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Investigating symmetries and conservation laws of PDEs and systems

P. Kabi-Nejad[a]

[a] *School of Mathematics, Iran University of Science and Technology, Narmak-16, Tehran, Iran.*

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Abstract. The main purpose of this article is to investigate some special kinds of symmetries and conservation laws of some partial differential equations (PDEs) and systems which play a significant role in physics. In fact, we review Lie point symmetry, approximate and generalized symmetry, Hamiltonian symmetry, μ-symmetry and different approaches for evaluating conservation laws of PDEs and systems. Additionally, we discuss the effect of the change of variables on the bi-Hamiltonian structure of some equations and obtain the corresponding Hamiltonian formalism of the transformed equation.

Keywords: Approximate symmetry, generalized symmetry, Hamiltonian symmetry, μ-symmetry, conservation law.

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1. Introduction

The symmetry group method has an important role in the analysis of differential equations. Also, the theory of Lie symmetry groups of differential equations was first developed by Lie. The symmetry group of PDEs system is the largest local Lie group of point transformations of both the independent and dependent variables of differential equations with the outstanding property of conserving the set of solutions. The symmetry group in Lie theory include the class of geometric transformations which act on the solutions by transforming their graphs. The Lie group method is a useful approach for constructing exact solutions of differential equations and implies many properties for both of the system and their solutions. In addition, many other types of exact solutions of PDEs can be obtained via the Lie group method. Classification of the group invariant solutions, detection of linearizing transformations, reduction of the order of ordinary

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E-mail address: parastookabinejad@iust.ac.ir (P. Kabi-Nejad).

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differential equations (ODEs) and mapping solutions to other solutions are other important applications of Lie groups in the theory of differential equations. Further, based on the Lie symmetry method, a PDE class which is invariant under a given group of transformations can be derived. For many other applications of Lie symmetries, we refer to [1, 20]. On the other hand, the concept of a conservation law and the relationship between symmetries and conservation laws, which are mathematical formulations of the familiar physical laws of conservation of energy, conservation of momentum and so on, play a significant key in the analysis of basic properties of the solutions. Asystematic way for the determination of conservation laws associated with variational symmetries for systems of Euler-Lagrange equations is indeed the famous Noether theorem. This theorem requires a Lagrangian.There are approaches that don't need a Lagrangian or even assume the existence of a Lagrangian for differential equations [2, 4]. In this paper, we review the conservation laws of some equations via different methods.

2. Symmetries

In this section, we recall some symmetries.

2.1 *Approximate symmetry*

First, we provide previous definitions and results in approximate symmetry of [12] that will be needed. If a function $f(x,\varepsilon)$ satisfies the condition $\lim \frac{f(x,\varepsilon)}{\varepsilon^p} = 0$, it is written $f(x,\varepsilon) = o(\varepsilon^p)$ and f is named of order less than ε^p. If $f(x,\varepsilon) - g(x,\varepsilon) = o(\varepsilon^p)$, f and g are said to be approximately equal (with an error $o(\varepsilon^p)$) and written as $f(x,\varepsilon) = g(x,\varepsilon)+ o(\varepsilon^p)$, briefly $f \approx g$. The approximate equality defines an equivalence relation, and we join functions into equivalence classes by considering $f(x,\varepsilon)$ and $g(x,\varepsilon)$ as members of the same class iff $f \approx g$. Given a function $f(x,\varepsilon)$, presume

$$f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x) \tag{1}$$

is the approximating polynomial of degree p in ε obtained via Taylor series expansion of $f(x,\varepsilon)$ in powers of ε around $\varepsilon = 0$. Then any function $g \approx f$ (in particular, f) has the form $g(x,\varepsilon) = f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^p f_p(x) + o(\varepsilon^p)$. Consequently, (1) is named a canonical representative of the equivalence class of functions containing f Thus, the equivalence class of functions $g(x,\varepsilon)f(x,\varepsilon)$ is determined by the ordered set of $p+1$ functions $f_0(x), \cdots, f_p(x)$. In the theory of approximate transformation groups, one can consider ordered sets of smooth vector-functions depending on x's and a group parameter a: $f_0(x,a), \cdots, f_p(x,a)$ with coordinates $f_0^i(x,a), \cdots, f_p^i(x,a)$ for $i = 1, \cdots, n$. Let's define the one-parameter family G of approximate transformations

$$\bar{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a) + \cdots + \varepsilon^p f_p^i(x,a),\ i = 1, \cdots, n \tag{2}$$

of points $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$ into points $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^n) \in \mathbb{R}^n$ as the class of invertible transformations

$$\bar{x} = f(x,a,\epsilon) \tag{3}$$

with vector-functions $f = (f^1, \cdots, f^n)$ such that

$$f^i(x,a,\epsilon) \approx f_0^i(x,a) + \epsilon f_1^i(x,a) + \cdots + \varepsilon^p f_p^i\ (x,a),\ i = 1, \cdots, n.$$

Here, a is a real parameter and the condition is $f(x, 0, \epsilon) \approx x$.

Definition 2.1 The set of (2) is a one-parameter approximate transformation group if $f(f(x,a,\varepsilon),b,\epsilon) \approx f(x,a+b,\varepsilon)$ for all transformations (3).

Definition 2.2 Presume G is a one-parameter approximate transformation group:

$$\bar{z}^i \approx f(z,a,\varepsilon) \equiv f_0^i(z,a) + \varepsilon f_1^i(z,a),\ i = 1, \cdots, N. \tag{4}$$

An approximate equation

$$F(z,\varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0 \tag{5}$$

is called approximately invariant regarding G, or admits G if

$$F(\bar{z},\varepsilon) \approx F(f(z,a,\varepsilon),\varepsilon) = o(\varepsilon)$$

whenever $z = (z^l, \cdots, z^N)$ satisfies (5).

If $z = (x, u, u_{(1)}, \cdots, u_{(k)})$, then (5) becomes an approximate differential equation of order k and G is an approximate symmetry group of the differential equation.

