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Thermal Boundary Layer Analysis over a Flat Plate with a Convective Surface Boundary Condition Using the Homotopy Perturbation Method condition

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ABSTRACT

This study presents an analytical investigation of the energy and momentum (Navier–Stokes) equations governing convection heat transfer over a flat plate in an incompressible flow regime. The original multivariable partial differential equations are first transformed into their non-dimensional forms using appropriate dimensionless parameters. Given the linear nature of the non-dimensional energy equation and the nonlinear character of the momentum equation, distinct solution methods are adopted for each. The energy equation is solved directly by applying the physical boundary conditions, while the nonlinear momentum equation is addressed using the Homotopy Perturbation Method (HPM), with an initial approximation derived from the problem's boundary conditions. The analytical results are presented through tables and graphs, illustrating temperature distributions, stream functions, velocity profiles, and skin friction coefficients. Additionally, the influence of non-dimensional parameters such as η, a, and Pr on temperature and heat transfer behavior is examined in detail.

Keywords: Analytical solution, Homotopy Perturbation Method (HPM), Thermal boundary layer, Convection boundary layer, Flat plate

1. INTRODUCTION

Heat transfer is always very important in industry so that many researchers have conducted many studies in this field. The discussion of the equations governing heat transfer at various surfaces such as vertical, horizontal, sloping flat surfaces, and spherical and cylindrical objects, blades and fins and etc. in the industry, refineries and power plants has attracted the attention of the scientists. Accordingly, the analytical solution of the equations governing heat transfer has been considered in this study in which the linear part of the governing equations (energy equation) has been solved by the direct method and their non - linear part (momentum equation) has been done by the Homotopy Perturbation Method (HPM) and the extensive researches have been conducted in this field around the world.

Raju et al. (2014) have conducted the analytical solution of MHD of free convective of dissipative boundary layer flow past a porous vertical surface in the presence of thermal radiation, chemical reaction and constant suction [1]. In 2014, Nofel has studied the application of the Homotopy Perturbation Method to nonlinear heat conduction [2]. In 2012, Yildirim and his colleagues have studied the analytical solution of MHD stagnation point flow in porous media by means of the Homotopy Perturbation Method [3]. Rashidi and his colleague (2011) have considerd the analytic approximate solutions for heat transfer of a micropolar fluid through a porous medium with radiation [4].

When the fluid with a specified temperature begins to move over a solid surface whose temperature is different from that of the fluid, convection heat transfer occurs due to the difference in temperature between solid surface and fluid. If the temperature difference between the moving fluid and the solid surface is high, the thermal boundary layer develops, and the fluid particles in contact with the surface reach a thermal equilibrium with the surface. The fluid in contact with the surface exchanges energy with the adjacent fluid and then it leads to the creation of a temperature gradient and this process continues with the formation of a temperature profile. Scientists and researchers have done a variety of research in this regard. For example, yacob et al. (2011) have examined boundary layer flow past a stretching/shrinking surface beneath an external uniform shear flow with a convective surface boundary condition in a nanofluid. In this study, the fluid flow is steady and the research has also been solved numerically using the Runge-Kutta method of Order 4 [5]. Makinde (2011) has studied the similarity solution for natural convection from a moving vertical plate with internal heat generation and a convective boundary condition [6]. Ishak (2010) has studied the similarity solutions for flow and heat transfer over a permeable surface with convective boundary condition [7]. Aziz (2009) has done a similarity solution for laminar thermal boundary layer over a flat plate with a convection surface boundary condition. This research has been solved numerically using the Runge-Kutta method of Order 4 [8]. In this research, it has been considered the analytical solution of two equations, one momentum (Navier-Stokes) equation and another energy equation. The heat transfer is a forced convective and the desired fluid is steady. Homotopy perturbation (HPM) and direct methods have been used to solve the momentum (Navier-Stokes) equation and energy equation, respectively. At the end, the results of the solution are displayed in the form of tables and graphs.

