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Perfect Reduced Equilibrium Equations and Cross-sectional Stress Resultants for Small Elastic Orientation in Nonlinear Spatial Euler-Bernoulli Beam

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ABSTRACT

Using variational indication and minimizing strain energy of a geometrically-nonlinear linearly-elastic isotropic homogeneous spatial Euler-Bernoulli beam, the equilibrium governing formulations are found. Therefore, the weak form of the strain energy variation was derived for large elastic orientation and on the basis of which the perfect weak form of the strain energy variation was extracted for small elastic orientation. The obtained perfect equilibrium equations include the balance of axial forces, torsional moment and two perpendicular flexural moments. The obtained cross-sectional stress resultants include axial force, two perpendicular shearing forces, torsional moment and two perpendicular flexural moments. The assumption of small elastic orientation is imposed to the perfect weak form of strain energy variation in the present manuscript, but it is imposed to imperfect strain, and then to imperfect nonweak form of strain energy variation in the classical nonlinear 3-D Euler-Bernoulli beam theory. Imperfect strain causes strain variation and strain energy variation to miss a few terms. Moreover, imperfect nonweak form of strain energy variation causes the weak form of the strain energy variation to miss many terms. As a result, in this theory equilibrium equations and cross-sectional stress resultants suffer from the lack of missed terms. The lost terms are revealed in this manuscript.

KEYWORDS:

Perfect Reduced Weak Form of Strain Energy Variation, Perfect Reduced Equilibrium Equations, Perfect Reduced Cross-sectional Stress Resultants, Nonlinear Spatial Euler-Bernoulli Beam, Elastic Orientation

1. INTRODUCTION

Few references concentrate on extracting perfect governing equations of beams undergoing small elastic orientation. However, there are many references that try to solve the available governing equations without regard to the perfect extracting of them. Relatively old papers consider isotropic beams. The major of the recent papers considers beams with a more complicated material than isotropic one.

Azartash et al [1] have derived perfect governing equations of planar shear deformable beam with small elastic orientation. *Zohoor et al* [2] have derived perfect governing equations of nonlinear spatial Euler-Bernoulli beam with small elastic orientation similar to those of the current manuscript. *Zohoor and Khorsandijou* have derived perfect governing equations of nonlinear spatial Euler-Bernoulli beams with large and small elastic orientation [3] and those of beams with large elastic orientation [4]. They [5-7] have derived perfect strain variation and enhanced nonlinear spatial Euler-Bernoulli beam governing equations less perfect than those of Reference [2] and more perfect than those of Reference [8]. *Nayfeh and Pai* [8] have derived the governing equation of planar and spatial, linear and nonlinear Euler-Bernoulli straight and curved beams undergoing cross-sectional warpings. *Tan et al* [9] have derived the governing equations of a rotating one-end-fixed Euler-Bernoulli beam under base excitation.

Thin prismatic rod has been investigated by *Novozhilov* [10] who reduced cross-sectional element displacement vector before deriving strain tensor. As a result, the obtained strain is not perfect. *Antman* [11] has proposed a general theory for nonlinearly-elastic rods capable of bending, twisting, extending and shear deformating complying with the Kirchhoff theory. *Green and Laws* [12] have considered rod as a curve at every point of which either one or two directors are attached. *Green et al* [13] have considered linear and nonlinear elastic rods as a continuum. *Whitman and DeSilva* [14] have considered a spatial shear deformable extensible elastic rod with large elastic orientation. *O'Reilly* and *Turcotte* [15] have developed and analyzed a model for the deformation of a rotating prismatic rod-like body. *Steigmann and Faulkner* [17] have simplified spatial rod study by means of assumed mode method.

Applying newmark method, *Azartash et al* [1] have solved complete motion equations of a geometrically-nonlinear Shear-deformable beam whose material is porous functionally grated. Using Ritz method, *Xie et al* [16] have solved nonlinear free vibration problem of a shear deformable beam whose material is functionally graded. Using finite element method, *Esen* [17] have solved the motion equations of a Timoshenko beam subjected to a moving mass. The beam is located on an elastic foundation, and the material of which is functionally graded. Using newmark method, *Wang et al* [18] have solved the motion equations of a high order shear deformable beam whose material is a functionally graded graphene nano plate let reinforced composite. The beam is subjected to two successive moving masses. Using finite element method, *Esen et al* [19] have solved the motion equations of a Timoshenko beam subjected to accelerating and decelerating massed separately. The beam material is functionally graded.

In the present manuscript, the weak form of strain energy variation is derived for a geometrically-nonlinear linearly-elastic isotropic spatial Euler-Bernoulli beam under large elastic orientation. Moreover, the perfect weak form of strain energy variation, perfect equilibrium equations and perfect cross-sectional stress resultants of the beam are extracted under small elastic orientation, and are compared with those of the classical nonlinear 3-D Euler-Bernoulli beam theory, and the cause of the differences is explained. This manuscript might be considered as a survey on perfect derivation of beam governing formulations under small elastic orientation.

2. DEFORMATION

The dependent field variables $u(s)$, $v(s)$, $w(s)$, $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ being functions of the Lagrangian coordinate s in ***Figure 1***, illustrate the elastic coordinates of deformation in spatial nonlinear Euler-Bernoulli beam by means of deformed and undeformed cross-sections. Cross-sectional reference frames in ***Figure 1*** and in other coming figures are denoted by F_{S0} before deformation and by F_S after deformation. The cross-section location before deformation is found by s whose domain is $[0 \quad L]$, where L is the beam length before deformation. The three field variables, viz $u(s)$, $v(s)$ and $w(s)$ are elastic axial, and two perpendicular lateral deflection of the beam cross-section.

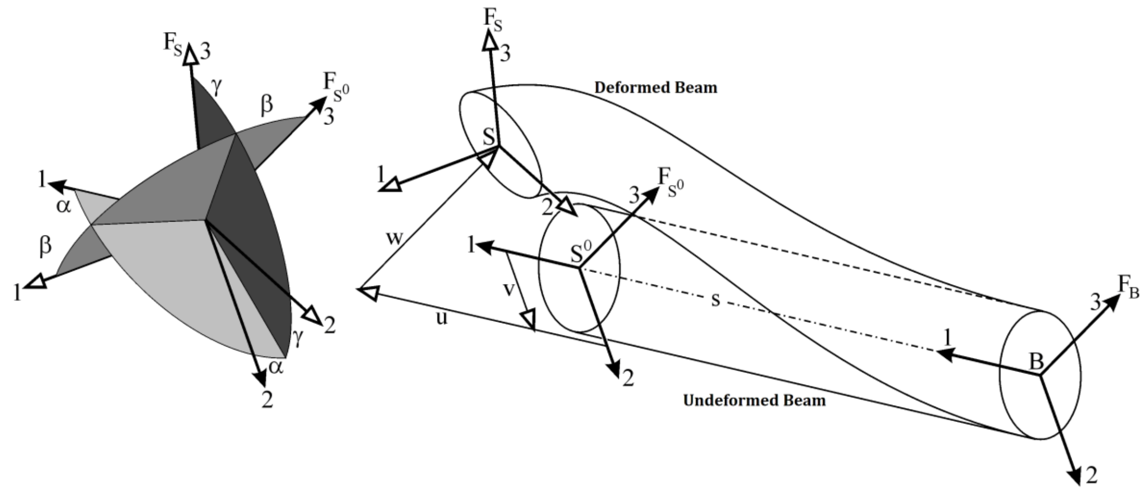

Figure 1. Elastic Coordinates **[5]**

The curve of beam central cord is assumed tangent to the first axis of cross-sectional reference frames. The field variables $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are the elastic Euler angles constructing rotation transformation from F_{S^0} to F_S in accordance with the ***Expression (1)*** with in which the sequence of the angles is capable of being interpreted in four different manners. Theses angles are called Bryant angles in some references. The rotation transformation of ***Expression (1)*** shows the elastic orientation of the beam cross-section.

$$R_{SS^0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & s\gamma \\ 0 & -s\gamma & c\gamma \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

The deformed cross-section is assumed not to undergo inplane and out-of-plane warpings and remains perpendicular to the beam central cord. Complying with these assuptions, the cross-section is considered a circle in order not to cause out-of-plane warpings due to torsion according to Saint Venant theory. Neglecting the inplane warping implies that the Poission's ratio is assumed zero.

3. STRUCTURAL HOLONOMIC CONSTRAINS

The dependency of the six elastic coordinates, viz $u(s)$, $v(s)$, $w(s)$, $\alpha(s)$, $\beta(s)$ and $\gamma(s)$, is understood from considering the rectangular triangles in ***Figure 2*** and is revealed by the two structural holonomic constraints defined by ***Equations (2)-(3)***.

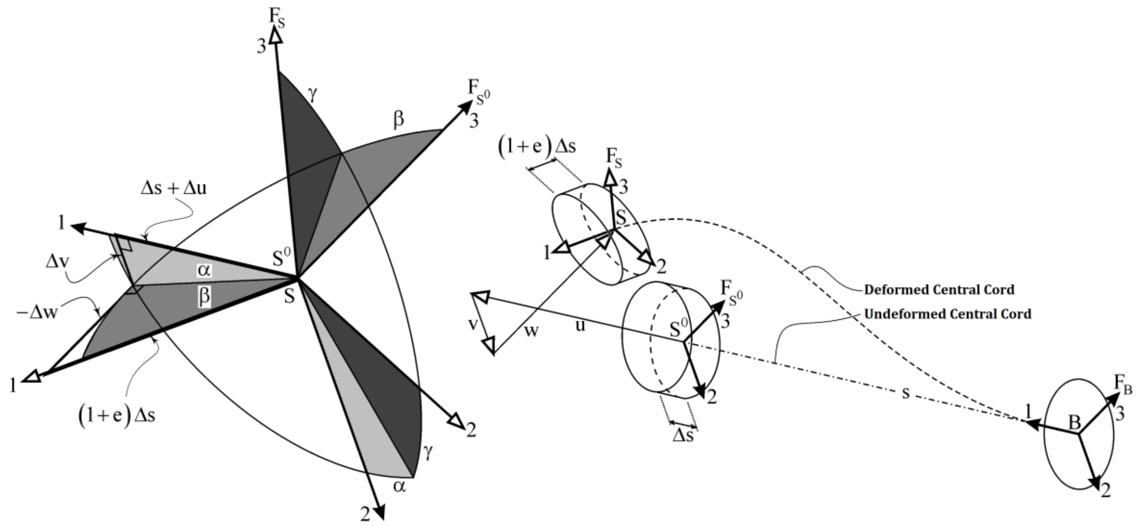

Figure 2. Structural Holonomic constraints **[5]**

$$\alpha = \lim_{\Delta s \to 0} \mathrm{Arc\,tan}\left(\frac{\Delta v}{\Delta s + \Delta u}\right) = \mathrm{Arc\,tan}\left(\frac{v'}{1+u'}\right) \tag{2}$$

$$\beta = \lim_{\Delta s \to 0} \mathrm{Arc\,tan}\frac{-\Delta w}{\sqrt{(\Delta s + \Delta u)^2 + \Delta v^2}} = -\mathrm{Arc\,tan}\frac{w'}{\sqrt{(1+u')^2 + v'^2}} \tag{3}$$

Because of the fact that the constraints of ***Equations (2)-(3)*** are holonomic, they might be eliminated along with two arbitrarily-chosen superfluous elastic coordinates. The remained four elastic coordinates are called elastic degrees of freedom. In the current research, the elastic degrees of freedom are chosen as $u(s)$, $v(s)$, $w(s)$ and $\gamma(s)$ that respectively express axial, lateral, another lateral and twisting deflections of the cross-section located at s before deformation.

4. AXIAL STRAIN

In the right hand side of ***Figure 2***, the beam element length is Δs before deformation and is $(1+e)\Delta s$ after deformation. There e denotes the axial strain. If the origins of F_S and F_{S^0} were translationally coincided with each other, the center position components of the deformed beam element flat face being parallel to the other flat face involving the origin of F_S would be involved by $\left[(1+e)\Delta s \quad 0 \quad 0\right]^T$ with respect to F_S, and would simultaneously be involved by $\left[(\Delta s+\Delta u) \quad \Delta v \quad \Delta w\right]^T$ with respect to F_{S^0}. As a result, the ***Equation (4)*** giving e implicitly is concluded.

$$\left[(\Delta s+\Delta u) \quad \Delta v \quad \Delta w\right] R_{S^0 S} = \left[(1+e)\Delta s \quad 0 \quad 0\right] \tag{4}$$

From solving ***Equation (4)*** for e, ***Expression (5)*** giving the axial strain explicitly is found.

$$e = \lim_{\Delta s \to 0} \frac{\sqrt{(\Delta s+\Delta u)^2 + \Delta v^2 + \Delta w^2}}{\Delta s} - 1 = \sqrt{(1+u')^2 + v'^2 + w'^2} - 1 \tag{5}$$

Two agent variables, viz h and r, are respectively defined by ***Expressions (6)-(7)*** are used in the formulae of the current manuscript.

$$h \equiv 1 + u' \tag{6}$$

$$r \equiv \sqrt{(1+u')^2 + v'^2} \tag{7}$$

5. NORMALIZED CURVATURE

The rotation transformation from F_{S^0} to F_S of ***Expression 1*** might be simplified to ***Expression 8***.

$$\underset{SS^0}{R} = \begin{bmatrix} \frac{h}{e+1} & \frac{v'}{e+1} & \frac{w'}{e+1} \\ -\frac{v'\cos\gamma}{r} - \frac{w'h\sin\gamma}{r(e+1)} & \frac{h\cos\gamma}{r} - \frac{w'v'\sin\gamma}{r(e+1)} & \frac{r\sin\gamma}{e+1} \\ \frac{v'\sin\gamma}{r} - \frac{w'h\cos\gamma}{r(e+1)} & -\frac{h\sin\gamma}{r} - \frac{w'v'\cos\gamma}{r(e+1)} & \frac{r\cos\gamma}{e+1} \end{bmatrix} \tag{8}$$

Orthogonal virtual rotation of the beam cross-section is denoted by the imperfect differential $\delta\Pi^S$ where Π^S is a quasicoordinate. Orthogonal virtual elastic rotation components are involved in ***Expression (9)*** that has been derived from ***Expression (8)***.

$$\delta\Pi^S = \begin{bmatrix} \delta\gamma \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{w'v'}{r^2(e+1)} & \frac{hw'}{r^2(e+1)} & 0 \\ \frac{-v'(e+1)\sin\gamma + hw'\cos\gamma}{r(e+1)^2} & \frac{h(e+1)\sin\gamma + v'w'\cos\gamma}{r(e+1)^2} & -\frac{r\cos\gamma}{(e+1)^2} \\ \frac{-v'(e+1)\cos\gamma - hw'\sin\gamma}{r(e+1)^2} & \frac{h(e+1)\cos\gamma - v'w'\sin\gamma}{r(e+1)^2} & \frac{r\sin\gamma}{(e+1)^2} \end{bmatrix} \begin{bmatrix} \delta u' \\ \delta v' \\ \delta w' \end{bmatrix} \tag{9}$$

The derivative of the imagined quasicoordinate Π^S with respect to s is the normalized curvature components and is given by ***Expressions (10)***. The real curvature components would be derived by differentiating Π^S with respect to $(1+e)s$.

$$\kappa^S = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} = \begin{bmatrix} \gamma' \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{w'v'}{r^2(e+1)} & \frac{hw'}{r^2(e+1)} & 0 \\ \frac{-v'(e+1)\sin\gamma + hw'\cos\gamma}{r(e+1)^2} & \frac{h(e+1)\sin\gamma + v'w'\cos\gamma}{r(e+1)^2} & -\frac{r\cos\gamma}{(e+1)^2} \\ \frac{-v'(e+1)\cos\gamma - hw'\sin\gamma}{r(e+1)^2} & \frac{h(e+1)\cos\gamma - v'w'\sin\gamma}{r(e+1)^2} & \frac{r\sin\gamma}{(e+1)^2} \end{bmatrix} \begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} \tag{10}$$

The normalized curvature skew-symmetric tensor components matrix is useful for deriving the derivative of elastic rotation tensor components matrix $\underset{SS^0}{R'}$ with respect to s, that is equal to $-\overset{S}{\kappa}\,\underset{SS^0}{R}$.