Theorem 2.3 (5) is approximately invariant under the approximate transformation group (4) with the generator

$$X = X_0 + \varepsilon X_1 \equiv \xi_0^i(z)\frac{\partial}{\partial z^i} + \varepsilon \xi_1^i \frac{\partial}{\partial z^i}, \tag{6}$$

iff

$$[X^{(k)}F(z,\varepsilon)]_{F\approx 0} = o(\varepsilon) \tag{7}$$

or

$$[X_0^{(k)}F_0(z) + \varepsilon(X_1^{(k)}F_0(z) + X_0^{(k)}F_1(z))]_{(2.5)} = o(\varepsilon), \tag{8}$$

where $X^{(k)}$ is the prolongation of X of order k.

The operator (6) satisfying (8) is called an infinitesimal approximate symmetry of, or an approximate operator admitted by (5). Accordingly, (8) is termed the determining equation for approximate symmetries.

Theorem 2.4 If (5) admits an approximate transformation group with a generator $X = X_0 + \varepsilon X_1$ $(X_0 \neq 0)$, then $X_0 = \xi_0^i(z)\frac{\partial}{\partial z^i}$ is an exact symmetry of

$$F_0(z) = 0. \tag{9}$$

(5) and (9) are termed a perturbed equation and an unperturbed equation, respectively. Under the conditions of Theorem 2.3, X_0 is a stable symmetry of (9). The corresponding

approximate symmetry generator $X = X_0 + \varepsilon X_1$ for (5) is called a deformation of the infinitesimal symmetry X_0 of (9) caused by the perturbation $\varepsilon F_1(z)$. In particular, if the most general symmetry Lie algebra of (9) is stable, we say (5) inherits the symmetries of the unperturbed equation.

2.2 *Generalized symmetries*

Take a system of n-th order differential equations in p independent and q dependent variables as follows:

$$\triangle_\upsilon(x, u^{(n)}) = 0, \quad \upsilon = 1, \cdots, N \tag{10}$$

involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$, and the derivatives of u with respect to x up to order n. A generalized vector field is an expression of

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha[u] \frac{\partial}{\partial u^\alpha} \tag{11}$$

in which coefficient functions ξ^i, ϕ_α depend on x, u and derivatives of u. By the prolongation formula of Theorem 2.36 in [20], we can define the prolonged generalized vector field $\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{\sharp J \leqslant n} \phi_\alpha^J[u] \frac{\partial}{\partial u_J^\alpha}$, where its coefficients is determined by

$$\phi_\alpha^J = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha. \tag{12}$$

Since all the prolongation of $\mathbf{v}$ have the same general expression for their coefficient functions ϕ_α^J, it is helpful to pass to the infinite prolongation, and take care of all the derivatives at once. Given a generalized vector field $\mathbf{v}$, its infinite prolongation is the infinite sum

$$\mathrm{pr}\mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha}, \tag{13}$$

where each ϕ_α^J is given by (12), and the sum in (13) now extends over all multi-indices $J = (j_1, \cdots, j_k)$ for $k \geqslant 0,\ 1 \leqslant j_k \leqslant p$. By the infinitesimal symmetry criterion in Theorem 2.72 of [20], we can state the following result.

A generalized vector field $\mathbf{v}$ is a generalized infinitesimal symmetry of a system of (10) iff

$$\mathrm{pr}\mathbf{v}[\triangle_\upsilon] = 0, \quad \upsilon = 1, \cdots, l \tag{14}$$

for each smooth solution $u = f(x)$. Among all the generalized vector fields, those in which the coefficients $\xi^i[u]$ of the $\frac{\partial}{\partial x^i}$ are zero have a important role. Assume $Q[u] = (Q_1[u], \cdots, Q_q[u]) \in \mathcal{A}^q$ is a q-tuple of differential functions. The generalized vector field $\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}$ is an evolutionary vector field and Q is its characteristic. Note by (12) that the prolongation of an evolutionary vector field takes a simple form

$\mathrm{pr}\mathbf{v}_Q = \sum_{\alpha,J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}$. Any generalized vector field $\mathbf{v}$ as in (11) has an associated evolutionary representative $\mathbf{v}_Q$ in which Q has entries $Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$ for $\alpha = 1, \cdots, q$, where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$. These two generalized vector fields have the same symmetry. In fact, the computation of generalized symmetries of a given system of differential equations proceeds in the same way as the earlier computations of geometrical symmetries, but with the following added features:

First, we should put the symmetry in evolutionary form $\mathbf{v}_Q$, which has the effect of reducing the number of unknown functions from $p+q$ to just q, while simultaneously simplifying the computation of the prolongation $\mathrm{pr}\mathbf{v}_Q$. One must fix the order of derivatives on which the characteristic $Q(x, u^{(m)})$ may depend. So, by taking m not too large will yield important information on the general form of the symmetries. Finally one should deal with the occurrence of trivial symmetries; the easiest way to handle these is to eliminate any superfluous derivatives in Q by substitution using the prolongation of the system.

