_{1,5}=0.5$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $d_{1,4}=4$, $d_{1,5}=5$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$, $e_{1,4}=4$, $e_{1,5}=5$; in the center for $t=10$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $b_{1,4,0}=4$, $b_{1,5,0}=5$, $g_{1,1}=0.1$, $g_{1,2}=0.2$, $g_{1,3}=0.3$, $g_{1,4}=0.4$, $g_{1,5}=0.5$, $h_{1,1}=0.1$, $h_{1,2}=0.2$, $h_{1,3}=0.3$, $h_{1,4}=0.4$, $h_{1,5}=0.5$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $d_{1,4}=4$, $d_{1,5}=5$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$, $e_{1,4}=4$, $e_{1,5}=5$; on the right for $t=10^3$, $a_1=1$, $b_{1,1,0}=1$, $b_{1,2,0}=2$, $b_{1,3,0}=3$, $b_{1,4,0}=4$, $b_{1,5,0}=5$, $g_{1,1}=0.1$, $g_{1,2}=0.2$, $g_{1,3}=0.3$, $g_{1,4}=0.4$, $g_{1,5}=0.5$, $h_{1,1}=0.1$, $h_{1,2}=0.2$, $h_{1,3}=0.3$, $h_{1,4}=0.4$, $h_{1,5}=0.5$, $d_{1,1}=1$, $d_{1,2}=2$, $d_{1,3}=3$, $d_{1,4}=4$, $d_{1,5}=5$, $e_{1,1}=1$, $e_{1,2}=2$, $e_{1,3}=3$, $e_{1,4}=4$, $e_{1,5}=5$.

8. Conclusion

Using an extended Darboux transform, we succeed to construct multi-parametric solutions to the KPI equation. These solutions depend on the order of determinant N, the degree of summation S and the degree of derivation D. In the case general case the solutions to the KPI equation can be expressed as a second derivative with respect to x of a logarithm of a determinant of order N, we obtain solutions depending on $N(S(D+5)+1)$ real parameters.
We only give the expressions of the solution in the simple case of order $N=1$.
All these solutions are different from these given in another approach given par the present author [18–24].
In the frame of plasmas, the KP equation is derived [25] to describe certain particular solutions called dust acoustic waves. The KP equation in [25] could be renormalized to use the classical KP equation. The solutions presented in [25] are different from those built in this study. Multi-lum solutions with 0 degree derivation are exact theoretical solutions to the KP equation. It will be interesting to observe them in plasmas.
It would important to study in more details this first order with the degree of derivation non equal to 0 and try to classify these solutions.
We postpone to another publication the study of the solutions with a non zero degree of derivation.
It would be also relevant to study the cases of order greater or equal to 2 and to realize an exhaustive classification of the solutions to the KPI equation.

9. appendix 1

$n(x,y,t)$ and $d(x,y,t)$ related to Eq. 8;

$$n(x,y,t) =$$
$$-b_{1,1,0}\exp[ig_{1,1}x-h_{1,1}x-iyg_{1,1}{}^2+2yg_{1,1}h_{1,1}+iyh_{1,1}{}^2$$
$$+itg_{1,1}{}^3-3tg_{1,1}{}^2h_{1,1}-3itg_{1,1}h_{1,1}{}^2+th_{1,1}{}^3+d_{1,1}+ie_{1,1}]g_{1,1}^2a_1$$
$$+b_{1,1,0}\exp[-h_{1,2}x-h_{1,1}x+ig_{1,1}x+ig_{1,2}x-3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2+ie_{1,1}-3itg_{1,2}h_{1,2}{}^2+iyh_{1,1}{}^2+d_{1,1}-3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2+th_{1,1}{}^3+itg_{1,2}{}^3+2yg_{1,2}h_{1,2}+ie_{1,2}+2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2}+itg_{1,1}{}^3+th_{1,2}{}^3+d_{1,2}-iyg_{1,1}{}^2]h_{1,1}{}^2b_{1,2,0}$$
$$-b_{1,2,0}\exp[ig_{1,2}x-h_{1,2}x-iyg_{1,2}{}^2+2yg_{1,2}h_{1,2}+iyh_{1,2}{}^2$$
$$+itg_{1,2}{}^3-3tg_{1,2}{}^2h_{1,2}-3itg_{1,2}h_{1,2}{}^2+th_{1,2}{}^3+d_{1,2}+ie_{1,2}]g_{1,2}{}^2a_1$$
$$+b_{1,2,0}\exp[-h_{1,2}x-h_{1,1}x+ig_{1,1}x+ig_{1,2}x-3itg_{1,1}h_{1,1}{}^2-iyg_{1,2}{}^2$$
$$+ie_{1,1}-3itg_{1,2}h_{1,2}{}^2+iyh_{1,1}{}^2+d_{1,1}-3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2+th_{1,1}{}^3+itg_{1,2}{}^3+2yg_{1,2}h_{1,2}+ie_{1,2}+2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2}+itg_{1,1}{}^3+th_{1,2}{}^3+d_{1,2}-iyg_{1,1}{}^2]h_{1,2}{}^2b_{1,1,0}$$
$$-2ib_{1,2,0}\exp[ig_{1,2}x-h_{1,2}x-iyg_{1,2}{}^2+2yg_{1,2}h_{1,2}+iyh_{1,2}{}^2+itg_{1,2}{}^3$$