6. DISPLACEMENT

Beam element displacement with respect to undeformed cross-sectional reference frame is given ***Expression (11)*** by considering ***Figure 3***.

$$\Delta = \begin{bmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \underset{S^0S}{R}\, p - p = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \underset{S^0S}{R} \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \tag{11}$$

Beam element position with respect to the undeformed and deformed cross-sectional reference frames is constant, because of neglecting in plane and out-of-plane warpings. The components of the beam element position are involved in $p \equiv \begin{bmatrix} 0 & y & z \end{bmatrix}^T$ as shown in ***Figure 3***.

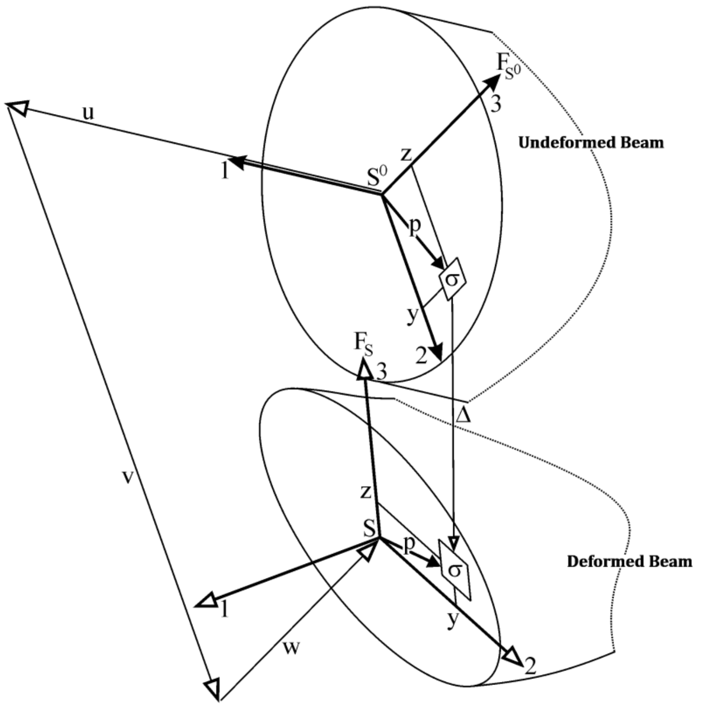

Figure 3. Beam Element Displacement with respect to Undeformed Cross-section **[5]**

7. CROSS-SECTIONAL AREA 1ST AND 2ND MOMENTS OF INERTIAS

Beam element position components with respect to the cross-sectional reference frame, ie F_S, is given by $p \equiv \begin{bmatrix} 0 & y & z \end{bmatrix}^T$ as shown in ***Figure 3***. If the origin of F_S is coincided with the centroid of the cross-sectional area, then the cross-sectional area 1st moment of inertia vanishes, that is $\int_A p dA = [0]_{3\times1}$, $\int_A p^T dA = [0]_{1\times3}$ and $\int_A p \, dA = [0]_{3\times3}$. Components matrix of cross-sectional area 2nd moments of inertia tensor with respect to F_S at centroid, ie $[J_S] \equiv \int_A - p\, p \, dA$, reduces to $J \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for circular cross-section. Cross-sectional area 2nd moment and 2nd polar moment of inertias are denoted by J and $2J$, respectively.

8. STRAIN

The Green's strain tensor was introduced by Green and Saint Venant in Lagrangian coordinates. Green's Infinitesimal Strain components are given by $\varepsilon_{nk}=\varepsilon_{kn}=\partial\Delta_n/2\partial X_k+\partial\Delta_k/2\partial X_n$ where $X_n, X_k\in\{X,Y,Z\}$. The derivatives of the beam element displacement, ie Δ, with respect to the Lagrangian coordinates $X\equiv s$, Y and Z given by ***Expressions (12)*** are useful for deriving the Green's strain components.

$$\frac{\partial\Delta}{\partial X}=\frac{\partial\Delta}{\partial s}=\begin{bmatrix}u'\\v'\\w'\end{bmatrix}+\underset{s^0s}{R}\begin{bmatrix}-y\kappa_z+z\kappa_y\\-z\kappa_x\\y\kappa_x\end{bmatrix},\ \frac{\partial\Delta}{\partial Y}=\begin{bmatrix}0&0&0\end{bmatrix}^T,\ \frac{\partial\Delta}{\partial Z}=\begin{bmatrix}0&0&0\end{bmatrix}^T \qquad (12)$$

Considering ***Expressions (12)***, the strain components are found as given by ***Expressions (13).***

$$\varepsilon_{xx}=\frac{\partial\Delta_x}{\partial X}=u'+y\left[\frac{v'\kappa_x}{r}\sin\gamma-\frac{1+u'}{1+e}\left(\kappa_z+\frac{w'\kappa_x}{r}\cos\gamma\right)\right]+z\left[\frac{v'\kappa_x}{r}\cos\gamma+\frac{1+u'}{1+e}\left(\kappa_y+\frac{w'\kappa_x}{r}\sin\gamma\right)\right]$$

$$\varepsilon_{xy}=\varepsilon_{yx}=\frac{1}{2}\frac{\partial\Delta_y}{\partial X}=\frac{v'}{2}-\frac{y}{2}\left[\frac{1+u'}{r}\kappa_x\sin\gamma+\frac{v'}{1+e}\left(\kappa_z+\frac{w'\kappa_x}{r}\cos\gamma\right)\right]-\frac{z}{2}\left[\frac{1+u'}{r}\kappa_x\cos\gamma-\frac{v'}{1+e}\left(\kappa_y+\frac{w'\kappa_x}{r}\sin\gamma\right)\right]$$

$$\varepsilon_{xz}=\varepsilon_{zx}=\frac{1}{2}\frac{\partial\Delta_z}{\partial X}=\frac{w'}{2}+\frac{y}{2}\left[\frac{-w'\kappa_z+r\kappa_x\cos\gamma}{1+e}\right]+\frac{z}{2}\left[\frac{w'\kappa_y-r\kappa_x\sin\gamma}{1+e}\right]$$

$$\varepsilon_{n_{yy}}=\varepsilon_{n_{zz}}=\varepsilon_{n_{yz}}=0 \qquad (13)$$

9. STRESS

The beam material is assumed linearly elastic, homogeneous and isotropic whose Young's modulus is denoted by E. Stress strain relationship for linearly-elastic isotropic material According to the Hooke's law is $\tau_{nk}=2G\,\varepsilon_{nk}+\lambda\,\varepsilon_{mm}\delta_{nk}$ where $\lambda=\nu E/(1+\nu)(1-2\nu)$ and $G=E/2(1+\nu)$. Since the present manuscript neglects the Poisson's ratio, ie $\nu=0$, the Hooke's law reduces to $\tau_{nk}=E\,\varepsilon_{nk}$. Stress components are given by ***Expresstions (14)*** for large elastic orientation.

$$\tau_{xx}=E\left\{u'+\hat{y}\left[\frac{v'\kappa_x}{r}\sin\gamma-\frac{1+u'}{1+e}\left(\kappa_z+\frac{w'\kappa_x}{r}\cos\gamma\right)\right]+\hat{z}\left[\frac{v'\kappa_x}{r}\cos\gamma_n+\frac{1+u'}{1+e}\left(\kappa_y+\frac{w'\kappa_x}{r}\sin\gamma\right)\right]\right\}$$

$$\tau_{xy}=\tau_{yx}=E\left\{\frac{v'}{2}-\frac{\hat{y}}{2}\left[\frac{1+u'}{r}\kappa_x\sin\gamma+\frac{v'}{1+e}\left(\kappa_z+\frac{w'\kappa_x}{r}\cos\gamma\right)\right]\right.$$
$$\left.-\frac{\hat{z}}{2}\left[\frac{1+u'}{r}\kappa_x\cos\gamma-\frac{v'}{1+e}\left(\kappa_y+\frac{w'\kappa_x}{r}\sin\gamma\right)\right]\right\}$$

$$\tau_{xz}=\tau_{zx}=E\left\{\frac{w'}{2}+\frac{\hat{y}}{2}\left[\frac{-w'\kappa_z+r\kappa_x\cos\gamma}{1+e}\right]+\frac{\hat{z}}{2}\left[\frac{w'\kappa_y-r\kappa_x\sin\gamma}{1+e}\right]\right\}$$

$$\tau_{yy}=\tau_{zz}=\tau_{yz}=\tau_{zy}=0 \qquad (14)$$

10. EQUILIBRIUM

Elastic virtual work is the variation of strain energy of the beam, ie δU, as shown by ***Expression (15)*** in this manuscript. The beam equilibrium is explained by the equation $\delta U = 0$.

$$\delta U = \int_0^L \int_A \left\{ \tau_{xx}\delta\varepsilon_{xx} + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{xz}\delta\varepsilon_{xz} \right\} dAds \tag{15}$$

Variations of the strain components, ie $\delta\varepsilon_{xx}$, $\delta\varepsilon_{xy}$ and $\delta\varepsilon_{xz}$, used in ***Expression (15)*** are given in term of the variations of curvature components, field variable derivatives and agent variables in ***Appendix A***. Variation of the normalized curvature components, ie $\delta\kappa_x$, $\delta\kappa_y$ and $\delta\kappa_z$, are given in terms, $\delta u'$, $\delta v'$, $\delta w'$, $\delta\gamma$, $\delta u''$, $\delta v''$, $\delta w''$, $\delta\gamma'$ in ***Appendix B***. Variations of the agent variables r and e are $\delta r = \frac{(1+u')\delta u' + v'\delta v'}{r}$ and $\delta e = \frac{r\delta r + w'\delta w'}{e+1}$. As a result, on the basis of the variations of strain components, the long variation of strain energy, ie δU, is given in ***Appendix C*** in terms of the variations of the elastic degrees of freedom and whose derivatives, ie $\delta\gamma$, $\delta\gamma'$, $\delta u'$, $\delta v'$, $\delta w'$, $\delta u''$, $\delta v''$, $\delta w''$. By means of integration by part the too long weak form of δU of ***Appendix C*** is given in ***Appendix D*** in terms of the variations of the elastic degrees of freedom, ie $\delta\gamma$, δu, δv, δw for large elastic orientation. It is obvious that the integration by part involves differentiation operation.

11. PERFECT REDUCTION FOR SMALL ELASTIC ORIENTATION

The formulae in ***Appendices C-D*** might be reduced for small elastic orientation by assuming that the elastic Euler angles, ie α, β and γ, are small. In this situation, one requires the reduced ***Formulae (16)***.

$$\alpha \approx 0,\quad \beta \approx 0,\quad \gamma \approx 0,\quad v' \approx 0,\quad w' \approx 0,\quad e \approx u',\quad r \approx 1+u',\quad \kappa_x \approx \gamma',\quad \kappa_y \approx \frac{-w''}{1+u'},\quad \kappa_z \approx \frac{v''}{1+u'},\quad \delta e \approx \delta u',$$

$$\delta\Pi^S \approx [\delta\gamma \quad \delta\beta \quad \delta\alpha]^T,\quad \delta\kappa_x \approx \frac{v''}{(1+u')^2}\delta w' + \delta\gamma',\quad \delta\kappa_y \approx \frac{v''}{1+u'}\delta\gamma + \frac{w''}{(1+u')^2}\delta u' + \frac{u''}{(1+u')^2}\delta w' - \frac{1}{1+u'}\delta w'',$$

$$\delta\kappa_z \approx \frac{w''}{1+u'}\delta\gamma - \frac{v''}{(1+u')^2}\delta u' - \frac{u''}{(1+u')^2}\delta v' + \frac{1}{1+u'}\delta v'',\qquad \underset{s^0 s}{R} \approx \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\delta \underset{s^0 s}{R} \approx \begin{bmatrix} 0 & -\delta\alpha & \delta\beta \\ \delta\alpha & 0 & -\delta\gamma \\ -\delta\beta & \delta\gamma & 0 \end{bmatrix} \tag{16}$$

The perfect reduced strain energy variation is given by ***Expression (17)*** that is found by reducing the formula of ***Appendix C*** for small elastic orientation.

$$\delta U = EJ\int_0^L \left\{ \gamma'\delta\gamma' + \left[\frac{A}{J}u' - \frac{v''^2 + w''^2}{(1+u')^3}\right]\delta u' - \left[\frac{\gamma' w''}{2(1+u')^2} + \frac{u''v''}{(1+u')^3}\right]\delta v' \right.$$
$$\left. + \left[\frac{3\gamma' v''}{2(1+u')^2} - \frac{u''w''}{(1+u')^3}\right]\delta w' + \frac{v''}{(1+u')^2}\delta v'' + \frac{w''}{(1+u')^2}\delta w'' \right\} ds \quad (17)$$

Expression (17) has been simplified to the form of ***Expression (18)*** in order to be more conveniently compared with the strain energy variation of the classical nonlinear 3D Euler-Bernoulli beam.

$$\delta U = EJ\int_0^L \left\{ \frac{A}{J} e\delta e + \kappa_z\delta\kappa_z + \kappa_y\delta\kappa_y + \kappa_x\delta\kappa_x + \frac{\kappa_x}{2\sqrt{(1+u')^2 + v'^2}}\left(\kappa_y\delta v' + \kappa_z\delta w'\right) \right\} ds \quad (18)$$

The weak form of the perfect reduced strain energy variation is given by ***Expression (19)*** that is found by reducing the formula of ***Appendix D*** for small elastic orientation.