2. Statement of the problem

In this problem, there is an incompressible two-dimensional steady flow over a flat plate. It is assumed that the desired flat plate is vertical, the thermal boundary layer is laminar and convective boundary condition is superficial. x and y axes have been considered in the direction of the plate and perpendicular to it, respectively. The cold fluid with temperature T_∞ and constant velocity U_∞ is moving over a horizontal flat plate with temperature Tf. Heat transfer takes place due to the difference in temperature between the plate and the fluid ($T_f > T_\infty$). According to the non-slip principle, when a fluid moves over the desired surface, a layer of fluid which is in contact with the solid surface, is motionless and we assume that its velocity is zero and hence the heat transfer in this thin layer takes place only as a conduction, so: $\dot{q}\ cond = \dot{q}\ conv$. Heat transfer in the higher layers occurs as convection and the fluid properties have been assumed constant.

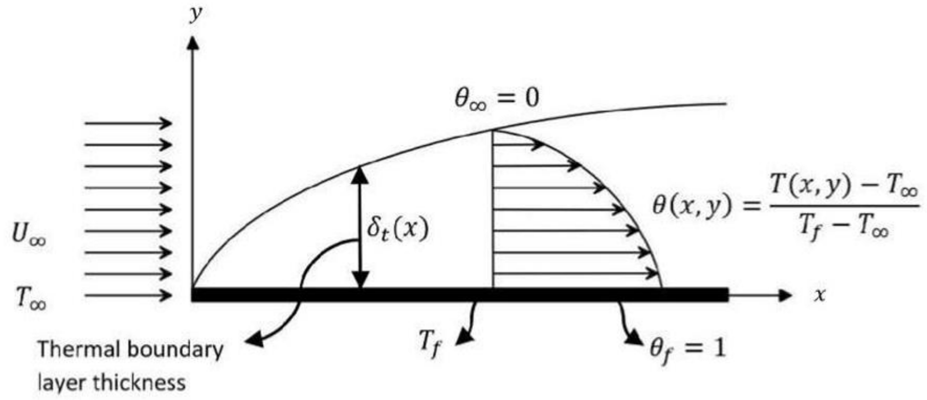

Fig. 1. Thermal boundary layer on the flat plate under the free stream U∞ , T∞

The equations governing the problem include three equations of continuity, momentum (Navier-Stokes) and energy that are as follows:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Momentum (Navier-Stokes) equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

Here, u and v are the speed components along x and y, respectively; T is temperature; ν is fluid kinematic viscosity (velocity penetration factor); and α is thermal diffusivity of the fluid.

Speed and temperature boundary conditions are as follows:

$$u(x,0) = v(x,0) = 0$$
$$u(x,\infty) = U_\infty$$
$$T(x,\infty) = T_\infty \qquad (4)$$

Using dimensionless quantities, the equations with partial multivariate derivatives are converted to ordinary univariate differential equations. The dimensionless quantities are:

$$\eta = y\left(\frac{U_\infty}{\nu x}\right)^{\frac{1}{2}}$$
$$f(\eta) = \frac{\psi}{U_\infty\sqrt{\nu x / U_\infty}}$$
$$\theta = \frac{T - T_\infty}{T_f - T_\infty} \qquad (5)$$

Here, η is independent variable, f is a variable dependent on the flow function ψ, and θ is dimensionless temperature. Also, θ_∞ is dimensionless temperature of fluid free flow and θ_f is dimensionless temperature of flat plate.