$$-3tg_{1,2}{}^2h_{1,2} - 3itg_{1,2}h_{1,2}{}^2 + th_{1,2}{}^3 + d_{1,2} + ie_{1,2}]g_{1,2}h_{1,2}a_1$$
$$+2ib_{1,1,0}\exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2}$$
$$+2yg_{1,1}h_{1,1} - 3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2}$$
$$-iyg_{1,1}{}^2]h_{1,1}b_{1,2,0}g_{1,2} + 2b_{1,1,0}\exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x$$
$$+ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2 - iyg_{1,2}{}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}{}^2$$
$$+iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]g_{1,1}b_{1,2,0}g_{1,2}$$
$$-2ib_{1,1,0}\exp[ig_{1,1}x - h_{1,1}x - iyg_{1,1}{}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}{}^2$$
$$+itg_{1,1}{}^3 - 3tg_{1,1}{}^2h_{1,1} - 3itg_{1,1}h_{1,1}{}^2 + th_{1,1}{}^3 + d_{1,1} + ie_{1,1}]$$
$$g_{1,1}h_{1,1}a_1 - 2ib_{1,2,0}\exp[-h_{1,2}x - h_{1,1}x$$
$$+ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2 - iyg_{1,2}{}^2 + ie_{1,1}$$
$$-3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]g_{1,2}h_{1,2}b_{1,1,0}$$
$$-b_{1,1,0}\exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]g_{1,1}{}^2b_{1,2,0}$$
$$+2ib_{1,1,0}exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]g_{1,1}b_{1,2,0}h_{1,2}$$
$$-b_{1,2,0}\exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]g_{1,2}^2b_{1,1,0}$$
$$-2ib_{1,1,0}\exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2 + ie_{1,1} - 3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]g_{1,1}h_{1,1}b_{1,2,0}$$
$$-2b_{1,1,0}\exp[-h_{1,2}x - h_{1,1}x + ig_{1,1}x + ig_{1,2}x - 3itg_{1,1}h_{1,1}{}^2$$
$$-iyg_{1,2}{}^2 + is_{1,1} - 3itg_{1,2}h_{1,2}{}^2 + iyh_{1,1}{}^2 + d_{1,1} - 3tg_{1,1}{}^2h_{1,1}]$$
$$\times\exp[+iyh_{1,2}{}^2 + th_{1,1}{}^3 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2} + ie_{1,2} + 2yg_{1,1}h_{1,1}$$
$$-3tg_{1,2}{}^2h_{1,2} + itg_{1,1}{}^3 + th_{1,2}{}^3 + d_{1,2} - iyg_{1,1}{}^2]h_{1,1}b_{1,2,0}h_{1,2}$$
$$+b_{1,1,0}\exp[ig_{1,1}x - h_{1,1}x - iyg_{1,1}{}^2 + 2yg_{1,1}h_{1,1} + iyh_{1,1}{}^2$$
$$+itg_{1,1}{}^3 - 3tg_{1,1}{}^2h_{1,1} - 3itg_{1,1}h_{1,1}{}^2 + th_{1,1}{}^3 + d_{1,1} + ie_{1,1}]h_{1,1}^2a_1$$
$$+b_{1,2,0}\exp[ig_{1,2}x - h_{1,2}x - iyg_{1,2}{}^2 + 2yg_{1,2}h_{1,2} + iyh_{1,2}{}^2$$
$$+itg_{1,2}{}^3 - 3tg_{1,2}{}^2h_{1,2} - 3itg_{1,2}h_{1,2}{}^2 + th_{1,2}{}^3 + d_{1,2} + ie_{1,2}]h_{1,2}{}^2a_1$$

and

$$d(x,y,t) =$$
$$b_{1,1,0}\exp[-h_{1,1}x + ig_{1,1}x - 3tg_{1,1}{}^2h_{1,1}$$
$$-3itg_{1,1}h_{1,1}{}^2 + itg_{1,1}{}^3 + 2yg_{1,1}h_{1,1}$$
$$+iyh_{1,1}{}^2 + th_{1,1}{}^3 - iyg_{1,1}{}^2 + d_{1,1} + ie_{1,1}]$$
$$+b_{1,2,0}\exp[-h_{1,2}x + ig_{1,2}x - 3tg_{1,2}{}^2h_{1,2}$$
$$-3itg_{1,2}h_{1,2}{}^2 + itg_{1,2}{}^3 + 2yg_{1,2}h_{1,2}$$
$$+iyh_{1,2}{}^2 + th_{1,2}{}^3 - iyg_{1,2}{}^2 + d_{1,2} + ie_{1,2}] + a_1$$

Conflict of interest statement:
The authors declare that they have no conflict of interest.

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