Expressions (17)-(19) are all the same and represent the perfect reduced strain energy variation for small elastic orientation. However, the perfect reduced strain energy variation of ***Expressions (17)-(18)*** cannot, and that of ***Formula (19)*** can produce perfect reduced equilibrium equations and perfect reduced cross-sectional stress resultants and perfect reduced boundary conditions. It is because of the fact that the perfect reduced ***Expression (19)*** is a weak form and does not require differentiation operation. However, the perfect reduced ***Expressions (17)-(18)*** are not weak forms and require integration by part that involves differentiation operation. Because of the fact that ***Expressions (17)-(18)*** are reductions for small elastic orientation, the differentiation operation cannot have access to many terms of the strain energy variation for producing perfect reduced equilibrium equations, cross-sectional stress resultants and boundary conditions for small elastic orientation.

$$\delta U = EJ\int_0^L \left\{ \left[-\gamma'' + \frac{-v''w''}{(1+u')^2} \right]\delta\gamma \right.$$
$$+\left[\frac{-A}{J}u'' + \frac{-2v'''v'' - 2w'''w'' - \gamma'v''^2}{2(1+u')^3} + \frac{2u''v''^2 + 2u''w''^2}{(1+u')^4} \right]\delta u$$
$$+\left[\frac{-A}{2J}v'' + \frac{2v'''' + 4\gamma'w''' + 7\gamma''w'' - \gamma'^2 v''}{2(1+u')^2} + \frac{-3u'''v'' - 4u''v''' - 6\gamma'u''w''}{(1+u')^3} \right.$$
$$\left. + \frac{-5v''^3 + 2v''w''^2 + 16u''^2 v''}{2(1+u')^4} \right]\delta v$$
$$+\left[\frac{-A}{2J}w'' + \frac{2w'''' - 4\gamma'v''' - 5\gamma''v'' - \gamma'^2 w''}{2(1+u')^2} + \frac{-3u'''w'' - 4u''w''' + 6\gamma'u''v''}{(1+u'_n)^3} \right.$$
$$\left. + \frac{-5w''^3 + 2v''^2 w'' + 16u''^2 w'' - 12v''^2 w''}{2(1+u'_n)^4} \right]\delta w$$
$$\left. \vphantom{\int} \right\} ds + EJ\left\{ \gamma'\delta\gamma + \frac{A}{J}u'\,\delta u + \left[\frac{-v''' - 2\gamma'w''}{(1+u')^2} + \frac{2u''v''}{(1+u')^3} \right]\delta v \right. \tag{19}$$
$$\left. \left. + \left[\frac{-w''' + 2\gamma'v''}{(1+u')^2} + \frac{2u''w''}{(1+u')^3} \right]\delta w + \frac{v''}{(1+u')^2}\delta v' + \frac{w''}{(1+u')^2}\delta w' \right\} \right|_0^L$$

12. NONLINEAR 3D EULER-BERNOULLI BEAM

As a matter of fact, the strain components proposed in the classical nonlinear 3D Euler-Bernoulli beam theory [8] are infinitesimal Green's strains reduced for small elastic orientation. They are perfect reduced formulae and are given by ***Expressions (20)***.

$$\varepsilon_{XX} = e - y\kappa_z + z\kappa_y, \qquad \varepsilon_{XY} = -\frac{1}{2}\kappa_x z, \qquad \varepsilon_{XZ} = \frac{1}{2}\kappa_x y, \qquad \varepsilon_{YY} = \varepsilon_{ZZ} = \varepsilon_{YZ} = 0 \tag{20}$$

The strain variations in ***Expressions (21)*** are imperfect reduced formulae, because they are found on the basis of reduced formulae, ie ***Expressions (20)*** [8].

$$\delta\varepsilon_{XX} = \delta e - y\,\delta\kappa_z + z\,\delta\kappa_y, \quad \delta\varepsilon_{XY} = -\frac{1}{2}z\,\delta\kappa_x, \quad \delta\varepsilon_{XZ} = \frac{1}{2}y\,\delta\kappa_x, \quad \delta\varepsilon_{YY} = \delta\varepsilon_{ZZ} = \delta\varepsilon_{YZ} = 0 \tag{21}$$

On the basis of ***Expressions (21)***, the obtained strain energy variation in ***Expression (22)*** is an imperfect reduced formula. ***Expression (22)*** might be rewritten as given by ***Expression (23)***. It is obvious that ***Expression (23)*** is also imperfect reduced formula. Naturally, on the basis of ***Expression (23)***, the obtained weak form of the strain energy variation, ie ***Expression (24)***, and eventually the obtained equilibrium equations, cross-sectional stress resultants and boundary conditions would be imperfect reductions in the classical nonlinear 3D Euler-Bernoulli beam theory [8].

$$\delta U_{CLASSIC} = EJ\int_0^L \left\{ \frac{A}{J} e\delta e + \kappa_z\delta\kappa_z + \kappa_y\delta\kappa_y + \kappa_x\delta\kappa_x \right\} ds \tag{22}$$

The strain energy variation of the classical nonlinear 3D Euler-Bernoulli beam theory, ie ***Expression (22)***, is rewritten as what given by ***Expression (23)***.

$$\delta U_{CLASSIC} = EJ\int_0^L \left\{ \gamma'\delta\gamma' + \left[\frac{A}{J}u' - \frac{v''^2 + w''^2}{(1+u')^3} \right]\delta u' - \frac{u''v''}{(1+u')^3}\delta v' + \left[\frac{\gamma'v''}{(1+u')^2} - \frac{u''w''}{(1+u')^3} \right]\delta w' + \frac{v''}{(1+u')^2}\delta v'' + \frac{w''}{(1+u')^2}\delta w'' \right\} ds \tag{23}$$

The strain energy variation given by ***Expression (23)*** is an imperfect reduced formula and on the basis of which the obtained weak form of strain energy variation, ie ***Expression (24)***, is much more imperfect in the classical nonlinear 3D Euler-Bernoulli beam theory.

$$\begin{aligned}\delta U_{CLASSIC} = EJ\int_0^L \Bigg\{ & -\gamma''\delta\gamma + \left[-\frac{A}{J}u'' + \frac{2v'''v'' + 2w'''w''}{(1+u')^3} + \frac{-3u''v''^2 - 3u''w''^2}{(1+u')^4} \right]\delta u \\ & + \left[\frac{v''''}{(1+u')^2} + \frac{-u'''v'' - 3u''v'''}{(1+u')^3} + \frac{3u''^2v''}{(1+u')^4} \right]\delta v \\ & + \left[\frac{w'''' - \gamma''v'' - \gamma'v'''}{(1+u')^2} + \frac{-u'''w'' - 3u''w''' + 2\gamma'u''v''}{(1+u')^3} + \frac{3u''^2w''}{(1+u')^4} \right]\delta w \Bigg\} ds \\ + EJ\Bigg\{ & \gamma'\delta\gamma + \left[\frac{A}{J}u' - \frac{v''^2 + w''^2}{(1+u')^3} \right]\delta u + \left[\frac{-v'''}{(1+u')^2} + \frac{u''v''}{(1+u')^3} \right]\delta v + \left[\frac{-w''' + \gamma'v''}{(1+u')^2} + \frac{u''w''}{(1+u')^3} \right]\delta w \\ & + \frac{v''}{(1+u')^2}\delta v' + \frac{w''}{(1+u')^2}\delta w' \Bigg\}\Bigg|_0^L \end{aligned} \tag{24}$$

13. RESULTS, DISCUSSION AND VERIFICATION

Concealing the beam external loads, on the basis of the minimization of the strain energy, ie $\delta U = 0$, and the perfect reduced weak form of the strain energy variation for small elastic orientation, ie ***Expression (19)***, one might easily find the perfect reduced equilibrium equations, the perfect reduced cross-sectional stress resultants and the perfect reduced boundary conditions for a geometrically-nonlinear linearly-elastic isotropic homogeneous spatial Euler-Bernoulli beam.

13.1 EQUILIBRIUM EQUATIONS:

It should be noted that the external loads have not been considered in ***Equation (25)-(28)*** of equilibrium. The perfect reduced balance equation of twisting moments is given by ***Equation (25)*** where the algebraic term $EJ\frac{-v''w''}{(1+u')^2}$ has not been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory.

$$EJ\left[-\gamma''+\frac{-v''w''}{(1+u')^2}\right]=0 \tag{25}$$

The perfect reduced balance equation of axial forces is given by ***Equation (26)*** where the algebraic term $EJ\left[\frac{-6v'''v''-6w'''w''-\gamma'v''^2}{2(1+u')^3}+\frac{5u''v''^2+5u''w''^2}{(1+u')^4}\right]$ has not been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory.

$$EJ\left[\frac{-A}{J}u''+\frac{-2v'''v''-2w'''w''-\gamma'v''^2}{2(1+u')^3}+\frac{2u''v''^2+2u''w''^2}{(1+u')^4}\right]=0 \tag{26}$$

The perfect reduced balance equation of lateral forces along v axis is given by ***Equation (27)*** where the algebraic term $EJ\left[-\frac{A}{2J}v''+\frac{-\gamma'^2v''+4\gamma'w'''+7\gamma''w''}{2(1+u')^2}+\frac{-2u'''v''-u''v'''-6\gamma'u''w''}{(1+u')^3}+\frac{-5v''^3+10u''^2v''+2w''^2v''}{2(1+u')^4}\right]$ has not been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory.

$$EJ\left[\frac{-A}{2J}v''+\frac{2v''''+4\gamma'w'''+7\gamma''w''-\gamma'^2v''}{2(1+u')^2}+\frac{-3u'''v''-4u''v'''-6\gamma'u''w''}{(1+u')^3}\right.$$
$$\left.+\frac{-5v''^3+2v''w''^2+16u''^2v''}{2(1+u')^4}\right]=0 \tag{27}$$

The perfect reduced balance equation of lateral forces along w axis is given by ***Equation (28)*** where the algebraic term $EJ\left[-\frac{A}{2J}w''+\frac{-\gamma'^2w''-2\gamma'v'''-3\gamma''v''}{2(1+u')^2}+\frac{-2u'''w''-u''w'''+4\gamma'u''v''}{(1+u'_n)^3}+\frac{-5w''^3+10u''^2w''-10v''^2w''}{2(1+u'_n)^4}\right]$ has not been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory. It should be noted that the term $\frac{-6EJv''^2w''}{(1+u'_n)^4}$ given by ***Equation (28)*** does not have a counterpart in the corresponding ***Equation (27)***, because the elastic rotation transformation is antisymmetric.

$$\mathrm{EJ}\left[\frac{-A}{2J}w''+\frac{2w''''-4\gamma'v'''-5\gamma''v''-\gamma'^2w''}{2(1+u')^2}+\frac{-3u'''w''-4u''w'''+6\gamma'u''v''}{(1+u_n')^3}\right.$$
$$\left.+\frac{-5w''^3+2v''^2w''+16u''^2w''-12v''^2w''}{2(1+u_n')^4}\right]=0 \tag{28}$$

13.2 BOUNDARY CONDITIONS:

The beam boundary conditions is given by the rational expression ***Expression (29)*** being considered for each of the two end cross-sections located at $s=0$ and $s=L$. However, because of the lack of the missed terms, the following rational expression

$$(\mathrm{EJ}\gamma'=0\vee\gamma=0)\wedge\left(\mathrm{EJ}\left[\frac{A}{J}u'-\frac{v''^2+w''^2}{(1+u')^3}\right]=0\vee u=0\right)\wedge\left(\mathrm{EJ}\left[\frac{-v'''}{(1+u')^2}+\frac{u''v''}{(1+u')^3}\right]=0\vee v=0\right)\wedge$$
$$\left(\mathrm{EJ}\left[\frac{-w'''+\gamma'v''}{(1+u')^2}+\frac{u''w''}{(1+u')^3}\right]=0\vee w=0\right)\wedge\left(\mathrm{EJ}\frac{v''}{(1+u')^2}=0\vee v'=0\right)\wedge\left(\mathrm{EJ}\frac{w''}{(1+u')^2}=0\vee w'=0\right)$$

would be the boundary conditions in the classical nonlinear 3D Euler Bernoulli beam theory. The rational expression ***Expression (29)*** gives the perfect reduced boundary conditions.

$$(\mathrm{EJ}\gamma'=0\ \vee\ \gamma=0)\ \wedge\ (\mathrm{EA}u'=0\ \vee\ u=0)\ \wedge$$
$$\left(\mathrm{EJ}\left[\frac{-v'''-2\gamma'w''}{(1+u')^2}+\frac{2u''v''}{(1+u')^3}\right]=0\ \vee\ v=0\right)\ \wedge\ \left(\mathrm{EJ}\frac{v''}{(1+u')^2}=0\ \vee\ v'=0\right)\ \wedge \tag{29}$$
$$\left(\mathrm{EJ}\left[\frac{-w'''+2\gamma'v''}{(1+u')^2}+\frac{2u''w''}{(1+u')^3}\right]=0\ \vee\ w=0\right)\ \wedge\ \left(\mathrm{EJ}\frac{w''}{(1+u')^2}=0\ \vee\ w'=0\right)$$

13.3 CROSS-SECTIONAL STRESS RESULTANTS:

The perfect reduced cross-sectional twisting moment is given by ***Expression (30)***

$$\mathrm{EJ}\gamma'=\mathrm{G}2\mathrm{J}\gamma' \tag{30}$$

The perfect reduced cross-sectional axial force is given by ***Expression (31)*** where the algebraic term $\mathrm{EJ}\left[\frac{v''^2+w''^2}{(1+u')^3}\right]$ has not algebraicly been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory.

$$\mathrm{EA}u' \tag{31}$$

The perfect reduced cross-sectional shearing force along v axis is given by ***Expression (32)*** where the algebraic term $EJ\left[\frac{-2\gamma' w''}{(1+u')^2}+\frac{u''v''}{(1+u')^3}\right]$ has not been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory.

$$EJ\left[\frac{-v'''-2\gamma' w''}{(1+u')^2}+\frac{2u''v''}{(1+u')^3}\right] \tag{32}$$

The perfect reduced cross-sectional shearing force along w axis is given by ***Expression (33)*** where the algebraic term $EJ\left[\frac{\gamma' v''}{(1+u')^2}+\frac{u''w''}{(1+u')^3}\right]$ has not been lost, contrary to the classical nonlinear 3D Euler Bernoulli beam theory.

$$EJ\left[\frac{-w'''+2\gamma' v''}{(1+u')^2}+\frac{2u''w''}{(1+u')^3}\right] \tag{33}$$

The perfect reduced cross-sectional flexural moment about w axis is given by ***Expression (34)***.

$$EJ\frac{v''}{(1+u')^2} \tag{34}$$

The perfect reduced cross-sectional flexural moment about v axis is given by ***Expression (35)***.

$$EJ\frac{w''}{(1+u')^2} \tag{35}$$

13.4 NONWEAK FORM OF STRAIN ENERGY VARIATION:

The extracted nonweak form of strain energy variation under small elastic orientation, ie ***Expression (18)***, is perfect and has not missed any terms, because it is a reduction found from the nonweak form of strain energy variation under large elastic orientation proposed in Appendix C. However, in the classical nonlinear 3-D Euler-Bernoulli beam theory [8], the nonweak form of strain energy variation under small elastic orientation, ie ***Expression (22)***, is imperfect, because it has primarily missed the terms represented by either ***Expression (36)*** or ***(37).*** It has missed the terms, because it has been originated from the reduced strains, ie ***Expressions (20)***, within which the elastic Euler angles have been converted into zero, and as a result, the differentiation operation for finding strain variation cannot produce the whole of the terms.

$$\delta U-\underset{\text{CLASSIC}}{\delta U}=EJ\int_0^L\left\{\frac{\kappa_x}{2\sqrt{(1+u')^2+v'^2}}\left(\kappa_y\delta v'+\kappa_z\delta w'\right)\right\}ds \tag{36}$$

By substituting the reduced normalized curvature components for small elastic orientation into ***Expression (25)***, the missed terms are proposed in terms of the derivatives of the elastic degrees of freedom as shown by ***Expression (26)***.

$$\delta U - \underset{\text{CLASSIC}}{\delta U} = EJ\int_0^L \left\{ \frac{-\gamma' w''}{2(1+u')^2}\delta v' + \frac{\gamma' v''}{2(1+u')^2}\delta w' \right\} ds \tag{37}$$

13.5 WEAK FORM OF STRAIN ENERGY VARIATION:

The extracted weak form of strain energy variation under small elastic orientation, ie ***Expression (19)***, is perfect and has not missed any terms, because it is a reduction found from the weak form of strain energy variation under large elastic orientation proposed in Appendix D. However, in the classical nonlinear 3-D Euler-Bernoulli beam theory [8], the weak form of strain energy variation under small elastic orientation, ie ***Expression (24)***, is imperfect, because it has missed a considerable number of terms, ie ***Expressions (38)***. It has missed the terms**,** both because of the lack of the primarily missed terms and because it is originated from the imperfect reduced strains energy variation, ie ***Expressions (23)*** on the basis of which the differentiation operation in the by part integration cannot produce the whole of the terms.

$$\begin{aligned}
\delta U - \underset{\text{CLASSIC}}{\delta U} = EJ\int_0^L \Bigg\{ & +\frac{-v''w''}{(1+u')^2}\delta\gamma + \left[\frac{-6v'''v'' - 6w'''w'' - \gamma' v''^2}{2(1+u')^3} + \frac{5u''v''^2 + 5u''w''^2}{(1+u')^4} \right]\delta u \\
& + \left[-\frac{A}{2J}v'' + \frac{-\gamma'^2 v'' + 4\gamma' w''' + 7\gamma'' w''}{2(1+u')^2} + \frac{-2u'''v'' - u''v''' - 6\gamma' u'' w''}{(1+u')^3} \right. \\
& \left. + \frac{-5v''^3 + 10u''^2 v'' + 2v''w''^2}{2(1+u')^4} \right]\delta v \\
& + \left[-\frac{A}{2J}w'' + \frac{-\gamma'^2 w'' - 2\gamma' v''' - 3\gamma'' v''}{2(1+u')^2} + \frac{-2u'''w'' - u''w''' + 4\gamma' u'' v''}{(1+u'_n)^3} \right. \\
& \left. + \frac{-5w''^3 + 10u''^2 w'' - 10v''^2 w''}{2(1+u'_n)^4} \right]\delta w \\
\Bigg\} ds & + EJ\left\{ \left[\frac{v''^2 + w''^2}{(1+u')^3} \right]\delta u + \left[\frac{-2\gamma' w''}{(1+u')^2} + \frac{u''v''}{(1+u')^3} \right]\delta v + \left[\frac{\gamma' v''}{(1+u')^2} + \frac{u''w''}{(1+u')^3} \right]\delta w \right\}\Bigg|_0^L
\end{aligned} \tag{38}$$

The lack of the missed terms given by ***Equation (38***) causes equilibrium equations, cross-sectional stress resultants and boundary conditions to miss a few terms.