After non-dimensionalization, momentum (Navier-Stocks) equation and energy equation will be becoming as follows: (therefore, we have momentum equation (2) and energy equation (3) as dimensionless ones as follows):

$$2f''' + ff'' = 0 \tag{6}$$

$$\theta'' + \frac{1}{2}P_r f\theta' = 0 \tag{7}$$

The boundary conditions for two equations (momentum and energy) in the non dimensional mode are:

$$\begin{aligned} &f(0) = f'(0) = 0 \\ &f'(\infty) = 1 \\ &\theta'(0) = -a[1 - \theta(0)] \\ &\theta(\infty) = 0 \end{aligned} \tag{8}$$

3. Solving method

The equations governing the problem were equations with partial multivariate derivatives which have been converted to the ordinary univariate differential equations with orders higher than one derivative through non-dimensionalization. Due to nonlinearity, the dimensionless momentum (Navier-Stokes) equation (6) cannot be solved by direct method as a linear equation, but equation (7), which is a linear equation, can be solved directly. Given the homogeneity of these two equations, two general solutions will be obtained.

We will first solve the energy equation (7), which has a homogeneous boundary condition (9) and a non- homogeneous boundary condition (10).

$$\theta(\infty) = 0 \tag{9}$$

$$\theta'(0) = -a[1 - \theta(0)] \tag{10}$$

In order to solve the above equations in a direct way, the following solutions are guessed for θ, θ′ and θ″.

$$\theta = e^{\lambda\eta} \tag{11}$$

$$\theta' = \lambda e^{\lambda\eta} \tag{12}$$

$$\theta'' = \lambda^2 e^{\lambda\eta} \tag{13}$$

By applying the solutions (11)-(13), the energy equation governing the problem (7) will become as follow:

$$\lambda^2 e^{\lambda\eta} + \frac{1}{2}P_r f\,\lambda e^{\lambda\eta} = 0 \tag{14}$$

The energy equation (14) has two roots which one of them is zero and another is $-\frac{1}{2}\,P_r f$.; according to two different roots and the use of a direct solving method, we have:

$$\theta = C_1 e^{(0)\eta} + C_2 e^{(-\frac{1}{2}P_r f)\eta} \tag{15}$$

By applying the boundary conditions governing the obtained energy equation, the constant quantities C_1 and C_2 are obtained as follows:

$$C_1 = 0 \tag{16}$$

$$C_2 = \frac{a}{\frac{1}{2}P_r f + a} \tag{17}$$

Given the specified coefficients C_1 and C_2, the solution from the governing energy equation is as follows:

$$\theta = \frac{a}{\frac{1}{2}P_r f + a} e^{(-\frac{1}{2}P_r f)\eta} \tag{18}$$

By determining the solution from solving the energy equation, we try to solve the momentum (Navier-Stokes) equation.

According to nonlinearity, the momentum (Navier-Stokes) equation cannot be solved as the previous method. Therefore, Homotopy Perturbation Method (HPM) can be used to solve equation (6).

Homotopy Perturbation Method (HPM) is a method for solving nonlinear equations developed by J. H. He in 1999. This new method has high precision and convergence speed. This technique is frequently used in engineering science and it is a good alternative to the Forth Order Runge-Kutta method because of its simplicity and high speed in doing calculations.
We assume that the nonlinear differential equation is as follows:

$$A(u) - f(r) = 0, r \in \Omega \tag{19}$$

And, the boundary condition of equation (19) is as follows:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \tag{20}$$

B is a boundary operator and Γ is the boundary of the domain Ω.
In equation (19), we have:

A (u) =L (u) +N (u), where L (u) and N (u) are equal to linear and nonlinear parts, respectively. Also, f(r) is known as an analytic function and A(u) is general differential operator. According to the Homotopy method, the Homotopy v(r,P):Ω×[0,1]→R can be formed in such a way that the Homotopy Perturbation equation is as follows:

$$H(v, p) = (1 - P)[L(v) - L(u_0)] + P[A(v) - f(r)] = 0 \tag{21}$$

In equation (21), P is known as Perturbation variable which has a numerical value between zero and one (P ∈ [0,1]). The results from equation (21) are expressed as follows:

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{22}$$

$$H(v, 1) = A(v) - f(r) = 0 \tag{23}$$

According to the results from (22)-(23), if P changes from zero to one, v(r,P) converts from $u_0(r)$ to $u(r)$.[2]

