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On the Violation of Convexity and Returns to Scale Assumptions in Fixed Cost Allocation (FCA) within the DEA Framework

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Abstract

Fixed Cost Allocation (FCA) among Decision Making Units (DMUs) is one of the essential requirements in both private organizations and public sectors. Data Envelopment Analysis (DEA) has achieved remarkable success in this field and gained a distinguished position among researchers. On the other hand, one of the fundamental principles on which DEA is based is the principle of convexity and returns to scale. This principle has not been considered in FCA problems. In this paper, we demonstrate that in FCA, the principles of convexity and returns to scale change. The obtained results are illustrated by a numerical example.

Keywords: Fixed Cost Allocation, Data Envelopment Analysis, Returns to Scale, Convexity Principle.

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1. Introduction

DEA is a non-parametric optimization approach designed to assess the relative efficiency of comparable DMUs that operate with multiple inputs and outputs. Initially, this method was introduced as the CCR model under Constant Returns to Scale (CRS) by Charnes et al. [1] in 1978. Structural differences in organizations led to the development of the BCC model under Variable Returns to Scale (VRS) by Banker et al. [2] in 1984. Further research in DEA introduced the Decreasing Returns to Scale model (FG) by Färe and Grosskopf [3] and the Increasing Returns to Scale model (ST) by Seiford and Thrall [4] in 1985 and 1990, respectively.

A prominent application of DEA is its use in FCA problems. FCA among DMUs is a crucial requirement in private organizations and public sectors. This approach was first introduced by Cook and Kress [5] based on two principles: efficiency invariance and Pareto optimality. Cook and Zhu [6] extended it for practical applications. Their model was designed using input-oriented and output-oriented CCR models, and they recommended using the VRS framework for further development. The Pareto optimality principle introduced by Cook and Kress [5] was later evaluated as inappropriate by Lin and Chen [7], who suggested that it should be based on super-efficiency invariance and feasibility. Jahanshahloo et al. [8] demonstrated that the Pareto optimality principle in Cook and Kress [5] was incomplete and proposed a simplified model with fewer computations, independent of output.

Lin [9] proposed a new model considering efficiencies and input-output scales. Amirteimoori and Shafiei [10] introduced a DEA-based method for removing a fixed number of common resources among DMUs under the assumption that efficiency remains unchanged before and after elimination Mostafaee [11] proposed allocating fixed costs by jointly considering efficiency ratings and returns-to-scale groupings. Li et al. [12] designed an allocation procedure grounded in common weights and the efficiency invariance principle Amirteimoori and Kordrostami [13] also employed an efficiency-invariance framework with a common weight set, but Jahanshahloo et al. [14] later showed that efficiency preservation is not always guaranteed. Hosseinzadeh Lotfi et al. [15] adopted a goal programming approach, ensuring that efficiency levels after allocation were explicitly assigned to DMUs.

Moreover, FCA has been extended to two-stage network systems. Several researchers, such as Zhou et al. [16], Li et al. [17], Ding et al. [18], and Yu et al. [19], have proposed different resource allocation approaches in two-stage network DEA. However, reviewing most FCA studies reveals that the issue of changes in convexity and returns to scale before and after allocation has been less addressed. Although Qianzhi et al. [20] examined returns to scale when fixed costs were considered as complementary inputs, they studied the relationship between fixed costs and VRS using the super-BCC model and proposed a fixed-cost approach with two conditions: (1) the share of fixed cost allocated to inelastic DMUs must align with their input shares, and (2) the same degree of efficiency satisfaction should hold for all DMUs in the unique optimal allocation.

Variations in returns to scale (RTS) during the FCA process adversely affect the performance assessment of DMUs and undermine the efficiency invariance principle. Since changes in RTS shift the efficiency frontier, efficiency scores obtained prior to cost allocation differ from those calculated afterward. In this study, we demonstrate that, beyond the violation of the convexity assumption, the evaluation models before and after FCA are not identical under certain assumptions. Moreover, through a numerical example, we illustrate that the CCR model transforms into the ST model, while the BCC model converts into the FG model once FCA is applied. Therefore, the two conditions originally defined by Cook and Kress [5] are insufficient, and additional requirements must be considered: adherence to convexity and the invariance of returns to scale before and after FCA.

The rest of this paper is organized as follows. Section 2 introduces various DEA models. Section 3 presents the FCA method of Cook and Kress [5], simplified by Jahanshahloo et al. [8]. Section 4 compares changes in returns-to-scale spaces before and after FCA. Section 5 provides numerical results, and Section 6 concludes the paper with suggestions for future research.