13.6 REDUCIBILITY TO LINEAR BEAM FORMULATIONS:

Just by neglecting the nonlinear terms of ***Equations (25)-(28)*** and ***Expressions (30)-(35)*** and concealing external loads applied to beam, the famous linear equilibrium ***Equations (39)-(42)*** and ***Expressions (43)-(48)*** are found.

The balance equations of twisting moments, axial forces, lateral forces along v and w axes are found as given by ***Equations (39)-(42)***, respectively.

$$-EJ\gamma'' = 0 \quad (39)$$
$$-EJAu'' = 0 \quad (40)$$
$$-GAv'' + EJv'''' = 0 \quad (41)$$
$$-GAw'' + EJw'''' = 0 \quad (42)$$

The cross-sectional twisting moment, axial force, shearing force along v and w axes and flexural moment about w and v axes are found as given by ***Expressions (43)-(48))***, respectively.

$$2JG\gamma' \quad (43)$$
$$EAu' \quad (44)$$
$$-EJv''' \quad (45)$$
$$-EJw''' \quad (46)$$
$$EJv'' \quad (47)$$
$$EJw'' \quad (48)$$

CONCLUSIONS

In the present manuscript the reduction of a general formula has been called “reduced formula” with in which some field variables have been substituted by constants or zero. The derivative of a reduced formula that has been found by means of a mathematical differentiation operation is called an “imperfect reduced formula”. It is imperfect due to the fact that, the derivatives of some terms of the reduced formula have not come into existence, because the terms have been converted into constants or zero and cannot produce their derivatives under differentiation operation. It might be known as a fact that perfect reduction of governing equations and boundary conditions of a beam or other structures are achieved when the reduction is executed after, not before differentiation operation. In other words, for small elastic orientation the perfect reduced equilibrium equations, the perfect reduced cross-sectional stress resultants and the perfect reduced boundary conditions are achieved when the imposing of the small elastic orientation assumption takes place after executing differentiation operations in finding strain variation and in integrating by part in finding weak form of strain energy variation.

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Appendix A

Variations of strain components, ie $\delta\varepsilon_{xx}$, $\delta\varepsilon_{xz}$, $\delta\varepsilon_{xy}$, $\delta\varepsilon_{yy}$, $\delta\varepsilon_{zz}$ and $\delta\varepsilon_{yz}$, of geometrically-nonlinear linearly-elastic isotropic spatial Euler-Bernoulli beam under large elastic orientation are given by ***Expressions (A1)-(A6)*** in term of the variations of normalized curvature components, γ, field variable derivatives and agent variables [3].

$$
\begin{aligned}
\delta\varepsilon_{xx} =& \left[1-\frac{\hat{y}_n}{1+e_n}\left(\kappa_{n_z}+\frac{w'_n\kappa_{n_x}}{r_n}\cos\gamma_n\right)+\frac{\hat{z}_n}{1+e_n}\left(\kappa_{n_y}+\frac{w'_n\kappa_{n_x}}{r_n}\sin\gamma_n\right)\right]\delta u'_n \\
&+\left(\hat{y}_n\frac{\kappa_{n_x}}{r_n}\sin\gamma_n+\hat{z}_n\frac{\kappa_{n_x}}{r_n}\cos\gamma_n\right)\delta v'_n \\
&+\left[-\hat{y}_n\left(\frac{1+u'_n}{1+e_n}\right)\frac{\kappa_{n_x}}{r_n}\cos\gamma_n+\hat{z}_n\left(\frac{1+u'_n}{1+e_n}\right)\frac{\kappa_{n_x}}{r_n}\sin\gamma_n\right]\delta w'_n \\
&+\left\{\hat{y}_n\left[\frac{v'_n\kappa_{n_x}}{r_n}\cos\gamma_n+\left(\frac{1+u'_n}{1+e_n}\right)\frac{w'_n\kappa_{n_x}}{r_n}\sin\gamma_n\right]+\hat{z}_n\left[-\frac{v'_n\kappa_{n_x}}{r_n}\sin\gamma_n+\left(\frac{1+u'_n}{1+e_n}\right)\frac{w'_n\kappa_{n_x}}{r_n}\cos\gamma_n\right]\right\}\delta\gamma_n \\
&+\left\{\hat{y}_n\left[-\frac{v'_n\kappa_{n_x}}{r_n^{\,2}}\sin\gamma_n+\left(\frac{1+u'_n}{1+e_n}\right)\frac{w'_n\kappa_{n_x}}{r_n^{\,2}}\cos\gamma_n\right]+\hat{z}_n\left[-\frac{v'_n\kappa_{n_x}}{r_n^{\,2}}\cos\gamma_n-\left(\frac{1+u'_n}{1+e_n}\right)\frac{w'_n\kappa_{n_x}}{r_n^{\,2}}\sin\gamma_n\right]\right\}\delta r_n \\
&+\left\{\hat{y}_n\left[\frac{1+u'_n}{(1+e_n)^2}\left(\kappa_{n_z}+\frac{w'_n\kappa_{n_x}}{r_n}\cos\gamma_n\right)\right]+\hat{z}_n\left[-\frac{1+u'_n}{(1+e_n)^2}\left(\kappa_{n_y}+\frac{w'_n\kappa_{n_x}}{r_n}\sin\gamma_n\right)\right]\right\}\delta e_n \\
&+\left\{\hat{y}_n\left[\frac{v'_n}{r_n}\sin\gamma_n-\left(\frac{1+u'_n}{1+e_n}\right)\frac{w'_n}{r_n}\cos\gamma_n\right]+\hat{z}_n\left[\frac{v'_n}{r_n}\cos\gamma_n+\left(\frac{1+u'_n}{1+e_n}\right)\frac{w'_n}{r_n}\sin\gamma_n\right]\right\}\delta\kappa_{n_x} \\
&+\hat{z}_n\left(\frac{1+u'_n}{1+e_n}\right)\delta\kappa_{n_y}-\hat{y}_n\left(\frac{1+u'_n}{1+e_n}\right)\delta\kappa_{n_z}
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
\delta\varepsilon_{xz} =& \left[\frac{1}{2}-\hat{y}_n\frac{\kappa_{n_z}}{2(1+e_n)}+\hat{z}_n\frac{\kappa_{n_y}}{2(1+e_n)}\right]\delta w'_n+\left[-\hat{y}_n\frac{r_n\kappa_{n_x}\sin\gamma_n}{2(1+e_n)}-\hat{z}_n\frac{r_n\kappa_{n_x}\cos\gamma_n}{2(1+e_n)}\right]\delta\gamma_n \\
&+\left[\hat{y}_n\frac{\kappa_{n_x}\cos\gamma_n}{2(1+e_n)}-\hat{z}_n\frac{\kappa_{n_x}\sin\gamma_n}{2(1+e_n)}\right]\delta r_n+\left\{\hat{y}_n\left[\frac{w'_n\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n}{2(1+e_n)^2}\right]+\hat{z}_n\left[\frac{-w'_n\kappa_{n_y}+r_n\kappa_{n_x}\sin\gamma_n}{2(1+e_n)^2}\right]\right\}\delta e_n \\
&+\left[\hat{y}_n\frac{r_n\cos\gamma_n}{2(1+e_n)}-\hat{z}_n\frac{r_n\sin\gamma_n}{2(1+e_n)}\right]\delta\kappa_{n_x}+\hat{z}_n\frac{w'_n}{2(1+e_n)}\delta\kappa_{n_y}-\hat{y}_n\frac{w'_n}{2(1+e_n)}\delta\kappa_{n_z}
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
\delta\varepsilon_{xy} =& \left[-\hat{y}_n \frac{\kappa_{n_x}}{2r_n}\sin\gamma_n - \hat{z}_n \frac{\kappa_{n_x}}{2r_n}\cos\gamma_n\right]\delta u'_n \\
&+\left[\frac{1}{2}-\frac{\hat{y}_n}{2(1+e_n)}\left(\kappa_{n_z}+\frac{w'_n\kappa_{n_x}}{r_n}\cos\gamma_n\right)+\frac{\hat{z}_n}{2(1+e_n)}\left(\kappa_{n_y}+\frac{w'_n\kappa_{n_x}}{r_n}\sin\gamma_n\right)\right]\delta v'_n \\
&+\left[-\hat{y}_n\frac{\kappa_{n_x}v'_n}{2r_n(1+e_n)}\cos\gamma_n+\hat{z}_n\frac{\kappa_{n_x}v'_n}{2r_n(1+e_n)}\sin\gamma_n\right]\delta w'_n \\
&+\left\{-\hat{y}_n\left[\frac{1+u'_n}{2r_n}\kappa_{n_x}\cos\gamma_n-\frac{\kappa_{n_x}v'_n w'_n}{2r_n(1+e_n)}\sin\gamma_n\right]+\hat{z}_n\left[\frac{1+u'_n}{2r_n}\kappa_{n_x}\sin\gamma_n+\frac{\kappa_{n_x}v'_n w'_n}{2r_n(1+e_n)}\cos\gamma_n\right]\right\}\delta\gamma_n \\
&+\left\{\hat{y}_n\left[\frac{1+u'_n}{2r_n{}^2}\kappa_{n_x}\sin\gamma_n+\frac{\kappa_{n_x}v'_n w'_n}{2r_n{}^2(1+e_n)}\cos\gamma_n\right]-\hat{z}_n\left[-\frac{1+u'_n}{2r_n{}^2}\kappa_{n_x}\cos\gamma_n+\frac{\kappa_{n_x}v'_n w'_n}{2r_n{}^2(1+e_n)}\sin\gamma_n\right]\right\}\delta r_n \\
&+\left\{\hat{y}_n\frac{v'_n}{2(1+e_n)^2}\left(\kappa_{n_z}+\frac{w'_n\kappa_{n_x}}{r_n}\cos\gamma_n\right)-\hat{z}_n\frac{v'_n}{2(1+e_n)^2}\left(\kappa_{n_y}+\frac{w'_n\kappa_{n_x}}{r_n}\sin\gamma_n\right)\right\}\delta e_n \\
&+\left\{-\hat{y}_n\left[\frac{1+u'_n}{2r_n}\sin\gamma_n+\frac{v'_n}{2(1+e_n)}\left(\frac{w'_n}{r_n}\cos\gamma_n\right)\right]-\hat{z}_n\left[\frac{1+u'_n}{2r_n}\cos\gamma_n-\frac{v'_n}{2(1+e_n)}\left(\frac{w'_n}{r_n}\sin\gamma_n\right)\right]\right\}\delta\kappa_{n_x} \\
&+\hat{z}_n\frac{v'_n}{2(1+e_n)}\delta\kappa_{n_y}-\hat{y}_n\frac{v'_n}{2(1+e_n)}\delta\kappa_{n_z}
\end{aligned} \tag{A3}
$$

$$\delta\varepsilon_{yy}=0 \tag{A4}$$

$$\delta\varepsilon_{zz}=0 \tag{A5}$$

$$\delta\varepsilon_{yz}=0 \tag{A6}$$

Appendix B

Variation of the normalized curvature components, ie $\delta\kappa_x$, $\delta\kappa_y$ and $\delta\kappa_z$, of geometrically-nonlinear linearly-elastic isotropic spatial Euler-Bernoulli beam under large elastic orientation are given by ***Expressions (B1)-(B3)*** in term of $\delta u'$, $\delta v'$, $\delta w'$, $\delta\gamma$, $\delta u''$, $\delta v''$, $\delta w''$ and $\delta\gamma'$ [3].