The obtained solutions can be expressed as power series of P:

$$v = v_0 + Pv_1 + P^2v_2 + \cdots \tag{24}$$

Then, the approximate solution of equation (19) can be obtained with the following limit:

$$u = \lim_{P\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{25}$$

By replacing equation (6) in equation (21), equation (21) is expressed as follows:

$$(1-P)[2f'''-L(u_0)]+P[2f'''+ff'']=0 \tag{26}$$

In equation (26), $L(u_0)$ is equal to initial guess which is equal to $-2e^{-\eta}$ in this study. we put the expression "$f=f_0+Pf_1$"instead of " f " and extend the equation (26) based on the said assumptions and then arrange the sentences based on powers equal to P.

For p^0mode:

$$P^0:\ 2f_0'''=-2e^{-\eta}$$
$$f_0(0)=0, f_0'(0)=0, f_0'(\infty)=1 \tag{27}$$

For p^1 mode:

$$P^1:\ 2f_1'''=+2e^{-\eta}-f_0f_0''$$
$$f_1(0)=0, f_1'(0)=0, f_1'(\infty)=0 \tag{28}$$

According to the specified boundary conditions, the solutions from equations (27) and (28) are equal to:

$$f_0=e^{-\eta}+\eta-1 \tag{29}$$
$$f_1=0.0625e^{-2\eta}+0.5\eta e^{-\eta}-0.375\eta-0.0625 \tag{30}$$

After determining f_0 and f_1 from the above solutions, the equation " f " is expressed as follows:

$$f(\eta)=f_0(\eta)+f_1(\eta) \tag{31}$$

4. Results and discussion

In this section, we will examine the results of solving the momentum (Navier-Stokes) equation and the energy equation through tables and graphs.

Considering the solution of the momentum (Navier-Stokes) equation by HPM method, the flow, velocity and skin friction graphs are displayed in terms of the variable η for the equation (6) as follows:

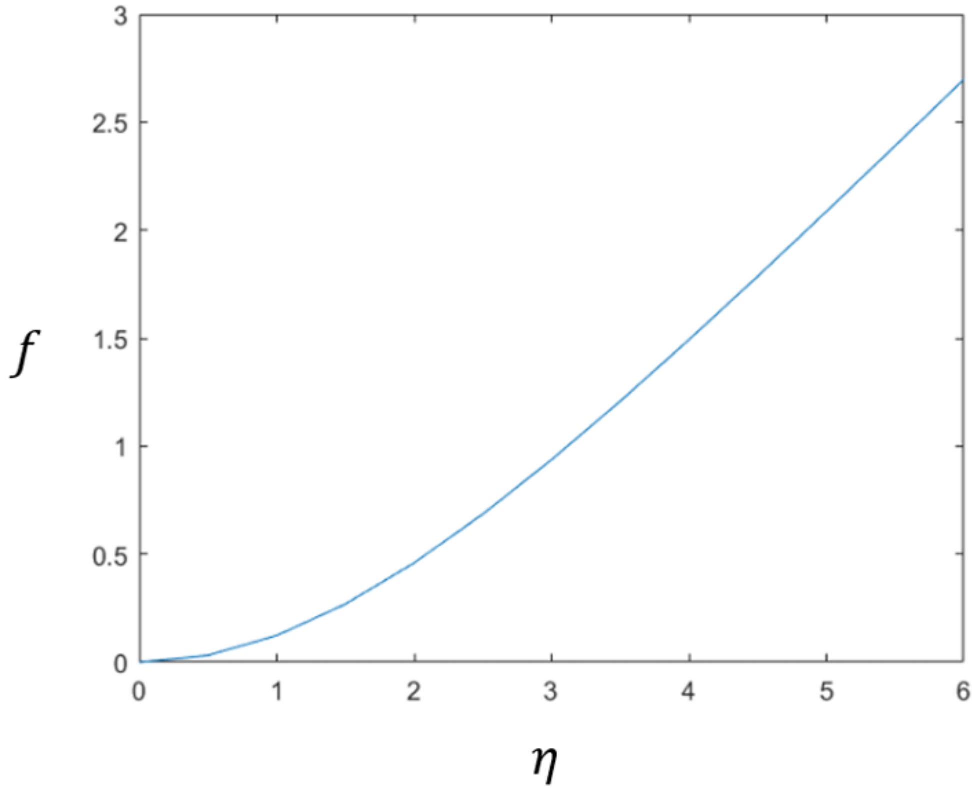

Fig. 2. Flow graph in η

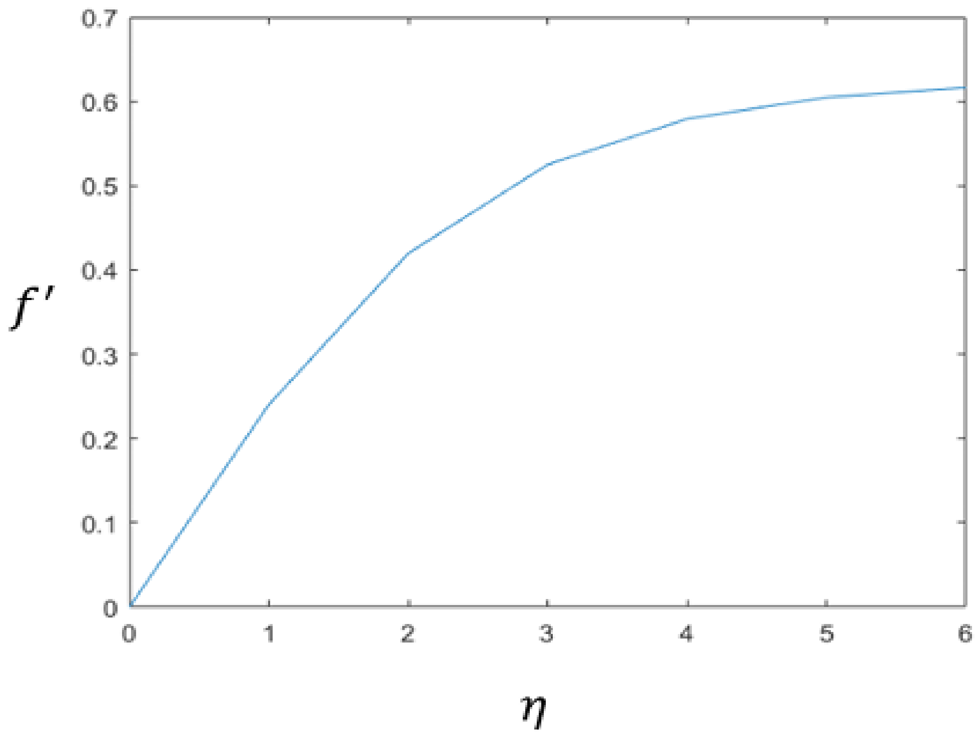

Fig. 3. Velocity graph in η

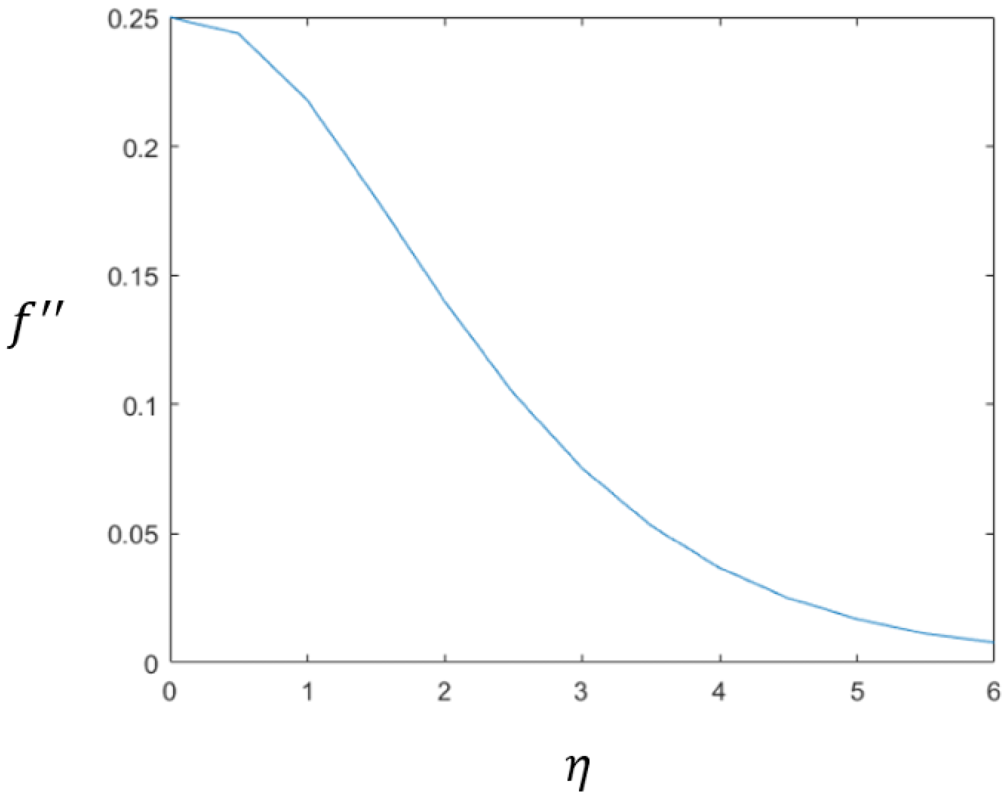

Fig. 4. Skin friction graph in η

Now, we review the tables and graphs related to the energy equation governing the problem using the specified momentum equations graphs.

The energy equation governing the problem has a known general solution (18) in which the effective parameters include Prandtl number (P_r), the variable dependent on the flow function (f), the independent variable of the problem (η) and α, in which the parameter α is equal to:

$$a = \frac{h_f}{k}\sqrt{\nu x \,/\, U_\infty} \tag{32}$$

As seen in this relation, h_f is equal to convection heat transfer factor and k is equal to conductive heat transfer factor.

In the below tables, the numerical values of $-\theta'(0)$, $\theta(0)$ have been given for three Prandtl numbers 0.1, 0.72 and 10 for different α values.

Table 1. Numerical values for Prandtl number 0.1

a	$-\theta'(0)$[19]	$\theta(0)$	$\theta(0)$[19]