2. DEA Models

Suppose we have n homogeneous DMUs, where each $DMU_j,\ j=1,\dots,n$ produces s outputs $Y_j=(y_{1j},\dots,y_{sj})^T$ by consuming m inputs $X_j=(x_{1j},\dots,x_{mj})^T$. The output-oriented CCR model [1] is formulated as follows:

$$
\begin{aligned}
Min \quad & \sum_{i=1}^{m} v_i x_{ih} \qquad (1)\\
s.t \quad & -\sum_{t=1}^{s} u_t y_{tj} + \sum_{i=1}^{m} v_j x_{ij} \ge 0, \quad j=1,2,\dots,n,\\
& \sum_{t=1}^{s} u_t y_{th} = 1\\
& v_i \ge 0, \quad i=1,2,\dots,m,\\
& u_t \ge 0, \quad t=1,2,\dots,s.
\end{aligned}
$$

The dual form of Model (1) is expressed as:

$$
\begin{aligned}
Max \quad & \gamma_h \qquad (2)\\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ih}, \quad i=1,2,\dots,m,\\
& \sum_{j=1}^{n} \lambda_j y_{tj} \ge \gamma_k y_{rh}, \quad t=1,2,\dots,s,\\
& \lambda_j \ge 0, \quad j=1,2,\dots,n.
\end{aligned}
$$

A DMU_h is considered efficient in Model (2) if and only if $\gamma^* = 1$ and all slack variables are equal to zero.

The output-oriented BCC model [2] is given as:

$$
\begin{aligned}
Max \quad & \delta_h \qquad (3)\\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ih}, \quad i=1,2,\dots,m,\\
& \sum_{j=1}^{n} \lambda_j y_{tj} \ge \delta_h y_{th}, \quad t=1,2,\dots,s,\\
& \sum_{j=1}^{n} \lambda_j = 1\\
& \lambda_j \ge 0, \quad j=1,2,\dots,n.
\end{aligned}
$$

The output-oriented FG model [3] is defined as:

$$
\begin{aligned}
Max \quad & w_h \qquad (4)\\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ih}, \quad i=1,2,\dots,m,\\
& \sum_{j=1}^{n} \lambda_j y_{tj} \ge w_h y_{th}, \quad t=1,2,\dots,s,\\
& \sum_{j=1}^{n} \lambda_j \le 1\\
& \lambda_j \ge 0, \quad j=1,2,\dots,n.
\end{aligned}
$$

Finally, the output-oriented ST model [4] is expressed as:

$$
\begin{aligned}
&Max \quad w_k && (5)\\
&s.t \quad \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{ik}, \quad i=1,2,...,m,\\
&\sum_{j=1}^{n}\lambda_j y_{tj} \ge w_k y_{tk}, \quad t=1,2,...,s,\\
&\sum_{j=1}^{n}\lambda_j \ge 1\\
&\lambda_j \ge 0, \quad j=1,2,...,n.
\end{aligned}
$$

3. FCA with the Cook and Kress Method

Suppose we want to allocate a fixed cost L among n DMUs. Each DMU has its own unique performance measures. The costs should be allocated such that the share of each DMU_j is l_j, subject to the condition $\sum_{j=1}^{n} l_j = L$. Jahanshahloo et al. [8] proposed a simpler allocation method compared to the Cook and Kress [5] approach, based on the two conditions of efficiency invariance and Pareto minimality, as follows. The allocated costs for each DMU should be considered as a new input added to the objective function of model (1).

$$
\begin{aligned}
&Min \quad \sum_{i=1}^{m} v_i x_{ih} + \mu l_h && (6)\\
&s.t \quad -\sum_{t=1}^{s} u_t y_{tj} + \sum_{i=1}^{m} v_i x_{ij} + \mu l_j \ge 0, \; j=1,2,...,n,\\
&\sum u_t y_{th} = 1,\\
&u_t \ge 0, \quad t=1,2,...,s,\\
&v_i \ge 0, \quad i=1,2,...,m,\\
&\mu \ge 0.
\end{aligned}
$$

If the efficiency of any DMU changes after allocation, then the equitable cost allocation is violated. Thus, the efficiency of DMUs should remain unchanged after FCA. The dual of model (6) is formulated as follows:

$$
\begin{aligned}
&Max \quad \gamma'_h && (7)\\
&s.t, \quad \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{ih}, i=1,2,...,m,\\
&\sum_{j=1}^{n}\lambda_j l_j \le l_h \qquad (**)\\
&\sum_{j=1}^{n}\lambda_j y_{tj} \ge \gamma'_h y_{th}, \quad t=1,2,...,s,\\
&\lambda_j \ge 0, \quad j=1,2,...n.
\end{aligned}
$$

The efficiency of model (2) and model (7) are equal if and only if constraint (**) in model (7) is redundant (for proof, see [8]). Therefore, it can be concluded that:

$$
l_j = L \times \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}} \qquad (8)
$$

4. Our Method for Analyzing Changes in Returns to Scale under FCA

Theorem 1: Consider the output-oriented CCR model (1). We aim to allocate a fixed cost L among n DMUs, such that $\sum_{j=1}^{n} l_j = L$. After cost allocation, model (2) becomes equivalent to model (7), where the convexity condition no longer holds.