2.3 *Hamiltonian symmetries*

In this part, we will provide the results on Hamiltonian operators of [20]. Presume $x = (x^1, \cdots, x^p)$ is the spatial variables, and $u = (u^1, \cdots, u^q)$ the field variables (dependent variables) so that each u^α is a function of $x^1, \cdots, x^p$ and the time t. We take autonomous systems of evolution equations $u_t = K[u]$ in which $K[u] = (K_1[u], \cdots, K_q[u])$ is a q-tuple of differential functions, where the square brackets indicate each K_α is a function of x, u and finitely many partial derivatives of each u^α regarding $x^1, \cdots, x^p$. A system of evolution equations is Hamiltonian if it can be written by $u_t = \mathcal{D}.\mathrm{E}_u(H)$. Here $\mathcal{H}[u] = \int H[u]dx$ is the Hamiltonian functional, and Hamiltonian function $H[u]$ depends on x, u, and the derivatives of the u's with respect to the x's. $\mathrm{E}_u = (\mathrm{E}_1, \cdots, \mathrm{E}_q)$ denotes the Euler operator or variational derivative with respect to u. The Hamiltonian operator $\mathcal{D}$ is a $q \times q$ matrix differential operator, which may depend on both x, u, and derivatives of u (but not on t), and is required to be skew-adjoint relative to the L^2-inner product $< f, g >= \int f.gdx = \int \sum f^\alpha.g^\alpha dx$. Thus, $\mathcal{D}^* = -\mathcal{D}$, where $*$ is the formal L^2 adjoint of a differential operator. Moreover, $\mathcal{D}$ must satisfy a nonlinear Jacobi condition that the corresponding poisson bracket

$$\{\mathcal{P}, \mathcal{Q}\} = \int \mathrm{E}_u[P].\mathcal{D}\mathrm{E}_u[Q]dx, \quad \mathcal{P} = \int P[u]dx, \quad \mathcal{Q} = \int Q[u]dx, \tag{15}$$

satisfies the Jacobi identity. In the spatial case that $\mathcal{D}$ is a field-independent skew-adjoint differential operator, meaning that the coefficient of $\mathcal{D}$ do not depend on u or its derivatives (but may depend on x), the Jacobi conditions are automatically satisfied. For more general field-dependent operators, the complicated Jacobi conditions can be simplified by the functional multi-vector method described in [20]. Multi-vectors are the dual objects of differential forms. To preserve the notational distinction between the two, we use θ_J^α for the uni-vector corresponding to the one-form du_J^α. Hence, a vertical multi-vector is a finite sum of terms, which are the product of a differential function times a wedge product of the basic uni-vectors. The space of functional multi-vectors is the co-kernel of the total divergence so that two vertical multi-vectors determine the same functional multi-vector iff they differ from a total divergence. The functional multi-vector determined by $\hat{\Theta}$ is denoted by $\Theta = \int \hat{\Theta}dx$. In particular, $\int \hat{\Theta}dx = 0$ iff $\hat{\Theta} = \mathrm{Div}\hat{\Psi}$ for some vertical multi-vector $\hat{\Psi}$. This induces that we can integrate functional multi-vectors by parts $\int \hat{\Theta} \wedge (D_i\hat{\Psi})dx = -\int (D_i\hat{\Theta}) \wedge \hat{\Psi}dx$. The principal example of a bi-vector is a Hamiltonian differential operator $\mathcal{D}$, which is $\Theta_{\mathcal{D}} = \int \theta \wedge \mathcal{D}(\theta)dx$. Finally, we recall the formal prolonged vector field: $\mathrm{pr}\mathbf{v}_{\mathcal{D}\theta} = \sum\limits_{\alpha,J} D_J(\sum\limits_{\beta} \mathcal{D}_{\alpha\beta}\theta^{\beta})\frac{\partial}{\partial u_J^{\alpha}}$, which acts on differential functions to produce uni-vectors. Next, let $\mathrm{pr}\mathbf{v}_{\mathcal{D}}\theta$ act on vertical multi-vectors by wedging the result of its action on the coefficient differential functions with the product of the θ's. Since $\mathrm{pr}\mathbf{v}_{\mathcal{D}}\theta$ commutes with the total derivative, there is a well-defined action of $\mathrm{pr}\mathbf{v}_{\mathcal{D}}\theta$ on the space of functional multi-vectors. The following theorem is a criterion for showing a differential operator is genuinely Hamiltonian.

Theorem 2.5 Presume $\mathcal{D}$ is a skew-adjoint differential operator with corresponding bi-vector $\Theta_{\mathcal{D}}$. Then $\mathcal{D}$ is a Hamiltonian operator iff

$$\mathrm{pr}\mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0. \tag{16}$$

The proof that (16) is equivalent to the Jacobi identity for the poisson bracket determined by $\mathcal{D}$ can be found in [20].

2.4 *μ-symmetry*

PDEs with suitable solutions are one of the most important topics in various branches of mathematical physics. The most accurate methods for order reduction and computation conservation rules are the classical Lie theory, the general theorem [20], the direct method [20], the μ-symmetries method [5] and the Noether theorem [19]. As we know, using Lie transformation group theory for order reduction and constructing solutions of nonlinear PDEs with integer order or fractional order PDEs and ODEs is one of the most efficient fields of research in the theory of nonlinear PDEs. The λ-symmetry method has been presented by Muriel and Romero [14], which is a modern method to order reduction of ODEs. Gaeta and Morando [7, 8] have extended the λ-symmetries approach for ODEs to the μ-symmetries method for PDEs. In the sequel, we will compare different symmetries and approaches for evaluating conservation laws for PDEs.

3. Harry Dym equation

The nonlinear PDE $u_t = D_x^3(u^{-\frac{1}{2}})$ is known as the Harry Dym equation [20]. It has a bi-Hamiltonian structure, an infinite number of conservation laws and infinitely many symmetries. Under change of variables $v = u^{-\frac{1}{2}}$, this equation can be written in the form $v_t = -\frac{1}{2}v^3v_{xxx}$. The Hamiltonian structure of the changed Harry Dym equation and determine Hamiltonian operators of the evolution equation is investigated in [18].

3.1 *Approximate symmetries of the perturbed Harry Dym equation*

Consider perturbed Harry Dym equation

$$u_t + \frac{1}{2}u^3u_{xxx} + \varepsilon u_x = 0. \tag{17}$$

Using the method of approximate transformation groups, we provide the infinitesimal approximate symmetries (6) for the perturbed Harry Dym equation (17).