$$\begin{aligned}
\delta\kappa_{n_x} = & \frac{1}{r_n^{4}(e_n+1)^{3}} w_n' \left\{ \left[2(e_n+1)^2 + r_n^{2}\right] h_n v_n' u_n'' + \left\{ (e_n+1)^2 v_n'^{2} - h_n^{2}\left[(e_n+1)^2 + r_n^{2}\right] \right\} v_n'' \right\} \delta u_n' + \\
& - \frac{1}{r_n^{4}(e_n+1)^{3}} w_n' \left\{ \left[2(e_n+1)^2 + r_n^{2}\right] h_n v_n' v_n'' + \left\{ (e_n+1)^2 h_n^{2} - v_n'^{2}\left[(e_n+1)^2 + r_n^{2}\right] \right\} u_n'' \right\} \delta v_n' + \\
& + \frac{1}{(e_n+1)^3}\left[h_n v_n'' - v_n' u_n''\right] \delta w_n' + \delta\gamma_n' - \frac{w_n' v_n'}{r_n^{2}(e_n+1)} \delta u_n'' + \frac{w_n' h_n}{r_n^{2}(e_n+1)} \delta v_n''
\end{aligned} \tag{B1}$$

$$\begin{aligned}
\delta\kappa_{n_y} = & \frac{1}{r_n(e_n+1)^2} \left\{ \left(h_n v_n'' - v_n' u_n''\right)(e_n+1)\cos\gamma_n + \left[r_n^{2} w_n'' - w_n'\left(h_n u_n'' + v_n' v_n''\right)\right]\sin\gamma_n \right\} \delta\gamma_n + \\
& + \frac{1}{r_n^{3}(e_n+1)^4} \left\{ \begin{bmatrix} \left(v_n'^{4} - 2h_n^{4} - h_n^{2} v_n'^{2} + w_n'^{2} v_n'^{2}\right) w_n' u_n'' + \\ -\left(3r_n^{2} + w_n'^{2}\right) h_n w_n' v_n' v_n'' + \left(r_n^{2} - w_n'^{2}\right) h_n r_n^{2} w_n'' \end{bmatrix} \cos\gamma_n \right. \\
& \left. \qquad + \left[\left(2r_n^{2} + w_n'^{2}\right) h_n v_n' u_n'' + \left(v_n'^{4} - h_n^{4} + w_n'^{2} v_n'^{2}\right) v_n''\right](e_n+1)\sin\gamma_n \right\} \delta u_n' + \\
& + \frac{1}{r_n^{3}(e_n+1)^4} \left\{ \begin{bmatrix} \left(h_n^{4} - 2v_n'^{4} - h_n^{2} v_n'^{2} + w_n'^{2} h_n^{2}\right) w_n' v_n'' + \\ -\left(3r_n^{2} + w_n'^{2}\right) h_n w_n' v_n' u_n'' + \left(r_n^{2} - w_n'^{2}\right) v_n' r_n^{2} w_n'' \end{bmatrix} \cos\gamma_n \right. \\
& \left. \qquad + \left[-\left(2r_n^{2} + w_n'^{2}\right) h_n v_n' v_n'' + \left(v_n'^{4} - h_n^{4} - w_n'^{2} h_n^{2}\right) u_n''\right](e_n+1)\sin\gamma_n \right\} \delta v_n' + \\
& + \frac{1}{r_n(e_n+1)^4} \left\{ \begin{aligned} & \left[\left(r_n^{2} - w_n'^{2}\right)\left(h_n u_n'' + v_n' v_n''\right) + 2r_n^{2} w_n' w_n''\right]\cos\gamma_n + \\ & \qquad + \left(v_n' u_n'' - h_n v_n''\right)(e_n+1) w_n' \sin\gamma_n \end{aligned} \right\} \delta w_n' + \\
& + \frac{1}{r_n(e_n+1)^2}\left[h_n w_n' \cos\gamma_n - (e_n+1) v_n' \sin\gamma_n\right] \delta u_n'' + \\
& + \frac{1}{r_n(e_n+1)^2}\left[v_n' w_n' \cos\gamma_n + (e_n+1) h_n \sin\gamma_n\right] \delta v_n'' - \frac{r_n}{(e_n+1)^2} \cos\gamma_n \delta w_n''
\end{aligned} \tag{B2}$$

$$
\begin{aligned}
\delta\kappa_{n_z} =& \frac{1}{r_n(e_n+1)^2}\left\{-\left(h_n v_n'' - v_n' u_n''\right)(e_n+1)\sin\gamma_n + \left[r_n^2 w_n'' - w_n'\left(h_n u_n'' + v_n' v_n''\right)\right]\cos\gamma_n\right\}\delta\gamma_n + \\
&+\frac{1}{r_n^3(e_n+1)^4}\left\{-\left[\begin{array}{l}\left(v_n'^4 - 2h_n^4 - h_n^2 v_n'^2 + w_n'^2 v_n'^2\right) w_n' u_n'' + \\ \qquad -\left(3r_n^2 + w_n'^2\right) h_n w_n' v_n' v_n'' + \left(r_n^2 - w_n'^2\right) h_n r_n^2 w_n''\end{array}\right]\sin\gamma_n\right. \\
&\left. + \left[\left(2r_n^2 + w_n'^2\right) h_n v_n' u_n'' + \left(v_n'^4 - h_n^4 + w_n'^2 v_n'^2\right) v_n''\right](e_n+1)\cos\gamma_n\right\}\delta u_n' + \\
&+\frac{1}{r_n^3(e_n+1)^4}\left\{-\left[\begin{array}{l}\left(h_n^4 - 2v_n'^4 - h_n^2 v_n'^2 + w_n'^2 h_n^2\right) w_n' v_n'' + \\ \qquad -\left(3r_n^2 + w_n'^2\right) h_n w_n' v_n' u_n'' + \left(r_n^2 - w_n'^2\right) v_n' r_n^2 w_n''\end{array}\right]\sin\gamma_n\right. \\
&\left. + \left[-\left(2r_n^2 + w_n'^2\right) h_n v_n' v_n'' + \left(v_n'^4 - h_n^4 - w_n'^2 h_n^2\right) u_n''\right](e_n+1)\cos\gamma_n\right\}\delta v_n' + \\
&+\frac{1}{r_n(e_n+1)^4}\left\{\begin{array}{l}-\left[\left(r_n^2 - w_n'^2\right)\left(h_n u_n'' + v_n' v_n''\right) + 2r_n^2 w_n' w_n''\right]\sin\gamma_n + \\ \qquad + \left(v_n' u_n'' - h_n v_n''\right)(e_n+1) w_n' \cos\gamma_n\end{array}\right\}\delta w_n' + \\
&+\frac{1}{r_n(e_n+1)^2}\left[-h_n w_n' \sin\gamma_n - (e_n+1) v_n' \cos\gamma_n\right]\delta u_n'' + \\
&+\frac{1}{r_n(e_n+1)^2}\left[-v_n' w_n' \sin\gamma_n + (e_n+1) h_n \cos\gamma_n\right]\delta v_n'' + \\
&+\frac{r_n}{(e_n+1)^2}\sin\gamma_n \delta w_n''
\end{aligned} \tag{B3}
$$

Appendix C

Nonweak form of variation of strain energy, ie δU, of geometrically-nonlinear linearly-elastic isotropic spatial Euler-Bernoulli beam under large elastic orientation is given by ***Expression (C1)*** in terms of the variations of the elastic degrees of freedom and whose derivatives, ie $\delta\gamma$, $\delta\gamma'$, $\delta u'$, $\delta v'$, $\delta w'$, $\delta u''$, $\delta v''$ and $\delta w''$ [3].

$$\delta U=\int_0^{L_n}\left\{\left\langle \frac{(1-\nu)EA_n}{(1+\nu)(1-2\nu)}u_n'+\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\right.\right.\right.$$

$$+\frac{1}{r_n^2(1+e_n)}\left[v_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)+\frac{h_n}{1+e_n}\left[\begin{matrix} w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)\\ +2r_n w_n'\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\end{matrix}\right]\right]+$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{h_n}{r_n}\left\{-v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{matrix} v_n' r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ -\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_z}\cos\gamma_n+\kappa_{n_y}\sin\gamma_n\right)\right]\end{matrix}\right]\right\}+$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{h_n}{e_n+1}\left[v_n' r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)-\frac{h_n}{1+e_n}\left[\begin{matrix} w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_z}^2+\kappa_{n_y}^2\right)\\ +2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\end{matrix}\right]\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{matrix} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n' u_n''+\left[(e_n+1)^2 v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2 w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_n v_n' u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2 w_n''\right]\sin\gamma_n\right.$$

$$\left.\left.+\left[\left(2r_n^2+w_n'^2\right)h_n v_n' u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\right\}\right\}+$$

$$+GJ\left\{\frac{\kappa_{n_x}}{r_n^2}\left[h_n\kappa_{n_x}+\frac{v_n'}{1+e_n}r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\right]+\right.$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{h_n}{r_n}\left\{-h_n^2\kappa_{n_x}-\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{\kappa_{n_x}}{(1+e_n)^2}\frac{h_n}{r_n}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{h_n}{e_n+1}\left[\begin{array}{l}h_n r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ -\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$-\frac{1}{(1+e_n)^3}\frac{h_n}{e_n+1}\left[r_n^2\kappa_{n_x}^2+w_n'^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)-2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n'u_n''+\left[(e_n+1)^2v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n'u_n''+\left[(e_n+1)^2v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_n v_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$

$$\cdot\left\{\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_n v_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2w_n''\right]\sin\gamma_n\right.$$

$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_n v_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\cdot\left\{-\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2w_n''\right]\sin\gamma_n\right.$$

$$\left.\left.+\left[\left(2r_n^2+w_n'^2\right)h_n v_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\right\}\right\}\Bigg\rangle\delta u_n'+$$

$$+\Bigg\langle GA_n v_n' + \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\Bigg\{ \frac{\kappa_{n_x}}{r_n^{2}}\left[v_n'\kappa_{n_x} + \frac{h_n r_n}{1+e_n}\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right)\right] +$$

$$+\frac{\kappa_{n_x}}{r_n^{3}}\frac{v_n'}{r_n}\left\{ -v_n'^{2}\kappa_{n_x} + \frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ -\dfrac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_z}\cos\gamma_n + \kappa_{n_y}\sin\gamma_n\right)\right]\end{array}\right]\right\} +$$

$$+\frac{h_n}{r_n^{2}(1+e_n)^{2}}\frac{v_n'}{e_n+1}\left[\begin{array}{l} v_n' r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ -\dfrac{h_n}{1+e_n}\left[w_n'^{2}\kappa_{n_x}^{2} + r_n^{2}\left(\kappa_{n_z}^{2} + \kappa_{n_y}^{2}\right) + 2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right] +$$

$$+\frac{1}{r_n^{2}}\frac{-1}{r_n^{4}(e_n+1)^{3}} w_n'\left\{ v_n'^{2}\kappa_{n_x} + \frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ +\dfrac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^{2} + r_n^{2}\right] h_n v_n' v_n'' + \left[(e_n+1)^{2} h_n^{2} - v_n'^{2}\left[(e_n+1)^{2} + r_n^{2}\right]\right] u_n''\right\} +$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^{3}(e_n+1)^{4}}\left[v_n'\kappa_{n_x}\cos\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_y} + w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{\Big[\left(h_n^{4} - 2v_n'^{4} - h_n^{2}v_n'^{2} + w_n'^{2}h_n^{2}\right) w_n' v_n'' - \left(3r_n^{2} + w_n'^{2}\right) h_n w_n' v_n' u_n'' + \left(r_n^{2} - w_n'^{2}\right) v_n' r_n^{2} w_n''\Big]\cos\gamma_n$$

$$+\Big[-\left(2r_n^{2} + w_n'^{2}\right) h_n v_n' v_n'' + \left(v_n'^{4} - h_n^{4} - w_n'^{2}h_n^{2}\right) u_n''\Big](e_n+1)\sin\gamma_n\Big\} +$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^{3}(e_n+1)^{4}}\left[-v_n'\kappa_{n_x}\sin\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_z} + w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{-\Big[\left(h_n^{4} - 2v_n'^{4} - h_n^{2}v_n'^{2} + w_n'^{2}h_n^{2}\right) w_n' v_n'' - \left(3r_n^{2} + w_n'^{2}\right) h_n w_n' v_n' u_n'' + \left(r_n^{2} - w_n'^{2}\right) v_n' r_n^{2} w_n''\Big]\sin\gamma_n$$

$$+\Big[-\left(2r_n^{2} + w_n'^{2}\right) h_n v_n' v_n'' + \left(v_n'^{4} - h_n^{4} - w_n'^{2}h_n^{2}\right) u_n''\Big](e_n+1)\cos\gamma_n\Big\}\Bigg\} +$$

$$+GJ\left\{\frac{1}{r_n^{2}(1+e_n)}\left[\begin{array}{l} h_n\kappa_{n_x} r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ +\dfrac{v_n'}{1+e_n}\left[r_n^{2}\left(\kappa_{n_y}^{2} + \kappa_{n_z}^{2}\right) + w_n'^{2}\kappa_{n_x}^{2} + 2w_n'\kappa_{n_x} r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right] +\right.$$

$$+\frac{\kappa_{n_x}}{r_n^{3}}\frac{v_n'}{r_n}\left\{ -h_n^{2}\kappa_{n_x} - \frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ +\dfrac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\} +$$

$$+\frac{\kappa_{n_x}}{(1+e_n)^2}\frac{v_n'}{r_n}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{v_n'}{e_n+1}\left[\begin{matrix}h_n r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\-\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]+$$

$$-\frac{1}{(1+e_n)^3}\frac{v_n'}{e_n+1}\left[r_n^2\kappa_{n_x}^2+w_n'^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)-2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{1}{r_n^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{matrix}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n'v_n''+\left[(e_n+1)^2h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n'v_n''+\left[(e_n+1)^2h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$

$$\cdot\left\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\sin\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\cdot\left\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\sin\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}\ \Bigg\}\ \Bigg\rangle\delta v_n'$$

$$+\Bigg\langle GA_n w_n' + \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)} \Bigg\{ \frac{h_n^2 \kappa_{n_x}}{r_n^2 (1+e_n)^2} \Big[w_n' \kappa_{n_x} + r_n \big(\kappa_{n_y} \sin\gamma_n + \kappa_{n_z}\cos\gamma_n \big)\Big] +$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{w_n'}{e_n+1}\Bigg[v_n' r_n \kappa_{n_x}\big(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\big) - \frac{h_n}{1+e_n}\Big[w_n'^2\kappa_{n_x}^2 + r_n^2\big(\kappa_{n_z}^2+\kappa_{n_y}^2\big) + 2w_n' r_n \kappa_{n_x}\big(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\big)\Big] \Bigg] +$$

$$+\frac{1}{r_n^2}\frac{1}{(e_n+1)^3}\big(h_n v_n'' - v_n' u_n''\big) \Bigg\{ v_n'^2 \kappa_{n_x} + \frac{h_n}{1+e_n} \Bigg[v_n' r_n\big(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\big) + \frac{h_n w_n'}{1+e_n}\Big[w_n'\kappa_{n_x} + r_n\big(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\big)\Big] \Bigg] \Bigg\} +$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\Big[v_n'\kappa_{n_x}\cos\gamma_n + \frac{h_n}{1+e_n}\big(r_n\kappa_{n_y} + w_n'\kappa_{n_x}\sin\gamma_n\big)\Big] \cdot$$

$$\cdot\Big\{\Big[\big(r_n^2 - w_n'^2\big)\big(h_n u_n'' + v_n' v_n''\big) + 2r_n^2 w_n' w_n''\Big]\cos\gamma_n + \big(v_n'u_n'' - h_n v_n''\big)(e_n+1)w_n'\sin\gamma_n\Big\} +$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\Big[-v_n'\kappa_{n_x}\sin\gamma_n + \frac{h_n}{1+e_n}\big(r_n\kappa_{n_z} + w_n'\kappa_{n_x}\cos\gamma_n\big)\Big]\cdot$$

$$\cdot\Big\{-\Big[\big(r_n^2-w_n'^2\big)\big(h_nu_n''+v_n'v_n''\big)+2r_n^2w_n'w_n''\Big]\sin\gamma_n + \big(v_n'u_n''-h_nv_n''\big)(e_n+1)w_n'\cos\gamma_n\Big\} \Bigg\} +$$

$$+GJ\Bigg\{ \frac{1}{(1+e_n)^2}\Bigg[w_n'\Bigg[\frac{v_n'^2}{r_n^2}\kappa_{n_x}^2 + \big(\kappa_{n_y}^2+\kappa_{n_z}^2\big)\Bigg] - \frac{h_n^2\kappa_{n_x}}{r_n}\big(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\big)\Bigg] +$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{w_n'}{e_n+1}\Bigg[h_n r_n\kappa_{n_x}\big(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\big) - \frac{v_n'}{1+e_n}\Big[w_n'^2\kappa_{n_x}^2 + r_n^2\big(\kappa_{n_y}^2+\kappa_{n_z}^2\big) + 2w_n'r_n\kappa_{n_x}\big(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\big)\Big]\Bigg] +$$

$$-\frac{1}{(1+e_n)^3}\frac{w_n'}{e_n+1}\Big[r_n^2\kappa_{n_x}^2 + w_n'^2\big(\kappa_{n_y}^2+\kappa_{n_z}^2\big) - 2w_n'r_n\kappa_{n_x}\big(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\big)\Big] +$$

$$+\frac{1}{r_n^2}\frac{1}{(e_n+1)^3}\big(h_nv_n''-v_n'u_n''\big)\Bigg\{h_n^2\kappa_{n_x} + \frac{v_n'}{1+e_n}\Bigg[h_nr_n\big(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\big) + \frac{v_n'w_n'}{1+e_n}\Big[w_n'\kappa_{n_x}+r_n\big(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\big)\Big]\Bigg]\Bigg\} +$$