0.05	0.0373	0.2545	0.2536
0.1	0.0594	0.4065	0.4046
0.2	0.848	-3.2520	0.5761
0.4	0.1076	0.7313	0.7310
0.6	0.1182	0.8032	0.8030
0.8	0.1243	0.8448	0.8446
1	0.1283	0.8718	0.8717
5	0.1430	0.9714	0.9714
10	0.1450	0.9855	0.9855
20	0.1461	0.9927	0.9927

Table 2. Numerical values for Prandtl number 0.72

a	$-\theta'(0)$ [19]	$\theta(0)$	$\theta(0)$[19]
0.05	0.0428	0.1441	0.1447
0.1	0.0747	0.2530	0.2528
0.2	0.1193	0.4036	0.4035
0.4	0.1700	0.5751	0.5750
0.6	0.1981	0.6700	0.6699
0.8	0.2159	0.7302	0.7302
1	0.2282	0.7719	0.7718
5	0.2791	0.9442	0.9441
10	0.2871	0.9713	0.9713
20	0.2913	0.9854	0.9854

Table 3. Numerical values for Prandtl number 10

a	$-\theta'(0)$ [19]	$\theta(0)$	$\theta(0)$[19]
0.05	0.0468	0.0641	0.0643
0.1	0.0879	0.1212	0.1208
0.2	0.1569	0.2162	0.2155
0.4	0.2582	0.3556	0.3546
0.6	0.3289	0.4528	0.4518
0.8	0.3812	0.5246	0.5235
1	0.4213	0.5797	0.5787
5	0.6356	0.8734	0.8729
10	0.6787	0.9324	0.9321
20	0.7026	0.9650	0.9649

In the above tables, the numerical value of θ (0) obtained from analytical solution has been compared with the value from numerical solution [19].