3.2 *Exact symmetries*

Take the approximate group generators in the form

$$X = X_0 + \varepsilon X_1 = (\xi_0 + \varepsilon\xi_1)\frac{\partial}{\partial x} + (\tau_0 + \varepsilon\tau_1)\frac{\partial}{\partial t} + (\phi_0 + \varepsilon\phi_1)\frac{\partial}{\partial u},$$

where $r\xi_i, \tau_i$ and ϕ_i for $i = 0, 1$ are unknown functions ofx, t and u. Solving the determining equation $X_0^{(3)}(u_t - \frac{1}{2}u^3 u_{xxx}) \mid_{u_t - \frac{1}{2}u^3 u_{xxx}=0} = 0$ for the exact symmetries X_0 of the unperturbed equation, we obtain

$$\xi_0 = (A_1 + A_2 x + \frac{A_3}{2}x^2),\ \tau_0 = (A_4 + A_5 t),\ \phi_0 = (A_2 - \frac{1}{3A_5} + xA_3)u,$$

where $A_1, \cdots, A_5$ are arbitrary constants. Hence,

$$X_0 = (A_1 + A_2 x + \frac{A_3}{2}x^2)\frac{\partial}{\partial x} + (A_4 + A_5 t)\frac{\partial}{\partial t} + ((A_2 - \frac{1}{3A_5} + xA_3)u)\frac{\partial}{\partial u}. \tag{18}$$

Therefore, the unperturbed Harry Dym equation, admits the five-dimensional Lie algebra with the basis

$$X_0^1 = \frac{\partial}{\partial x}, X_0^2 = \frac{\partial}{\partial t}, X_0^3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, X_0^4 = 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, X_0^5 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}. \tag{19}$$

3.3 *Approximate symmetries*

At first, we need to determine the auxiliary function H by virtue of (5), (7) and (8), i.e., by

$$H = \frac{1}{\varepsilon}[X_0^{(k)}(F_0(z) + \varepsilon F_1(z)) \mid_{F_0(z)+\varepsilon F_1(z)=0}]. \tag{20}$$

Substituting (18) of the generator X_0 into (20), we obtain the auxiliary function $H = u_x(A_5 - A_2) + A_3(u - xu_x)$. Now, calculate X_1 by solving the inhomogeneous determining equation for deformations $X_1^{(k)}F_0(z) \mid_{F_0(z)=0} + H = 0$. So, the determining equation for this equation is

$$X_1^{(3)}(u_t + \frac{1}{2}u^3 u_{xxx}) \mid_{u_t + \frac{1}{2}u^3 u_{xxx}=0} + u_x(A_5 - A_2) + A_3(u - xu_x) = 0.$$

Solving the determining equation yields

$$\begin{aligned}
\xi_1 &= (A_5 - A_2)t - A_3 xt + C_4 x - C_5 + \frac{C_3}{2}x^2,\\
\tau_1 &= (C_1 t + C_2),\\
\phi_1 &= (-A_3 t + C_4 + C_3 x + \frac{C_1}{3})u,
\end{aligned}$$

where $C_1, \cdots, C_5$ are arbitrary constants. Thus, we have the following approximate symmetries of the perturbed Harry Dym equation:

$$\begin{aligned} &\mathbf{v_1} = \frac{\partial}{\partial x}, && \mathbf{v}_6 = \varepsilon\frac{\partial}{\partial x}, \\ &\mathbf{v}_2 = \frac{\partial}{\partial t}, && \mathbf{v}_7 = \varepsilon\frac{\partial}{\partial t}, \\ &\mathbf{v}_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, && \mathbf{v}_8 = \varepsilon(x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}), \\ &\mathbf{v}_4 = 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, && \mathbf{v}_9 = \varepsilon(3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}), \\ &\mathbf{v}_5 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}, && \mathbf{v}_{10} = \varepsilon(x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}). \end{aligned} \tag{21}$$

The following table of commutators, evaluated in the first-order of precision, shows that the operators (21) span an ten-dimensional approximate Lie algebra, and hence, generate an ten-parameter approximate transformations group. (21) show that all symmetries (19) of the Harry Dym equation are stable. Hence, the perturbed equation (17) inherits the symmetries of the unperturbed equation. The structure of the Lie algebra of symmetries of the perturbed Harry Dym equation is evaluated in [15].

Table 1. Approximate Commutators of approximate symmetry of perturbed Harry Dym equation

	$\mathbf{v_1}$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v_1}$	0	0	$\mathbf{v_1}$	0	$2\mathbf{v}_3$	0	0	$\mathbf{v}_6$	0	$2\mathbf{v}_8$
$\mathbf{v}_2$	0	0	0	$12\mathbf{v}_2$	0	0	0	0	$3\mathbf{v}_7$	0
$\mathbf{v}_3$	$-\mathbf{v}_1$	0	0	0	$\mathbf{v}_5$	$-\mathbf{v}_6$	0	0	0	$\mathbf{v}_{10}$
$\mathbf{v}_4$	0	$-12\mathbf{v}_2$	0	0	0	0	$-3\mathbf{v}_7$	0	0	0
$\mathbf{v}_5$	$-2\mathbf{v}_3$	0	$-\mathbf{v}_5$	0	0	$-2\mathbf{v}_8$	0	$-\mathbf{v}_{10}$	0	0
$\mathbf{v}_6$	0	0	$\mathbf{v}_6$	0	$2\mathbf{v}_8$	0	0	0	0	0
$\mathbf{v}_7$	0	0	0	$3\mathbf{v}_7$	0	0	0	0	0	0
$\mathbf{v}_8$	$-\mathbf{v}_6$	0	0	0	$\mathbf{v}_{10}$	0	0	0	0	0
$\mathbf{v}_9$	0	$-3\mathbf{v}_7$	0	0	0	0	0	0	0	0
$\mathbf{v}_{10}$	$-2\mathbf{v}_8$	0	$-\mathbf{v}_{10}$	0	0	0	0	0	0	0

4. Camassa-Holm equation

In this section, we consider the Camassa-Holm equation

$$u_t - u_{tx^2} + ku_x + 3uu_x = 2u_x u_{x^2} + uu_{x^3}, \quad k \in \mathbb{R}. \tag{22}$$