$$+\frac{r_n}{(1+e_n)^2}\frac{1}{(e_n+1)^3}\big(h_nv_n''-v_n'u_n''\big)\Big[r_n\kappa_{n_x} - w_n'\big(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\big)\Big] +$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\Big[-h_n\kappa_{n_x}\cos\gamma_n + \frac{v_n'}{1+e_n}\big(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\big)\Big]\cdot$$

$$\cdot\Big\{\Big[\big(r_n^2 - w_n'^2\big)\big(h_n u_n'' + v_n' v_n''\big) + 2r_n^2 w_n' w_n''\Big]\cos\gamma_n + \big(v_n'u_n'' - h_n v_n''\big)(e_n+1)w_n'\sin\gamma_n\Big\} +$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^4}\big(w_n'\kappa_{n_y} - r_n\kappa_{n_x}\sin\gamma_n\big)\cdot$$

$$\cdot\Big\{\Big[\big(r_n^2 - w_n'^2\big)\big(h_n u_n'' + v_n' v_n''\big) + 2r_n^2 w_n' w_n''\Big]\cos\gamma_n + \big(v_n'u_n'' - h_n v_n''\big)(e_n+1)w_n'\sin\gamma_n\Big\} +$$

$$+\frac{v_n{}'}{r_n\left(1+e_n\right)}\frac{1}{r_n\left(e_n+1\right)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n{}'}{1+e_n}\left(r_n\kappa_{n_z}+w_n{}'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(r_n{}^2-w_n{}'^2\right)\left(h_nu_n{}''+v_n{}'v_n{}''\right)+2r_n{}^2w_n{}'w_n{}''\right]\sin\gamma_n+\left(v_n{}'u_n{}''-h_nv_n{}''\right)\left(e_n+1\right)w_n{}'\cos\gamma_n\right\}+$$

$$+\frac{w_n{}'}{\left(1+e_n\right)^2}\frac{1}{r_n\left(e_n+1\right)^4}\left(w_n{}'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\left.\left.\cdot\left\{-\left[\left(r_n{}^2-w_n{}'^2\right)\left(h_nu_n{}''+v_n{}'v_n{}''\right)+2r_n{}^2w_n{}'w_n{}''\right]\sin\gamma_n+\left(v_n{}'u_n{}''-h_nv_n{}''\right)\left(e_n+1\right)w_n{}'\cos\gamma_n\right\}\right\}\right\rangle\delta w_n{}'+$$

$$+\left\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\frac{h_n\kappa_{n_x}}{r_n\left(1+e_n\right)}\left[\frac{h_n}{1+e_n}w_n{}'\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)-v_n{}'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+\right.\right.$$

$$+\frac{h_n}{r_n\left(1+e_n\right)}\frac{1}{r_n\left(e_n+1\right)^2}\left[v_n{}'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n{}'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left(h_nv_n{}''-v_n{}'u_n{}''\right)\left(e_n+1\right)\cos\gamma_n+\left[r_n{}^2w_n{}''-w_n{}'\left(h_nu_n{}''+v_n{}'v_n{}''\right)\right]\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n\left(1+e_n\right)}\frac{1}{r_n\left(e_n+1\right)^2}\left[-v_n{}'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n{}'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\left.\cdot\left\{-\left(h_nv_n{}''-v_n{}'u_n{}''\right)\left(e_n+1\right)\sin\gamma_n+\left[r_n{}^2w_n{}''-w_n{}'\left(h_nu_n{}''+v_n{}'v_n{}''\right)\right]\cos\gamma_n\right\}\right\}+$$

$$+GJ\left\{-\frac{h_n\kappa_{n_x}}{r_n\left(1+e_n\right)}\left[\frac{h_n}{1+e_n}w_n{}'\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)-v_n{}'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+\right.$$

$$+\frac{v_n{}'}{r_n\left(1+e_n\right)}\frac{1}{r_n\left(e_n+1\right)^2}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n{}'}{1+e_n}\left(r_n\kappa_{n_y}+w_n{}'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left(h_nv_n{}''-v_n{}'u_n{}''\right)\left(e_n+1\right)\cos\gamma_n+\left[r_n{}^2w_n{}''-w_n{}'\left(h_nu_n{}''+v_n{}'v_n{}''\right)\right]\sin\gamma_n\right\}+$$

$$+\frac{w_n{}'}{\left(1+e_n\right)^2}\frac{1}{r_n\left(e_n+1\right)^2}\left(w_n{}'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left\{\begin{array}{l}\left(h_nv_n{}''-v_n{}'u_n{}''\right)\left(e_n+1\right)\cos\gamma_n\\+\left[r_n{}^2w_n{}''-w_n{}'\left(h_nu_n{}''+v_n{}'v_n{}''\right)\right]\sin\gamma_n\end{array}\right\}+$$

$$+\frac{v_n{}'}{r_n\left(1+e_n\right)}\frac{1}{r_n\left(e_n+1\right)^2}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n{}'}{1+e_n}\left(r_n\kappa_{n_z}+w_n{}'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left(h_nv_n{}''-v_n{}'u_n{}''\right)\left(e_n+1\right)\sin\gamma_n+\left[r_n{}^2w_n{}''-w_n{}'\left(h_nu_n{}''+v_n{}'v_n{}''\right)\right]\cos\gamma_n\right\}+$$

$$+\frac{w_n{}'}{\left(1+e_n\right)^2}\frac{1}{r_n\left(e_n+1\right)^2}\left(w_n{}'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\left.\left.\cdot\left\{-\left(h_nv_n{}''-v_n{}'u_n{}''\right)\left(e_n+1\right)\sin\gamma_n+\left[r_n{}^2w_n{}''-w_n{}'\left(h_nu_n{}''+v_n{}'v_n{}''\right)\right]\cos\gamma_n\right\}\right\}\right\rangle\delta\gamma_n+$$

$$
+\left\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{1}{r_n^{\,2}}\left\{ v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.
$$

$$
+GJ\left\{\frac{1}{r_n^{\,2}}\left\{h_n^{\,2}\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.
$$

$$
\left.\left.+\frac{r_n}{(1+e_n)^2}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\right\}\right\rangle\delta\gamma_n'+
$$

$$
+\left\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\right\{
$$

$$
\frac{-w_n' v_n'}{r_n^{\,4}(e_n+1)}\left\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+
$$

$$
+\frac{h_n}{r_n^{\,2}(1+e_n)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+
$$

$$
\left.+\frac{h_n}{r_n^{\,2}(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]\right\}+
$$

$$
+GJ\left\{\frac{-w_n' v_n'}{r_n^{\,4}(e_n+1)}\left\{h_n^{\,2}\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.
$$

$$
+\frac{-w_n' v_n'}{r_n(1+e_n)^3}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+
$$

$$
+\frac{v_n'}{r_n^{\,2}(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+
$$

$$
+\frac{w_n'}{r_n(1+e_n)^3}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+
$$

$$
+\frac{v_n'}{r_n^{\,2}(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]+
$$

$$
\left.\left.+\frac{w_n'}{r_n(1+e_n)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]\right\}\right\rangle\delta u_n''+
$$

$$+\Bigg\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\Bigg\{ \frac{w_n' h_n}{r_n^4(e_n+1)}\Bigg\{ v_n'^2\kappa_{n_x} + \frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ + \frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right] \end{array}\right]\Bigg\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^2}\left[v_n'\kappa_{n_x}\cos\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right] \Bigg\}+$$

$$+GJ\Bigg\{\frac{1}{r_n^2}\frac{w_n'h_n}{r_n^2(e_n+1)}\Bigg\{h_n^2\kappa_{n_x} + \frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\Bigg\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{w_n'h_n}{r_n^2(e_n+1)}\left[r_n\kappa_{n_x} - w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n + \frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^2}\left(w_n'\kappa_{n_y} - r_n\kappa_{n_x}\sin\gamma_n\right)\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n + \frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z} - r_n\kappa_{n_x}\cos\gamma_n\right)\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right] \Bigg\}\Bigg\rangle \delta v_n'' +$$

$$+\Bigg\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\Bigg\{\frac{h_n}{r_n(1+e_n)}\frac{-r_n}{(e_n+1)^2}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n +$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{r_n}{(e_n+1)^2}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n \Bigg\}+$$

$$+GJ\Bigg\{\frac{v_n'}{r_n(1+e_n)}\frac{-r_n}{(e_n+1)^2}\left[-h_n\kappa_{n_x}\cos\gamma_n + \frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{-r_n}{(e_n+1)^2}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cos\gamma_n + \frac{w_n'}{(1+e_n)^2}\frac{r_n}{(e_n+1)^2}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\sin\gamma_n +$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{r_n}{(e_n+1)^2}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n \Bigg\}\Bigg\rangle\delta w_n'' \Bigg\} ds_n$$

(C1)

Appendix D

Weak form of variation of strain energy, ie δU, of geometrically-nonlinear linearly-elastic isotropic spatial Euler-Bernoulli beam under large elastic orientation is given by ***Expression (D1)*** in terms of the variations of the elastic degrees of freedom, ie $\delta\gamma$, δu, δv and δw [3].

$$
\begin{aligned}
\delta U = \int_0^{L_n} \Bigg\{ & -\Bigg\langle \frac{(1-\nu)EA_n}{(1+\nu)(1-2\nu)} u_n' + \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)} \Bigg\{ \\
& + \frac{1}{r_n^2(1+e_n)} \begin{bmatrix} v_n' r_n \kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ + \frac{h_n}{1+e_n}\left[w_n'^2 \kappa_{n_x}^2 + r_n^2\left(\kappa_{n_y}^2 + \kappa_{n_z}^2\right) + 2 r_n w_n' \kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right] \end{bmatrix} + \\
& + \frac{\kappa_{n_x}}{r_n^3}\frac{h_n}{r_n}\left\{ -v_n'^2 \kappa_{n_x} + \frac{h_n}{1+e_n} \begin{bmatrix} v_n' r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ - \frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_z}\cos\gamma_n + \kappa_{n_y}\sin\gamma_n\right)\right] \end{bmatrix} \right\} + \\
& + \frac{h_n}{r_n^2(1+e_n)^2}\frac{h_n}{e_n+1} \begin{bmatrix} v_n' r_n \kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ - \frac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2 + r_n^2\left(\kappa_{n_z}^2 + \kappa_{n_y}^2\right) + 2 w_n' r_n \kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right] \end{bmatrix} + \\
& + \frac{1}{r_n^2}\frac{1}{r_n^4(e_n+1)^3} w_n' \left\{ v_n'^2\kappa_{n_x} + \frac{h_n}{1+e_n} \begin{bmatrix} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ + \frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right] \end{bmatrix} \right\} \cdot \\
& \qquad \cdot\left\{ \left[2(e_n+1)^2 + r_n^2\right] h_n v_n' u_n'' + \left[(e_n+1)^2 v_n'^2 - h_n^2\left[(e_n+1)^2 + r_n^2\right]\right] v_n'' \right\} + \\
& + \frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_y} + w_n'\kappa_{n_x}\sin\gamma_n\right)\right] \cdot \\
& \cdot\Big\{ \Big[\left(v_n'^4 - 2h_n^4 - h_n^2 v_n'^2 + w_n'^2 v_n'^2\right) w_n' u_n'' - \left(3r_n^2 + w_n'^2\right) h_n w_n' v_n' v_n'' + \left(r_n^2 - w_n'^2\right) h_n r_n^2 w_n'' \Big]\cos\gamma_n \\
& \qquad + \left[\left(2r_n^2 + w_n'^2\right) h_n v_n' u_n'' + \left(v_n'^4 - h_n^4 + w_n'^2 v_n'^2\right) v_n'' \right](e_n+1)\sin\gamma_n \Big\} +
\end{aligned}
$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$
$$\cdot\left\{-\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_nr_n^2w_n''\right]\sin\gamma_n\right.$$
$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_nv_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\right\}\Bigg\}+$$

$$+GJ\left\{\frac{\kappa_{n_x}}{r_n^2}\left[h_n\kappa_{n_x}+\frac{v_n'}{1+e_n}r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\right]+\right.$$
$$+\frac{\kappa_{n_x}}{r_n^3}\frac{h_n}{r_n}\left\{-h_n^2\kappa_{n_x}-\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_nr_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+$$
$$+\frac{\kappa_{n_x}}{(1+e_n)^2}\frac{h_n}{r_n}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$
$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{h_n}{e_n+1}\left[\begin{array}{l}h_nr_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\-\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$
$$-\frac{1}{(1+e_n)^3}\frac{h_n}{e_n+1}\left[r_n^2\kappa_{n_x}^2+w_n'^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)-2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$
$$+\frac{1}{r_n^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_nr_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$
$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_nv_nu_n''+\left[(e_n+1)^2v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$
$$+\frac{r_n}{(1+e_n)^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\cdot$$
$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_nv_n'u_n''+\left[(e_n+1)^2v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$
$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$
$$\cdot\left\{\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_nr_n^2w_n''\right]\cos\gamma_n\right.$$
$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_nv_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\right\}+$$
$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$
$$\cdot\left\{\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_nr_n^2w_n''\right]\cos\gamma_n\right.$$
$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_nv_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{v_n'}{r_n\left(1+e_n\right)}\frac{1}{r_n^{3}\left(e_n+1\right)^{4}}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{-\Big[\left(v_n'^{4}-2h_n^{4}-h_n^{2}v_n'^{2}+w_n'^{2}v_n'^{2}\right)w_n'u_n''-\left(3r_n^{2}+w_n'^{2}\right)h_nw_n'v_n'v_n''+\left(r_n^{2}-w_n'^{2}\right)h_nr_n^{2}w_n''\Big]\sin\gamma_n$$

$$+\Big[\left(2r_n^{2}+w_n'^{2}\right)h_nv_n'u_n''+\left(v_n'^{4}-h_n^{4}+w_n'^{2}v_n'^{2}\right)v_n''\Big]\left(e_n+1\right)\cos\gamma_n\Big\}+$$

$$+\frac{w_n'}{\left(1+e_n\right)^{2}}\frac{1}{r_n^{3}\left(e_n+1\right)^{4}}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\cdot\Big\{-\Big[\left(v_n'^{4}-2h_n^{4}-h_n^{2}v_n'^{2}+w_n'^{2}v_n'^{2}\right)w_n'u_n''-\left(3r_n^{2}+w_n'^{2}\right)h_nw_n'v_n'v_n''+\left(r_n^{2}-w_n'^{2}\right)h_nr_n^{2}w_n''\Big]\sin\gamma_n$$

$$+\Big[\left(2r_n^{2}+w_n'^{2}\right)h_nv_n'u_n''+\left(v_n'^{4}-h_n^{4}+w_n'^{2}v_n'^{2}\right)v_n''\Big]\left(e_n+1\right)\cos\gamma_n\Big\}\Bigg\}\Bigg\rangle'\delta u_n+$$

$$+\Bigg\langle\frac{\left(1-\nu\right)EJ}{\left(1+\nu\right)\left(1-2\nu\right)}\Bigg\{\frac{-w_n'v_n'}{r_n^{4}\left(e_n+1\right)}\Bigg\{v_n'^{2}\kappa_{n_x}+\frac{h_n}{1+e_n}\Bigg[v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)+\frac{h_nw_n'}{1+e_n}\Big[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\Big]\Bigg]\Bigg\}+$$

$$+\frac{h_n}{r_n^{2}\left(1+e_n\right)^{3}}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_nw_n'\cos\gamma_n-\left(e_n+1\right)v_n'\sin\gamma_n\right]+$$