Proof: The convexity condition states that:

$$
\forall \begin{pmatrix} x \\ y \end{pmatrix} \in T_c \; and \; \forall \alpha \ge 0 \Rightarrow \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \in T_c \qquad (9)
$$

After FCA in model (7), we obtain:

$$
\begin{pmatrix} x + l_i \\ y \end{pmatrix} = \begin{pmatrix} x + \frac{L\sum_{i=1}^{m} x_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}} \\ y \end{pmatrix}_c \quad and \quad \forall \alpha \ge 0 \Rightarrow \begin{pmatrix} \alpha x + \alpha l_i \\ \alpha y \end{pmatrix}
$$

$$= \begin{pmatrix} \alpha x + \dfrac{L\sum_{i=1}^{m}\alpha x_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m}\alpha x_{ij}} \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha x + \dfrac{L\sum_{i=1}^{m} x_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}} \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha x + l_i \\ \alpha y \end{pmatrix} \notin T_c \quad (10)$$

The result shows that the convexity condition does not hold after FCA.

Theorem 2: In the CCR model, under a specific condition, after cost allocation, this model reduces to the FG model.

Proof: Suppose the total input of all DMUs is a constant K. That is:

$$\sum_{i=1}^{m} x_{ij} = K \qquad j = 1, \dots, n, \quad (11)$$

By substituting relation (11) into constraint (**) of model (7), we have:

$$l_j = L \times \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}} = L \times \frac{K}{nK} = \frac{1}{n} L \quad (12)$$

Replacing relation (12) in model (7),

$$\begin{aligned} Max \quad & \gamma'_k && (13) \\ s.t, \quad & \sum_{j=1}^{n} \lambda_j \, x_{ij} \le x_{ih}, i = 1, 2, \dots, m, \\ & \sum_{j=1}^{n} \lambda_j \le 1, \quad (***) \\ & \sum_{j=1}^{n} \lambda_j y_{tj} \ge \gamma'_h y_{th}, t = 1, 2, \dots, s, \\ & \lambda_j \ge 0, \qquad j = 1, 2, \dots n. \end{aligned}$$

we obtain model (13), which is exactly the FG model. That is, the CRS technology after FCA transforms into DRS.

Another observation in FCA is that returns to scale change after FCA. That is, the CCR model after FCA becomes the ST model, and the BCC model after FCA becomes the FG model. We could not find a rigorous mathematical proof for this, but the numerical results in the next section confirm this claim.

5. Numerical Example

In this section, we use the dataset provided by Cook and Kress [5].

Table 1: Dataset from Cook and Kress [5]

DMUs	Input 1	Input 2	Input 3	Output 1	Output2
1	350	39	9	67	751
2	298	26	8	73	611
3	422	31	7	75	584
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	17	83	1070
7	540	18	10	72	457
8	267	33	5	74	590
9	323	25	5	75	1074
10	444	64	6	74	1072
11	323	25	5	25	350
12	444	64	6	104	1199

Table 2: Efficiency Comparison (CCR vs ST model after FCA)

DMUs model	γ_h	New Input	γ'_h	ST
1	1.29822	11.25	1.29822	1.29822
2	1.05497	9.39	1.05497	1.05497
3	1.33564	13.00	1.33564	1.33564
4	1.00000	8.65	1.00000	1.00000
5	1.00000	9.13	1.00000	1.00000
6	1.03446	11.47	1.03446	1.03446
7	1.16224	16.06	1.16224	1.16224
8	1.00000	8.88	1.00000	1.00000
9	1.00000	9.99	1.00000	1.00000
10	1.00000	14.53	1.00000	1.00000
11	3.00000	9.98	3.00000	3.00000
12	1.00000	14.53	1.00000	1.00000

Table 3: Efficiency Computation for BCC Model vs FG Model after FCA

DMUs	δ_h	New Input	δ'_h	FG model
1	1.26225	11.25	1.29822	1.29822
2	1.03466	9.39	1.03466	1.03466
3	1.12101	13.00	1.12101	1.12101
4	1.00000	8.65	1.00000	1.00000
5	1.00000	9.13	1.00000	1.00000
6	1.00000	11.47	1.00000	1.00000
7	1.05876	16.06	1.05876	1.05876
8	1.00000	8.88	1.00000	1.00000
9	1.00000	9.99	1.00000	1.00000
10	1.11847	14.53	1.11847	1.11847
11	3.00000	9.98	3.00000	3.00000
12	1.00000	14.53	1.00000	1.00000

6. Conclusion and Suggestions

In FCA problems with the Cook and Kress method, the results show that the principles of convexity and returns to scale change. In a special case, the CCR model after FCA reduces to the FG model. Numerical results also demonstrate that the CCR model transforms into the ST model, and the BCC model becomes the FG model after FCA. Therefore, it is of particular importance that in FCA problems, convexity and returns to scale principles should be explicitly considered.

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