4.1 *Infinitesimal generalized symmetries of Camassa-Holm equation*

(22) was first introduced as a model describing propagation of unidirectional gravitational waves in shallow water approximation, with u representing the fluid velocity at time t in the x direction. Suppose $\mathbf{v}_Q = Q[u]\partial_u$ is a generalized symmetry in evolutionary form. Note that we can replace some derivatives of u occurring in Q by their corresponding expressions without changing the equivalence class of $\mathbf{v}$. For instance, u_{xxt} is replaced by

$u_t + ku_x + 3uu_x - 2u_xu_{xx} - uu_{xxx}$ and so on. Thus each symmetry is uniquely equivalent to one with characteristic $Q = Q(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xtt}, u_{ttt}, \cdots)$. The prolongation of $\mathbf{v}_Q$ is given by

$$\mathrm{pr}\mathbf{v}_Q = Q\partial_u + D_xQ\partial_{u_x} + D_tQ\partial_{u_t} + D_x^2Q\partial_{u_{xx}} + D_xD_tQ\partial_{u_{xt}} + D_t^2Q\partial_{u_{tt}} + \cdots .$$

The infinitesimal condition (14) for invariance is

$$D_tQ - D_x^2D_tQ + kD_xQ + 3(u_xQ + uD_Q) = 2(u_{xx}D_xQ + u_xD_x^2Q) + u_{xxx}Q + uD_x^3Q, \tag{23}$$

which must be satisfied for all solutions. To calculate third order symmetries, we require $Q = Q(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xtt}, u_{ttt})$ So, upon substituting for $u_{xxt}, u_{xxxt}, u_{xxtt}, \cdots$ in (23) according to the equation and after eliminating any dependence among the derivatives of the function u, we are left to a complete system of determining PDEs. Therefore, the most general third-order characteristic function Q is

$$\begin{aligned} Q = {} & (C_1t - \frac{3}{2}C_2u^2 + \frac{1}{2}((2u_{xx} - 2k)C_2 - 2C_3)u + C_3u_{xx} + \frac{1}{2}u_x^2C_2 + C_4)u_t \\ & + u_{xt^2}C_3 - \frac{1}{3}C_2u_{ttt} - \frac{kt}{2}C_1u_x + (-\frac{1}{3}C_2u^3 - u^2C_3 + \frac{C_5}{(k + 2u - 2u_{xx})^{\frac{3}{2}}})u_{xxx} \\ & - \frac{1}{2}C_3u_x^3 + \frac{1}{2}C_1k + C_1u + \frac{1}{2}u_x(\frac{2C_5}{(k + 2u - 2u_{xx})^{\frac{3}{2}}} + \frac{8}{3}C_2u^3 \\ & + ((-2u_{xx} + k)C_2 + 9C_3)u^2 + 4(k - \frac{3}{2}u_{xx})C_3u + 2C_6), \end{aligned}$$

where $C_1, \cdots, C_6$ are arbitrary constants.

Theorem 4.1 The most general third-order infinitesimal generalized symmetries of the Camassa-Holm equation is a $\mathbb{R}$-linear combination of following six vector fields

$$\begin{aligned} Q_1 &= u_x \\ Q_2 &= u_t \\ Q_3 &= -\frac{1}{2}ktu_x + tu_t + u + \frac{1}{2}k \\ Q_4 &= (-\frac{3}{2}u^2 + \frac{1}{2}(2u_{xx} - 2k)u + \frac{1}{2}u_x^2)u_2 - \frac{1}{3}u_{ttt} - \frac{1}{3}u^3u_{xxx} \\ &\quad + \frac{1}{2}u_x(\frac{8}{3}u^3 + (-2u_{xx} + k)u^2) \\ Q_5 &= (-u + u_{xx})u_t + u_{xtt} - u^3u_{xxx} - \frac{1}{2}u_x(9u^2 + 4(k - \frac{3}{2}u_{xx})u) \\ Q_6 &= u_t^2 - \frac{1}{2}u_xkt + \frac{u_{xxx}}{(-2u_{xx} + k + 2u)^{\frac{3}{2}}} + \frac{1}{2}k + u - \frac{u_x}{(-2u_{xx} + k + 2u)^{\frac{3}{2}}} \end{aligned}$$

which $\mathbf{v}_{Q_1}, \mathbf{v}_{Q_2}, \mathbf{v}_{Q_3}$ form a three-dimensional Lie algebra $\mathbf{g}$ of symmetry group associated to the Camassa-Holm equation.

The symmetries $u_x\partial_u$ and $u_t\partial_u$ of the Camassa-Holm equation are just the evolutionary representative of the space and time translational symmetry generators. Similarly, the symmetry $\mathbf{v}_{Q_3}$ has geometric form $-kt\partial_x + 2t\partial_t + (-k-2u)\partial_u$. We call these evolutionary symmetries ($\mathbf{v}_{Q_1}, \mathbf{v}_{Q_2}$ and $\mathbf{v}_{Q_3}$) in geometric form with $Y_1 = \partial_x$, $Y_2 = \partial_t$ and $Y_3 = -kt\partial_x + 2t\partial_t + (-k-2u)\partial_u$, respectively.

Proposition 4.2 The one-parameter groups $g_i(t) : M \longrightarrow M$ generated by Y_i for $i = 1, 2, 3$ are given in the following table:

$$\begin{aligned}
g_1(s): &\quad (x,t,u) \longmapsto (x+s,t,u),\\
g_2(s): &\quad (x,t,u) \longmapsto (x,t+s,u),\\
g_3(s): &\quad (x,t,u) \longmapsto (-\frac{1}{2}kte^{2s} + \frac{1}{2}kt + x, te^{2s}, -\frac{1}{2}k + e^{-2s}(u+\frac{1}{2}k)),
\end{aligned}$$

where the entries give the transformed point $\exp(sY_i)(x,t,u) = (\bar{x}, \bar{t}, \bar{u})$.