$$+\frac{h_n}{r_n^{2}\left(1+e_n\right)^{3}}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_nw_n'\sin\gamma_n-\left(e_n+1\right)v_n'\cos\gamma_n\right]\Bigg\}+$$

$$+GJ\Bigg\{\frac{1}{r_n^{2}}\frac{-w_n'v_n'}{r_n^{2}\left(e_n+1\right)}\Bigg\{h_n^{2}\kappa_{n_x}+\frac{v_n'}{1+e_n}\Bigg[h_nr_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)+\frac{v_n'w_n'}{1+e_n}\Big[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\Big]\Bigg]\Bigg\}+$$

$$+\frac{r_n}{\left(1+e_n\right)^{2}}\frac{-w_n'v_n'}{r_n^{2}\left(e_n+1\right)}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^{2}\left(1+e_n\right)^{3}}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_nw_n'\cos\gamma_n-\left(e_n+1\right)v_n'\sin\gamma_n\right]+$$

$$+\frac{w_n'}{\left(1+e_n\right)^{2}}\frac{1}{r_n\left(e_n+1\right)^{2}}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[h_nw_n'\cos\gamma_n-\left(e_n+1\right)v_n'\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^{2}\left(1+e_n\right)^{3}}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_nw_n'\sin\gamma_n-\left(e_n+1\right)v_n'\cos\gamma_n\right]+$$

$$+\frac{w_n'}{r_n\left(1+e_n\right)^{4}}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-h_nw_n'\sin\gamma_n-\left(e_n+1\right)v_n'\cos\gamma_n\right]\Bigg\}\Bigg\rangle''\delta u_n+$$

$$-\left\langle GA_n v_n' + \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\right\{$$

$$+\frac{\kappa_{n_x}}{r_n^2}\left[v_n'\kappa_{n_x} + \frac{h_n r_n}{1+e_n}\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right)\right]+$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{v_n'}{r_n}\left\{-v_n'^2\kappa_{n_x} + \frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ -\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_z}\cos\gamma_n + \kappa_{n_y}\sin\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{v_n'}{e_n+1}\left[\begin{array}{l} v_n' r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ -\frac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2 + r_n^2\left(\kappa_{n_z}^2 + \kappa_{n_y}^2\right) + 2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$+\frac{-w_n'}{r_n^6(e_n+1)^3}\left\{v_n'^2\kappa_{n_x} + \frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2 + r_n^2\right]h_n v_n' v_n'' + \left[(e_n+1)^2 h_n^2 - v_n'^2\left[(e_n+1)^2 + r_n^2\right]\right]u_n''\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_y} + w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(h_n^4 - 2v_n'^4 - h_n^2 v_n'^2 + w_n'^2 h_n^2\right)w_n' v_n'' - \left(3r_n^2 + w_n'^2\right)h_n w_n' v_n' u_n'' + \left(r_n^2 - w_n'^2\right)v_n' r_n^2 w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2 + w_n'^2\right)h_n v_n' v_n'' + \left(v_n'^4 - h_n^4 - w_n'^2 h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_z} + w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(h_n^4 - 2v_n'^4 - h_n^2 v_n'^2 + w_n'^2 h_n^2\right)w_n' v_n'' - \left(3r_n^2 + w_n'^2\right)h_n w_n' v_n' u_n'' + \left(r_n^2 - w_n'^2\right)v_n' r_n^2 w_n''\right]\sin\gamma_n\right.$$

$$\left.\left.+\left[-\left(2r_n^2 + w_n'^2\right)h_n v_n' v_n'' + \left(v_n'^4 - h_n^4 - w_n'^2 h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}\right\}+$$

$$+GJ\left\{\frac{1}{r_n^2(1+e_n)}\left[\begin{array}{l} h_n\kappa_{n_x} r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ +\frac{v_n'}{1+e_n}\left[r_n^2\left(\kappa_{n_y}^2 + \kappa_{n_z}^2\right) + w_n'^2\kappa_{n_x}^2 + 2w_n'\kappa_{n_x} r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right.+$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{v_n'}{r_n}\left\{-h_n^2\kappa_{n_x} - \frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{\kappa_{n_x}}{(1+e_n)^2}\frac{v_n'}{r_n}\left[r_n\kappa_{n_x} - w_n'\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{v_n'}{e_n+1}\left[\begin{array}{l} h_n r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ -\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2 + r_n^2\left(\kappa_{n_y}^2 + \kappa_{n_z}^2\right) + 2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$-\frac{1}{(1+e_n)^3}\frac{v_n'}{e_n+1}\left[r_n^2\kappa_{n_x}^2+w_n'^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)-2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{1}{r_n^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n' v_n''+\left[(e_n+1)^2 h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n' v_n''+\left[(e_n+1)^2 h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n' v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' u_n''+\left(r_n^2-w_n'^2\right)v_n' r_n^2 w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n' v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$

$$\cdot\left\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n' v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' u_n''+\left(r_n^2-w_n'^2\right)v_n' r_n^2 w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n' v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n' v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' u_n''+\left(r_n^2-w_n'^2\right)v_n' r_n^2 w_n''\right]\sin\gamma_n\right.$$

$$\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n' v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\cdot\left\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n' v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' u_n''+\left(r_n^2-w_n'^2\right)v_n' r_n^2 w_n''\right]\sin\gamma_n\right.$$

$$\left.\left.\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n' v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}\right\}\right\rangle'\delta v_n$$

$$+\left\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\frac{w_n' h_n}{r_n^4(e_n+1)}\left\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\right.\right.+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n' w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]\Bigg\}+$$

$$+GJ\Bigg\{\frac{1}{r_n^2}\frac{w_n'h_n}{r_n^2(e_n+1)}\Bigg\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\Bigg[h_nr_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\Bigg]\Bigg\}\Bigg\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{w_n'h_n}{r_n^2(e_n+1)}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]\Bigg\}\Bigg\rangle''\delta v_n+$$

$$-\Bigg\langle GA_nw_n'+\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\Bigg\{\frac{h_n^2\kappa_{n_x}}{r_n^2(1+e_n)^2}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{w_n'}{e_n+1}\Bigg[v_n'r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)-\frac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_z}^2+\kappa_{n_y}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\Bigg]+$$

$$+\frac{1}{r_n^2}\frac{1}{(e_n+1)^3}\left(h_nv_n''-v_n'u_n''\right)\Bigg\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\Bigg[v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)+\frac{h_nw_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\Bigg]\Bigg\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(r_n^2-w_n'^2\right)\left(h_nu_n''+v_n'v_n''\right)+2r_n^2w_n'w_n''\right]\cos\gamma_n+\left(v_n'u_n''-h_nv_n''\right)(e_n+1)w_n'\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(r_n^2-w_n'^2\right)\left(h_nu_n''+v_n'v_n''\right)+2r_n^2w_n'w_n''\right]\sin\gamma_n+\left(v_n'u_n''-h_nv_n''\right)(e_n+1)w_n'\cos\gamma_n\right\}\Bigg\}+$$

$$+GJ\Bigg\{\frac{1}{(1+e_n)^2}\left[w_n'\left[\frac{v_n'^2}{r_n^2}\kappa_{n_x}^2+\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)\right]-\frac{h_n^2\kappa_{n_x}}{r_n}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{w_n'}{e_n+1}\left[\begin{array}{l} h_n r_n \kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right) \\ -\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^{\ 2}+r_n^2\left(\kappa_{n_y}^{\ 2}+\kappa_{n_z}^{\ 2}\right)+2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$-\frac{1}{(1+e_n)^3}\frac{w_n'}{e_n+1}\left[r_n^2\kappa_{n_x}^{\ 2}+w_n'^2\left(\kappa_{n_y}^{\ 2}+\kappa_{n_z}^{\ 2}\right)-2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{(e_n+1)^3}\left(h_n v_n''-v_n' u_n''\right)\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right) \\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{1}{(e_n+1)^3}\left(h_n v_n''-v_n' u_n''\right)\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(r_n^2-w_n'^2\right)\left(h_n u_n''+v_n' v_n''\right)+2r_n^2 w_n' w_n''\right]\cos\gamma_n+\left(v_n' u_n''-h_n v_n''\right)(e_n+1)w_n'\sin\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$

$$\cdot\left\{\left[\left(r_n^2-w_n'^2\right)\left(h_n u_n''+v_n' v_n''\right)+2r_n^2 w_n' w_n''\right]\cos\gamma_n+\left(v_n' u_n''-h_n v_n''\right)(e_n+1)w_n'\sin\gamma_n\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(r_n^2-w_n'^2\right)\left(h_n u_n''+v_n' v_n''\right)+2r_n^2 w_n' w_n''\right]\sin\gamma_n+\left(v_n' u_n''-h_n v_n''\right)(e_n+1)w_n'\cos\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\left.\left.\cdot\left\{-\left[\left(r_n^2-w_n'^2\right)\left(h_n u_n''+v_n' v_n''\right)+2r_n^2 w_n' w_n''\right]\sin\gamma_n+\left(v_n' u_n''-h_n v_n''\right)(e_n+1)w_n'\cos\gamma_n\right\}\ \right\}\ \right\rangle'\delta w_n+$$

$$+\left\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\begin{array}{l}\frac{-h_n}{(e_n+1)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n+ \\ +\frac{h_n}{(e_n+1)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n\end{array}\right\}+\right.$$

$$+GJ\left\{\begin{array}{l}\frac{-v_n'}{(e_n+1)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n \\ +\frac{v_n'}{(e_n+1)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n \\ +\frac{-r_n w_n'}{(1+e_n)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cos\gamma_n+\frac{r_n w_n'}{(1+e_n)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\sin\gamma_n\end{array}\right\}\left.\vphantom{\frac{1}{1}}\right\rangle''\delta w_n+$$

$$-\left\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{1}{r_n^2}\left\{ v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.$$

$$+GJ\left\{\frac{1}{r_n^2}\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.$$

$$\left.\left.+\frac{r_n}{(1+e_n)^2}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\right\}\right\rangle'\delta\gamma_n+$$

$$+\left\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\frac{h_n\kappa_{n_x}}{r_n(1+e_n)}\left[\frac{h_n}{1+e_n}w_n'\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)-v_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+\right.\right.$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^2}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left(h_n v_n''-v_n'u_n''\right)(e_n+1)\cos\gamma_n+\left[r_n^2 w_n''-w_n'\left(h_n u_n''+v_n'v_n''\right)\right]\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^2}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\left.\cdot\left\{-\left(h_n v_n''-v_n'u_n''\right)(e_n+1)\sin\gamma_n+\left[r_n^2 w_n''-w_n'\left(h_n u_n''+v_n'v_n''\right)\right]\cos\gamma_n\right\}\right\}+$$

$$+GJ\left\{-\frac{h_n\kappa_{n_x}}{r_n(1+e_n)}\left[\frac{h_n}{1+e_n}w_n'\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)-v_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+\right.$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^2}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left(h_n v_n''-v_n'u_n''\right)(e_n+1)\cos\gamma_n+\left[r_n^2 w_n''-w_n'\left(h_n u_n''+v_n'v_n''\right)\right]\sin\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^2}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$

$$\cdot\left\{\left(h_n v_n''-v_n'u_n''\right)(e_n+1)\cos\gamma_n+\left[r_n^2 w_n''-w_n'\left(h_n u_n''+v_n'v_n''\right)\right]\sin\gamma_n\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^2}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left(h_n v_n''-v_n'u_n''\right)(e_n+1)\sin\gamma_n+\left[r_n^2 w_n''-w_n'\left(h_n u_n''+v_n'v_n''\right)\right]\cos\gamma_n\right\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^2}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\left.\left.\cdot\left\{-\left(h_n v_n''-v_n'u_n''\right)(e_n+1)\sin\gamma_n+\left[r_n^2 w_n''-w_n'\left(h_n u_n''+v_n'v_n''\right)\right]\cos\gamma_n\right\}\right\}\right\rangle\delta\gamma_n$$

$$\left.\right\}ds_n+$$

$$+\left\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{ \frac{-w_n'v_n'}{r_n^4(e_n+1)}\left\{ v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l} v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right) \\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\right.\right.+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+$$

$$+\left.\frac{h_n}{r_n^2(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]\right\}+$$

$$+GJ\left\{ \frac{-w_n'v_n'}{r_n^4(e_n+1)}\left\{ h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right) \\ +\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\right.+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{-w_n'v_n'}{r_n^2(e_n+1)}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]+$$

$$\left.\left.+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]\right\}\right\rangle \delta u_n'\Big|_0^{L_n}+$$

$$+\left\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{ \frac{w_n'h_n}{r_n^4(e_n+1)}\left\{ v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l} v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right) \\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\right.\right.+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]\Bigg\}+$$

$$+GJ\Bigg\{\frac{1}{r_n^2}\frac{w_n'h_n}{r_n^2(e_n+1)}\Bigg\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_nr_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\Bigg\}+$$

$$+\frac{w_n'h_n}{r_n(e_n+1)^3}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]\Bigg\}\Bigg\rangle\delta v_n'\Bigg|_0^{L_n}+$$

$$+\Bigg\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{h_n}{(e_n+1)^3}\Bigg\{-\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n+$$

$$+\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n\Bigg\}+$$

$$+GJ\frac{1}{(e_n+1)^3}\Bigg\{v_n'\left[h_n\kappa_{n_x}\cos\gamma_n-\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n$$

$$+v_n'\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n$$

$$-\frac{r_nw_n'}{1+e_n}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cos\gamma_n+\frac{r_nw_n'}{1+e_n}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\sin\gamma_n\Bigg\}\Bigg\rangle\delta w_n'\Bigg|_0^{L_n}+$$

$$-\Bigg\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\Bigg\{\frac{-w_n'v_n'}{r_n^4(e_n+1)}\Bigg\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\+\frac{h_nw_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\Bigg\}+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_nw_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_nw_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]\Bigg\}+$$

$$+GJ\left\{\frac{-w_n'v_n'}{r_n^4(e_n+1)}\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.$$

$$+\frac{r_n}{(1+e_n)^2}\frac{-w_n'v_n'}{r_n^2(e_n+1)}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(1+e_n)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[h_n w_n'\cos\gamma_n-(e_n+1)v_n'\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]+$$

$$\left.\left.+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-h_n w_n'\sin\gamma_n-(e_n+1)v_n'\cos\gamma_n\right]\right\}\right\rangle'\delta u_n\Big|_0^{L_n}+$$

$$-\left\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\frac{w_n'h_n}{r_n^4(e_n+1)}\left\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\+\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.\right.$$

$$+\frac{h_n}{r_n^2(1+e_n)^3}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$\left.+\frac{h_n}{r_n^2(1+e_n)^3}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]\right\}+$$

$$+GJ\left\{\frac{w_n'h_n}{r_n^4(e_n+1)}\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\right.$$

$$+\frac{w_n'h_n}{r_n(1+e_n)^3}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\left[v_n'w_n'\cos\gamma_n+(e_n+1)h_n\sin\gamma_n\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]+$$

$$\left.\left.+\frac{w_n'}{r_n(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\left[-v_n'w_n'\sin\gamma_n+(e_n+1)h_n\cos\gamma_n\right]\right\}\right\rangle'\delta v_n\Big|_0^{L_n}+$$

$$-\left\langle \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{h_n}{(1+e_n)^3}\left\{ -\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n+ \right.\right.$$

$$\left. +\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n\right\}+$$

$$+GJ\left\{\frac{-v_n'}{(e_n+1)^3}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cos\gamma_n\right.$$

$$+\frac{v_n'}{(1+e_n)^3}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\sin\gamma_n$$