We know that Prandtl number (Pr) is a dimensionless number that represents the ratio of velocity penetration factor to thermal penetration factor. This dimensionless number is of orders 1, 10 and 10-2 for gases, water and liquid metals, respectively. For liquid metals, Prandtl number is smaller than one. Therefore, the heat is released very quickly and the thermal boundary layer δt is thicker than the velocity boundary layer δ (Pr <<1, δ < δt). While, if the Prandtl number is much larger than one, the momentum is released much faster than heat and the thermal boundary layer δt is thinner than the velocity boundary layer δ (Pr >>1, δ > δt).

Figures (5), (6) and (7) are the temperature graphs in terms of variable η for Prandtl numbers 0.1, 0.72 and 10, respectively.

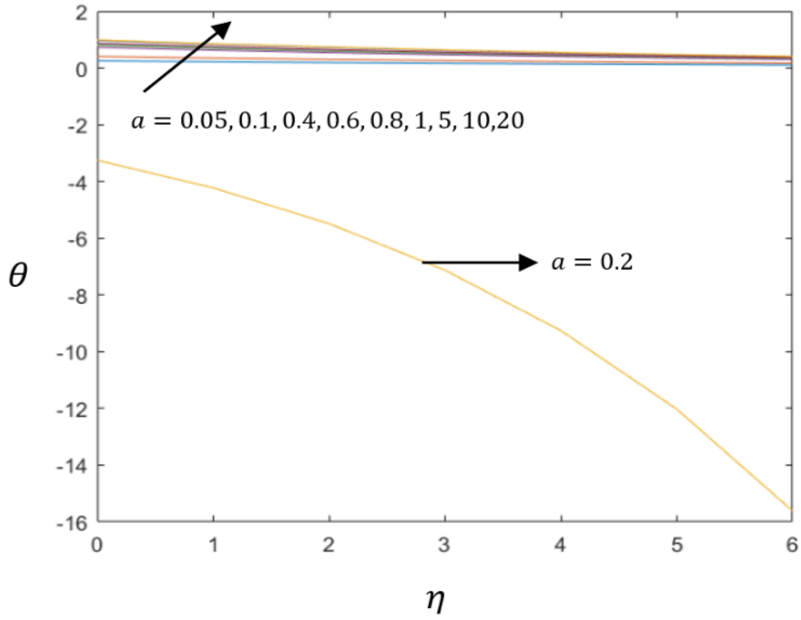

Fig. 5. Temperature graph in terms of η for Pr=0.1

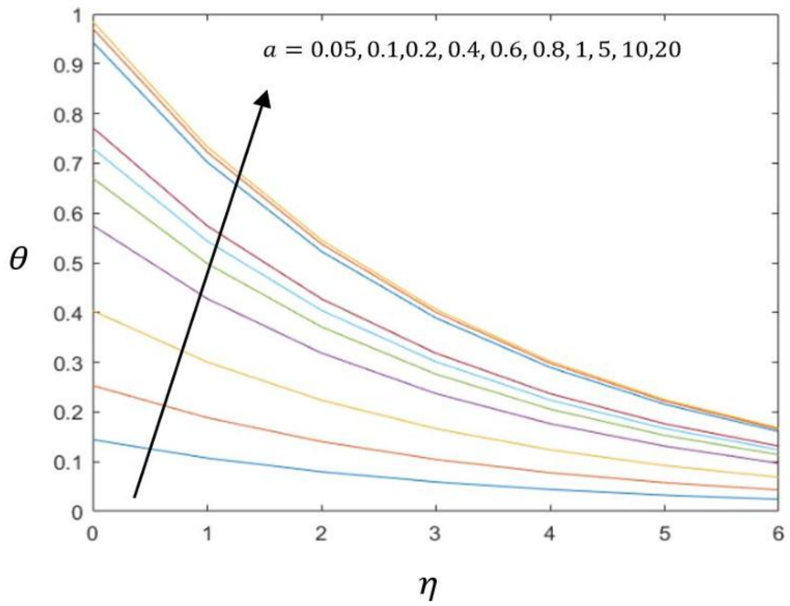

Fig. 6. Temperature graph in terms of η for Pr=0.72

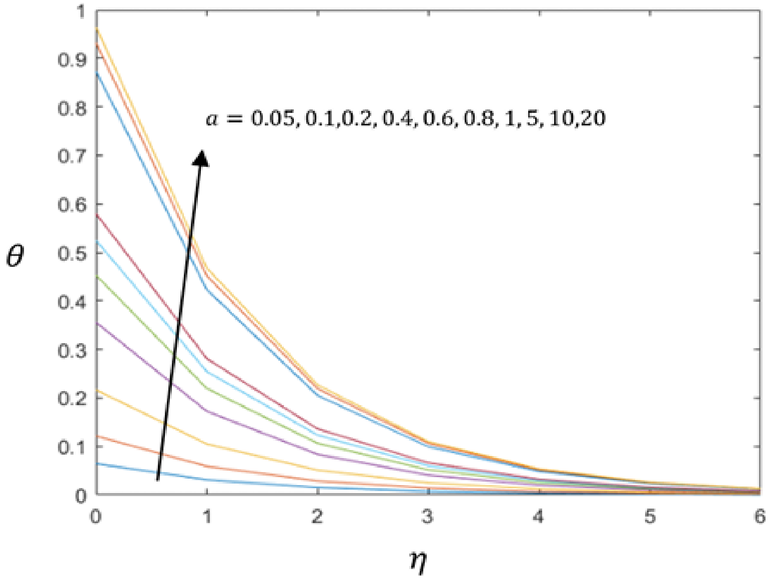

Fig. 7. Temperature graph in terms of η for Pr=10

5. Conclusions

Heat transfer equations including momentum (Navier-Stokes) equation and the energy equation have been analyzed through analytical solution. Below are the results of the analytical solution and its comparison with numerical solution.

I. As η increases, the flow velocity is increased toward one.
II. As η increases, skin friction coefficient is reduced sharply.