Consequently, we can state the following theorem:

Theorem 4.3 If $u = U(x,t)$ is a solution of (22), there are $u^i(x,t) = U(x,t)$ for $i = 1, 2, 3$ and $s \in \mathbb{R}$, where

$$u^1 = U(x+s,t),\ u^2 = U(x,t+s),\ u^3 = e^{2s}U\Big(\frac{kt}{2}(1-e^{2s}) + x, te^{2s}\Big) + \frac{k}{2}\Big(1 - e^{-2s}\Big).$$

4.2 *Higher-order conservation laws for Camassa-Holm equation*

Consider a system of N PDEs of order n with p independent variables $x = (x^1, \cdots, x^p)$ and q dependent variables $u = (u^1, \cdots, u^q)$, given by (10). A conservation law of (10) is a divergence expression

$$\mathrm{D}_1\mathrm{P}_1 + \cdots + \mathrm{D}_p\mathrm{P}_p = 0, \tag{24}$$

holding for all solutions $u = f(x)$ of the given system. In (24), $\mathrm{P}_i(x, u^{(r)})$ for $i = 1, \cdots, p$ are called the fluxes of the conservation law, and the highest-order derivative r in the fluxes is called the order of the conservation law. If one of the independent variables of (10) is time t, (24) takes the form $\mathrm{D}_t\mathrm{T} + \mathrm{Div}\mathrm{X} = 0$, where Div is the spatial divergence of X regarding the spatial variables $x = (x^1, \cdots, x^p)$. T is referred to a density and $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_p)$ as spatial fluxes of the conservation law (24). The conserved density T and the associated flux $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_p)$ are two functions of x, t, u and the derivatives of u regarding both x and t. Specially, each admitted conservation law arises from multipliers $\lambda^\nu(x, u^{(l)})$ so that $\lambda^\nu(x, u^{(l)})\Delta_\nu(x, u^{(n)}) = \mathrm{D}_i\mathrm{P}_i(x, u^{(r)})$ holds, where the summation convention is used whenever appropriate. The determining of conservation laws for a PDE system (10) reduces to finding sets of multipliers. The Euler operator regarding u_j is

$$\mathrm{E}_{u^j} = \frac{\partial}{\partial u_j} - \mathrm{D}_i\frac{\partial}{\partial u_i^j} + \cdots + (-1)^s\mathrm{D}_{i_1}\cdots\mathrm{D}_{i_s}\frac{\partial}{\partial u^j_{i_1\cdots i_s}} + \cdots. \tag{25}$$

Note that Euler operators (25) annihilate any divergence expression $\mathrm{D}_i\mathrm{P}_i(x, u^{(r)})$ and the identities $\mathrm{E}_{u^j}(\mathrm{D}_i\mathrm{P}_i(x, u^{(r)})) = 0$ hold for arbitrary function u and $j = 1, \cdots, q$.

The converse also holds. The following theorem is applied for connecting multipliers and conservation laws.

Theorem 4.4 A set of multipliers $\{\lambda^\nu(x, u^{(l)})\}_{\nu=1}^N$ yields a conservation law for (10) if the set of identities

$$\mathrm{E}_{u^j}(\lambda^\nu(x, u^{(l)})\Delta_\nu(x, u^{(n)})) = 0, \quad j = 1, \cdots, q \tag{26}$$

holds identically.

The set of equations (26) yields the set of linear determining equations to find all sets of conservation law multipliers of the PDE system (10) by considering multipliers of all orders. See [3] for more details.

Here, we review higher order conservation laws for Camassa-Holm equation. Consider the multipliers of $\lambda(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, u_{ttt})$ for (22). The determining equation for multipliers is $\mathrm{E}_u\big[\lambda(u_t - u_{tx^2} + ku_x + 3uu_x - 2u_xu_{x^2} - uu_{x^3})\big] = 0$ in which the standard Euler operator E_u is

$$\mathrm{E}_u = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_t\frac{\partial}{\partial u_t} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_t\frac{\partial}{\partial u_{xt}} + D_t^2\frac{\partial}{\partial u_{tt}} - \cdots,$$

where D_x and D_t are the total derivatives in respect of x and t. Hence,

$$\lambda = C_1u_{tt} + C_1u_xu_t + (-C_1u + C_2)(u_{xt} + \frac{1}{2}{u_x}^2) + \frac{2C_5}{\sqrt{-u_{xx} + u}} + (-2C_1u + C_2)uu_{xx} + \frac{5}{2}C_1u^3 - \frac{3}{2}C_2u^2 + C_3u + C_4,$$

where C_1, C_2, C_3, C_4 and C_5 are constants.

To calculate the conserved quantities T and X, we need to invert the total divergence operator. The homotopy operator is a powerful algorithmic tool originating from homological algebra and variational bi-complexes [10]. The conserved vectors are represented by two components T_1 and T_2 which are conserved density and flux, respectively. Thus, by using the 2-dimensional homotopy (integral) formula of Hereman et al. [11], we have computed conserved vectors in [17].

5. Whitham-Broer-Kaup equations

The system of equations WBK

$$u_t = uu_x + v_x - \frac{1}{2}u_{xx} \ \text{ and } \ v_t = (uv)_x + \frac{1}{2}v_{xx} \tag{27}$$

admits three Hamiltonian operators

$$\mathcal{D}_0 = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}, \qquad \mathcal{D}_1 = \begin{pmatrix} 2D_x & D_x.u - D_x^2 \\ uD_x + D_x^2 & 2vD_x + v_x \end{pmatrix},$$

$$\mathcal{D}_2 = \begin{pmatrix} 4uD_x + 2u_x & 4vD_x + 2v_x + D_x(D_x - u)^2 \\ 4vD_x + 2v_x + (D_x + u)^2D_x & (D_x + u)(2vD_x + v_x) - (2vD_x + v_x)(D_x - u) \end{pmatrix},$$

and so can be written in Hamiltonian form in three distinct ways [6]. The skew symmetry of these Hamiltonian structures is manifest. The proof of the Jacobi identity for this structures as well their compatibility can be shown by the standard method of functional multi vectors. As the coefficients of $\mathcal{D}_0$ do not depend on u or its derivatives, $\mathcal{D}_0$ is a Hamiltonian operator. For $\mathcal{D}_1$, it is enough to show $\mathrm{pr}\mathbf{v}_{\mathcal{D}_1\theta}(\Theta_{\mathcal{D}_1}) = 0$, where $\Theta_{\mathcal{D}_1}$ is the corresponding functional bi-vector and $\theta = (\theta, \zeta)$ so that θ and ζ are the basic uni-vectors corresponding to u and v, respectively.