$$\left.\left.+\frac{r_nw_n'}{(e_n+1)^4}\left(-w_n'\kappa_{n_y}+r_n\kappa_{n_x}\sin\gamma_n\right)\cos\gamma_n+\frac{r_nw_n'}{(1+e_n)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\sin\gamma_n\right\}\right\rangle'\delta w_n\Big|_0^{L_n}+$$

$$+\left\langle\frac{(1-\nu)EA_n}{(1+\nu)(1-2\nu)}u_n'+\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\left\{\right.\right.$$

$$+\frac{1}{r_n^2(1+e_n)}\left[\begin{array}{l}v_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\+\frac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+2r_nw_n'\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{h_n}{r_n}\left\{-v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n'r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\-\frac{h_nw_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_z}\cos\gamma_n+\kappa_{n_y}\sin\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{h_n}{e_n+1}\left[\begin{array}{l}v_n'r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\-\frac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_z}^2+\kappa_{n_y}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\+\frac{h_nw_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_nv_n'u_n''+\left[(e_n+1)^2v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_nr_n^2w_n''\right]\cos\gamma_n\right.$$

$$\left.+\left[\left(2r_n^2+w_n'^2\right)h_nv_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_nr_n^2w_n''\right]\sin\gamma_n\right.$$

$$+\left[\left(2r_n^2+w_n'^2\right)h_n v_n' u_n''+\left(v_n'^4-h_n^4+w_n'^2 v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\Big\}\Bigg\}+$$

$$+GJ\Bigg\{\frac{\kappa_{n_x}}{r_n^2}\left[h_n\kappa_{n_x}+\frac{v_n'}{1+e_n}r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\right]+$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{h_n}{r_n}\left\{-h_n^2\kappa_{n_x}-\frac{v_n'}{1+e_n}\left[\begin{array}{l} h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{\kappa_{n_x}}{(1+e_n)^2}\frac{h_n}{r_n}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{h_n}{e_n+1}\left[\begin{array}{l} h_n r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\ -\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$-\frac{1}{(1+e_n)^3}\frac{h_n}{e_n+1}\left[r_n^2\kappa_{n_x}^2+w_n'^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)-2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\ +\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n' u_n''+\left[(e_n+1)^2 v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{1}{r_n^4(e_n+1)^3}w_n'\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n' u_n''+\left[(e_n+1)^2 v_n'^2-h_n^2\left[(e_n+1)^2+r_n^2\right]\right]v_n''\right\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{\Big[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2 w_n''\Big]\cos\gamma_n$$

$$+\left[\left(2r_n^2+w_n'^2\right)h_n v_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\Big\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot$$

$$\cdot\Big\{\Big[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2 w_n''\Big]\cos\gamma_n$$

$$+\left[\left(2r_n^2+w_n'^2\right)h_n v_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\sin\gamma_n\Big\}+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{-\Big[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_n w_n' v_n' v_n''+\left(r_n^2-w_n'^2\right)h_n r_n^2 w_n''\Big]\sin\gamma_n$$

$$+\left[\left(2r_n^2+w_n'^2\right)h_n v_n' u_n''+\left(v_n'^4-h_n^4+w_n'^2 v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\Big\}+$$

$$+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot$$

$$\cdot\Big\{-\left[\left(v_n'^4-2h_n^4-h_n^2v_n'^2+w_n'^2v_n'^2\right)w_n'u_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'v_n''+\left(r_n^2-w_n'^2\right)h_nr_n^2w_n''\right]\sin\gamma_n$$

$$+\left[\left(2r_n^2+w_n'^2\right)h_nv_n'u_n''+\left(v_n'^4-h_n^4+w_n'^2v_n'^2\right)v_n''\right](e_n+1)\cos\gamma_n\Big\}\Bigg\}\Bigg\rangle\delta u_n\Bigg|_0^{L_n}+$$

$$+\Bigg\langle GA_nv_n'+\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\Bigg\{$$

$$+\frac{\kappa_{n_x}}{r_n^2}\left[v_n'\kappa_{n_x}+\frac{h_nr_n}{1+e_n}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\right]+$$

$$+\frac{\kappa_{n_x}}{r_n^3}\frac{v_n'}{r_n}\left\{-v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n'r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\-\dfrac{h_nw_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_z}\cos\gamma_n+\kappa_{n_y}\sin\gamma_n\right)\right]\end{array}\right]\right\}+$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{v_n'}{e_n+1}\left[\begin{array}{l}v_n'r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\-\dfrac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_z}^2+\kappa_{n_y}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+$$

$$+\frac{1}{r_n^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left\{v_n'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\+\dfrac{h_nw_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot$$

$$\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_nv_n'v_n''+\left[(e_n+1)^2h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\cos\gamma_n$$

$$+\left[-\left(2r_n^2+w_n'^2\right)h_nv_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\Big\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n+\frac{h_n}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\Big\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_nw_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\sin\gamma_n$$

$$+\left[-\left(2r_n^2+w_n'^2\right)h_nv_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\Big\}\Bigg\}+$$

$$+GJ\left\{\frac{1}{r_n^2(1+e_n)}\left[\begin{array}{l}h_n\kappa_{n_x}r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\dfrac{v_n'}{1+e_n}\left[r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+w_n'^2\kappa_{n_x}^2+2w_n'\kappa_{n_x}r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+\right.$$

$$
+\frac{\kappa_{n_x}}{r_n^3}\frac{v_n'}{r_n}\left\{-h_n^2\kappa_{n_x}-\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+
$$

$$
+\frac{\kappa_{n_x}}{(1+e_n)^2}\frac{v_n'}{r_n}\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+
$$

$$
+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{v_n'}{e_n+1}\left[\begin{array}{l}h_n r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\-\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2+r_n^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)+2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]+
$$

$$
-\frac{1}{(1+e_n)^3}\frac{v_n'}{e_n+1}\left[r_n^2\kappa_{n_x}^2+w_n'^2\left(\kappa_{n_y}^2+\kappa_{n_z}^2\right)-2w_n'r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]+
$$

$$
+\frac{1}{r_n^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left\{h_n^2\kappa_{n_x}+\frac{v_n'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n'w_n'}{1+e_n}\left[w_n'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}\cdot
$$

$$
\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n'v_n''+\left[(e_n+1)^2h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+
$$

$$
+\frac{r_n}{(1+e_n)^2}\frac{-1}{r_n^4(e_n+1)^3}w_n'\left[r_n\kappa_{n_x}-w_n'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\cdot
$$

$$
\cdot\left\{\left[2(e_n+1)^2+r_n^2\right]h_n v_n'v_n''+\left[(e_n+1)^2h_n^2-v_n'^2\left[(e_n+1)^2+r_n^2\right]\right]u_n''\right\}+
$$

$$
+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y}+w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot
$$

$$
\cdot\left\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\cos\gamma_n\right.
$$

$$
\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+
$$

$$
+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot
$$

$$
\cdot\left\{\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\cos\gamma_n\right.
$$

$$
\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\sin\gamma_n\right\}+
$$

$$
+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n^3(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n'}{1+e_n}\left(r_n\kappa_{n_z}+w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot
$$

$$
\cdot\left\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\sin\gamma_n\right.
$$

$$
\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}+
$$

$$
+\frac{w_n'}{(1+e_n)^2}\frac{1}{r_n^3(e_n+1)^4}\left(w_n'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot
$$

$$
\cdot\left\{-\left[\left(h_n^4-2v_n'^4-h_n^2v_n'^2+w_n'^2h_n^2\right)w_n'v_n''-\left(3r_n^2+w_n'^2\right)h_n w_n'v_n'u_n''+\left(r_n^2-w_n'^2\right)v_n'r_n^2w_n''\right]\sin\gamma_n\right.
$$

$$
\left.\left.+\left[-\left(2r_n^2+w_n'^2\right)h_n v_n'v_n''+\left(v_n'^4-h_n^4-w_n'^2h_n^2\right)u_n''\right](e_n+1)\cos\gamma_n\right\}\ \right\}\ \Bigg\rangle\delta v_n\Bigg|_0^{L_n}
$$

$$+\left\langle GA_n w_n' + \frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\right\{$$

$$+\frac{h_n^2\kappa_{n_x}}{r_n^2(1+e_n)^2}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{h_n}{r_n^2(1+e_n)^2}\frac{w_n'}{e_n+1}\left[\begin{matrix} v_n' r_n\kappa_{n_x}\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ -\frac{h_n}{1+e_n}\left[w_n'^2\kappa_{n_x}^2 + r_n^2\left(\kappa_{n_z}^2 + \kappa_{n_y}^2\right) + 2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{(e_n+1)^3}\left(h_n v_n'' - v_n' u_n''\right)\left\{v_n'^2\kappa_{n_x} + \frac{h_n}{1+e_n}\left[\begin{matrix} v_n' r_n\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ +\frac{h_n w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[v_n'\kappa_{n_x}\cos\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_y} + w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(r_n^2 - w_n'^2\right)\left(h_n u_n'' + v_n' v_n''\right) + 2r_n^2 w_n' w_n''\right]\cos\gamma_n + \left(v_n' u_n'' - h_n v_n''\right)(e_n+1)w_n'\sin\gamma_n\right\}+$$

$$+\frac{h_n}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[-v_n'\kappa_{n_x}\sin\gamma_n + \frac{h_n}{1+e_n}\left(r_n\kappa_{n_z} + w_n'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot$$

$$\cdot\left\{-\left[\left(r_n^2 - w_n'^2\right)\left(h_n u_n'' + v_n' v_n''\right) + 2r_n^2 w_n' w_n''\right]\sin\gamma_n + \left(v_n' u_n'' - h_n v_n''\right)(e_n+1)w_n'\cos\gamma_n\right\}\Big\}+$$

$$+GJ\left\{\frac{1}{(1+e_n)^2}\left[w_n'\left[\frac{v_n'^2}{r_n^2}\kappa_{n_x}^2 + \left(\kappa_{n_y}^2 + \kappa_{n_z}^2\right)\right] - \frac{h_n^2\kappa_{n_x}}{r_n}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\right.+$$

$$+\frac{v_n'}{r_n^2(1+e_n)^2}\frac{w_n'}{e_n+1}\left[\begin{matrix} h_n r_n\kappa_{n_x}\left(\kappa_{n_y}\cos\gamma_n - \kappa_{n_z}\sin\gamma_n\right) \\ -\frac{v_n'}{1+e_n}\left[w_n'^2\kappa_{n_x}^2 + r_n^2\left(\kappa_{n_y}^2 + \kappa_{n_z}^2\right) + 2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]+$$

$$-\frac{1}{(1+e_n)^3}\frac{w_n'}{e_n+1}\left[r_n^2\kappa_{n_x}^2 + w_n'^2\left(\kappa_{n_y}^2 + \kappa_{n_z}^2\right) - 2w_n' r_n\kappa_{n_x}\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{1}{r_n^2}\frac{1}{(e_n+1)^3}\left(h_n v_n'' - v_n' u_n''\right)\left\{h_n^2\kappa_{n_x} + \frac{v_n'}{1+e_n}\left[\begin{matrix} h_n r_n\left(\kappa_{n_z}\sin\gamma_n - \kappa_{n_y}\cos\gamma_n\right) \\ +\frac{v_n' w_n'}{1+e_n}\left[w_n'\kappa_{n_x} + r_n\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]\end{matrix}\right]\right\}+$$

$$+\frac{r_n}{(1+e_n)^2}\frac{1}{(e_n+1)^3}\left(h_n v_n'' - v_n' u_n''\right)\left[r_n\kappa_{n_x} - w_n'\left(\kappa_{n_y}\sin\gamma_n + \kappa_{n_z}\cos\gamma_n\right)\right]+$$

$$+\frac{v_n'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[-h_n\kappa_{n_x}\cos\gamma_n + \frac{v_n'}{1+e_n}\left(r_n\kappa_{n_y} + w_n'\kappa_{n_x}\sin\gamma_n\right)\right]\cdot$$

$$\cdot\left\{\left[\left(r_n^2 - w_n'^2\right)\left(h_n u_n'' + v_n' v_n''\right) + 2r_n^2 w_n' w_n''\right]\cos\gamma_n + \left(v_n' u_n'' - h_n v_n''\right)(e_n+1)w_n'\sin\gamma_n\right\}+$$

$$
\begin{aligned}
&+\frac{w_n{}'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^4}\left(w_n{}'\kappa_{n_y}-r_n\kappa_{n_x}\sin\gamma_n\right)\cdot\\
&\quad\cdot\left\{\left[\left(r_n{}^2-w_n{}'^2\right)\left(h_n u_n{}''+v_n{}'v_n{}''\right)+2r_n{}^2 w_n{}'w_n{}''\right]\cos\gamma_n+\left(v_n{}'u_n{}''-h_n v_n{}''\right)(e_n+1)w_n{}'\sin\gamma_n\right\}+\\
&+\frac{v_n{}'}{r_n(1+e_n)}\frac{1}{r_n(e_n+1)^4}\left[h_n\kappa_{n_x}\sin\gamma_n+\frac{v_n{}'}{1+e_n}\left(r_n\kappa_{n_z}+w_n{}'\kappa_{n_x}\cos\gamma_n\right)\right]\cdot\\
&\quad\cdot\left\{-\left[\left(r_n{}^2-w_n{}'^2\right)\left(h_n u_n{}''+v_n{}'v_n{}''\right)+2r_n{}^2 w_n{}'w_n{}''\right]\sin\gamma_n+\left(v_n{}'u_n{}''-h_n v_n{}''\right)(e_n+1)w_n{}'\cos\gamma_n\right\}+\\
&+\frac{w_n{}'}{(1+e_n)^2}\frac{1}{r_n(e_n+1)^4}\left(w_n{}'\kappa_{n_z}-r_n\kappa_{n_x}\cos\gamma_n\right)\cdot\\
&\cdot\left\{-\left[\left(r_n{}^2-w_n{}'^2\right)\left(h_n u_n{}''+v_n{}'v_n{}''\right)+2r_n{}^2 w_n{}'w_n{}''\right]\sin\gamma_n+\left(v_n{}'u_n{}''-h_n v_n{}''\right)(e_n+1)w_n{}'\cos\gamma_n\right\}\Bigg\}\Bigg\rangle\delta w_n\Bigg|_0^{L_n}+\\
&+\Bigg\langle\frac{(1-\nu)EJ}{(1+\nu)(1-2\nu)}\frac{1}{r_n{}^2}\left\{v_n{}'^2\kappa_{n_x}+\frac{h_n}{1+e_n}\left[\begin{array}{l}v_n{}'r_n\left(\kappa_{n_y}\cos\gamma_n-\kappa_{n_z}\sin\gamma_n\right)\\+\frac{h_n w_n{}'}{1+e_n}\left[w_n{}'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\\
&\quad+GJ\Bigg\{\frac{1}{r_n{}^2}\left\{h_n{}^2\kappa_{n_x}+\frac{v_n{}'}{1+e_n}\left[\begin{array}{l}h_n r_n\left(\kappa_{n_z}\sin\gamma_n-\kappa_{n_y}\cos\gamma_n\right)\\+\frac{v_n{}'w_n{}'}{1+e_n}\left[w_n{}'\kappa_{n_x}+r_n\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\end{array}\right]\right\}+\\
&\qquad\qquad+\frac{r_n}{(1+e_n)^2}\left[r_n\kappa_{n_x}-w_n{}'\left(\kappa_{n_y}\sin\gamma_n+\kappa_{n_z}\cos\gamma_n\right)\right]\Bigg\}\Bigg\rangle\delta\gamma_n\Bigg|_0^{L_n}
\end{aligned}
\tag{D1}
$$