III. As Prandtl number (Pr) increases from 0.1 to 10, the value of θ (0) for α=0.05 is
IV. reduced.
V. As Prandtl number (Pr) increases from 0.1 to 10, the value of θ (0) for α=20 is
VI. reduced
VII. As the value of α increases from 0.05 to 20, the value of θ (0) is increased.
VIII. The results obtained from the analytic solution in comparison with the numerical
IX. solution (fourth order Runge-Kutta) for the energy equation show that the
X. numbers obtained from the analytical solution vary with the numbers resulting
XI. from the numerical solution in thousandth and ten thousandths.

6. Appendix:

x	vector in the direction of the plate
y	vector perpendicular to the plate
u	velocity component in the direction of the plate
v	velocity component perpendicular to the plate
T_∞	Fluid temperature around the plate
U_∞	constant speed fluid
Tf	the desired plate temperature
T	temperature
ν	velocity penetration factor
α	The thermal penetration factor
η	problem variable
f	dimensionless velocity factor
θ	dimensionless temperature
Pr	dimensionless Prandtl number
P	perturbation variable
A (u)	general differential operator
L (u)	linear part
L (u0)	initial guess
N (u)	nonlinear part
F(r)	analytical function
B	boundary operator
hf	convection heat transfer factor
K	conductive heat transfer factor
δ	velocity boundary layer
δt	thermal boundary layer

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