5.1 *Conservation Laws*

Give a system of N PDEs of order n with p independent variables $x = (x^1, \cdots, x^p)$ and q dependent variables $u = (u^1, \cdots, u^q)$ by

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, N. \tag{28}$$

A conservation law of (28) is a divergence expression

$$\mathrm{D}_1\mathrm{P}_1 + \cdots + \mathrm{D}_p\mathrm{P}_p = 0 \tag{29}$$

holding for all solutions $u = f(x)$ of the given system. In (29), $\mathrm{P}_i(x, u^{(r)})$ for $i = 1, \cdots, p$ are named the fluxes of the conservation law, and the highest-order derivative r present in the fluxes is the order of the conservation law. If one of the independent variables of (28) is time t, the conservation law (29) takes the form $\mathrm{D}_t\mathrm{T} + \mathrm{Div}\mathrm{X} = 0$, where Div is the spatial divergence of X regarding the spatial variables $x = (x^1, \cdots, x^p)$. Here, T is referred to a density and $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_p)$ is spatial fluxes of the conservation law (29). The conserved density T and the associated flux $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_p)$ are two functions of x, t, u and the derivatives of u regarding both x and t. Specially, each admitted conservation law arises from multipliers $\lambda^\nu(x, u^{(l)})$ such that $\lambda^\nu(x, u^{(l)})\Delta_\nu(x, u^{(n)}) = \mathrm{D}_i\mathrm{P}_i(x, u^{(r)})$ holds identically, where the summation convention is used whenever appropriate. The determining of conservation laws for a given PDE system (28) reduces to finding sets of multipliers.

Here, one can show the Euler operators annihilate any divergence expression $\mathrm{D}_i\mathrm{P}_i(x, u^{(r)})$ and the identities $\mathrm{E}_{u^j}(\mathrm{D}_i\mathrm{P}_i(x, u^{(r)})) = 0$ hold for arbitrary function u and $j = 1, \cdots, q$. The converse also holds. Specifically, the only scalar expressions annihilated by Euler operators are divergence expressions. The following theorem is applied for connecting multipliers and conservation laws.

Theorem 5.1 A set of multipliers $\{\lambda^\nu(x, u^{(l)})\}_{\nu=1}^N$ yields a conservation law for the PDE system (28) iff

$$\mathrm{E}_{u^j}(\lambda^\nu(x, u^{(l)})\Delta_\nu(x, u^{(n)})) = 0, \qquad j = 1, \cdots, q. \tag{30}$$

holds identically.

The set of equations (30) yields the set of linear determining equations to find all sets of conservation law multipliers of (28) by considering multipliers of all orders. See [13] for more details.

Now, we recall conservation laws for (27). Consider the multipliers of the form

$$\lambda^1(x, t, u, u_x, u_{x^2}, u_{x^3}, v, v_x, v_{x^2}, v_{x^3}), \quad \lambda^2(x, t, u, u_x, u_{x^2}, u_{x^3}, v, v_x, v_{x^2}, v_{x^3})$$

for (22). The determining equations for multipliers is

$$\begin{aligned}\mathrm{E}_u\big[\lambda^1(u_t-uu_x+v_x-\frac{1}{2}u_{x^2})\big]&=0,\\ \mathrm{E}_u\big[\lambda^2(v_t-(uv)_x+\frac{1}{2}v_{x^2})\big]&=0,\\ \mathrm{E}_v\big[\lambda^1(u_t-uu_x+v_x-\frac{1}{2}u_{x^2}\big]&=0,\\ \mathrm{E}_v\big[\lambda^2(v_t-(uv)_x+\frac{1}{2}v_{x^2})\big]&=0.\end{aligned}$$

Hence,

$$\begin{aligned}\lambda^1&=(2c_2u+c_3)v^2+((v_x+\frac{1}{3}u_{x^2}+\frac{2}{3}u^3)c_2+c_3u^2+c_1t+c_5+2c_4u)v\\ &\quad+(\frac{1}{6}v_{x^3}+\frac{1}{3}u_xv_x+v_xu^2+\frac{2}{3}uv_{xx})c_2+c_4v_x+c_3(\frac{1}{3}v_{xx}+uv_x)+c_6,\\ \lambda^2&=-\frac{1}{6}c_2u^4+\frac{1}{3}c_3u^3+((2v-u_x)c_2+c_4)u^2+(\frac{2}{3}c_2u_{xx}+c_3(2v-u_x)+c_1t+c_5)u\\ &\quad+(\frac{1}{3}v_{x^2}-\frac{1}{6}u_{x^3}+v^2-u_xv+\frac{1}{2}{u_x}^2)c_2+c_1x+c_4(2v-u_x)+\frac{1}{3}c_3u_{x^2}+c_7\end{aligned}$$

where c_i for $i=1,\cdots,7$ are constants. Here, the first homotopy formula is reviewed to construct conservation laws of (22). It is described in [2, Bluman et al.]. Conserved vectors are represented by two components T_1 and T_2 which are conserved density and flux, respectively and computed in details in [16].

5.2 *Hamiltonian Symmetries*

The correspondence between Hamiltonian symmetry groups and conservation laws for systems of evolution equations in Hamiltonian form is known as Noether theorem, after the prototype [19]. This relationship has been discussed by Olver [20], Gelfand and Dikii [9]. Any conservation law of a system of evolution equations takes the form $\mathrm{D}_t\mathrm{T}+\mathrm{Div}\mathrm{X}=0$ in which Div denotes spatial divergence. Note that if $\mathrm{T}(x,t,u^{(n)})$ is each differential function and u is a solution to the evolutionary system $u_t=K[u]$, then $\mathrm{D}_t=\partial_t\mathrm{T}+\mathrm{pr}\mathbf{v}_K(\mathrm{T})$, where $\partial_t=\partial/\partial t$ denotes the partial t-derivative. Hence, T is the density for a conservation law of the system iff its associated functional $\mathcal{T}$ satisfies $\partial\mathcal{T}/\partial t+\mathrm{pr}\mathbf{v}_K(\mathcal{T})=0$. For Hamiltonian form of our system, the following proposition is used.

Proposition 5.2 [20] Assume $\mathcal{D}$ is a Hamiltonian operator with poisson bracket (15). For each $\mathcal{H}=\int H dx$, there is an evolutionary vector field $\hat{\mathbf{v}}_{\mathcal{H}}$ called the Hamiltonain vector field associated with $\mathcal{H}$ satisfying $\mathrm{pr}\hat{\mathbf{v}}_{\mathcal{H}}(\mathcal{P})=\{\mathcal{P},\mathcal{H}\}$ for all functionals $\mathcal{P}$. Indeed, $\hat{\mathbf{v}}_{\mathcal{H}}$ has characteristic $\mathcal{D}\delta\mathcal{H}=\mathcal{D}\mathrm{E}(H)$.

Hence, the bracket relation immediately leads to the Noether relation between Hamiltonian symmetries and conservation laws. So, for the system of equations WBK, generalized symmetries which are Hamiltonian can be deduced from conserved densities by the Hamiltonian operators.

Theorem 5.3 The system of WBK equations admits Hamiltonian symmetries with the following characteristics for Hamiltonian operator $\mathcal{D}_0$,

$$
\begin{aligned}
&Q_1^u = u_t,\ Q_1^v = v_t, \quad Q_2^u = u_x,\ Q_2^v = v_x \quad Q_3^u = tu_x + 1,\ Q_3^v = tv_x,\\
&Q_4^u = (-u_x + 2v)u_x - uu_{xx} + \frac{1}{3}u_{xxx} + 2uv_x,\ Q_4^v = v_x u_x + u^2 v_x + uv_{xx} + \frac{1}{3}v_{xxx},\\
&Q_5^u = \frac{5}{3}u_x u_{xx} + \frac{1}{3}{u_x}^2 - u_{xx}u^2 - \frac{2}{3}vu_{xx} + \frac{1}{3}uu_{xx} + \frac{2}{3}uu_{xxx} - \frac{1}{6}u_{xxxx} + 2u^2 v_x\\
&\qquad - \frac{2}{3}v_x u_x + \frac{1}{3}v_{xxx},\\
&Q_5^v = 2u_x v^2 + v_{xx}u_x - \frac{1}{3}v_x u_x + \frac{2}{3}v_x u_{xx} + \frac{1}{3}vu_{xxx} + \frac{2}{3}v_x u^3 + \frac{2}{3}{v_x}^2 + v_{xx}u^2\\
&\qquad + \frac{2}{3}vv_{xx} - \frac{1}{3}uv_{xx} + \frac{2}{3}uv_{xxx} + \frac{1}{6}v_{xxxx}.
\end{aligned}
$$

Also, generalized symmetries corresponding to the Hamiltonian operators $\mathcal{D}_1$ can be deduced from the conservation laws. Thus, the Hamiltonian symmetries relative to $\mathcal{D}_1$ are

$$
\begin{aligned}
&Q_1^u = u_t,\ Q_1^v = v_t,\\
&Q_2^u = 2tu_t + xu_x + u,\ Q_2^v = 2tv_t + xv_x + 2v,\\
&Q_3^u = 4uv_x + 4vu_x + u(-u_{xx} + 2v_x) + u^2 u_x + 2v - u_x + u_{xxx},\\
&Q_3^v = 2uv_t + 4v_x u_x + 2uv_{x^2} + v_{xxx} + 4vv_x + u^2 v_x + 2v - u_x,\\
&Q_4^u = 2u^2 v_x + \frac{2}{3}v_{xxx} - 2u{u_x}^2 + 4vuu_x - u_{xx}u^2 + \frac{4}{3}uu_{xxx} + 2u^2 v_x + \frac{1}{3}u_x u^3\\
&\qquad + \frac{10}{3}u_x u_{xx} - 2v_x u_x - 2vu_{xx} - \frac{1}{3}u_{xxxx},\\
&Q_4^v = u^3 v_x + v_{xx}u^2 + \frac{4}{3}uv_{xxx} + 2v_{xx}u_x + \frac{4}{3}v_x u_{xx} + v_{xx}u^2 + \frac{1}{3}v_{xxxx} - 2{u_x}^2 v\\
&\qquad + 4u_x v^2 - 2vuu_{xx} + \frac{2}{3}vu_{xxx} + 6vuv_x + \frac{1}{3}v_x u^3.
\end{aligned}
$$

6. Future works

It is well-known that Hamiltonian systems of differential equations are one of the most famous and significant concepts in physics. These important systems appear in the various fields of physics such as motion of rigid bodies, celestial mechanics, quantization theory, fluid mechanics, plasma physics, etc. Due to the significance of Hamiltonian structures, by applying the linear behavior of the Euler operator, characteristics, prolongation and Fréchet derivative of vector fields, we can extend approximate symmetry methods on the Hamiltonian and bi-Hamiltonian systems of evolution equations to investigate the interplay between approximate symmetry groups, approximate conservation laws and approximate recursion operators. Also, we can extend μ-symmetry methods on Hamiltonian systems to make new conservation laws.

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