Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.1, (2026), 54-71
DOI: https://doi.org/10.71602/tfss.2026.1209092

Article Type: Original Research Article

Domination in Inverse Fuzzy Mixed Graphs with Application

Rahul Mondal, **Ganesh Ghorai***

(Special Issue on "Dynamical Systems in Fuzzy Environments".)

Abstract. In the first part of our work, the concepts of inverse fuzzy mixed dominating set (IFMDS) and inverse fuzzy mixed domination number (IFMDN) have been explored for inverse fuzzy mixed graphs (IFMGs). Based on these ideas, the inequality $\mathcal{R}^{ir} \leq \mathcal{R}^{\gamma} \leq \mathcal{R}^{i} \leq \mathcal{R}^{\beta_0} \leq \mathcal{R}^{\Gamma} \leq \mathcal{R}^{IR}$ has been established for an IFMG $\mathcal{R}$. In the second part of our work, we have explored the concept of (g,h)-IFMDS, which is defined as a fuzzy subset of the membership function of nodes. Most importantly, the relation (g,h)-IN$(\mathcal{R}) \leq (g,h)$-DN$(\mathcal{R}) \leq (g,h)$-UDN$(\mathcal{R}) \leq (g,h)$-UIN$(\mathcal{R})$ has been established for an IFMG $\mathcal{R}$. At the end, a real-life application of the concept of (g,h)-IFMDS is given.

AMS Subject Classification 2020: 05C10; 05C72
Keywords and Phrases: Inverse fuzzy mixed dominating function, Inverse fuzzy mixed domination number, Upper inverse fuzzy mixed domination number, (g,h)- inverse fuzzy mixed dominating set.

1 Introduction

The concept of fuzzy graphs (FGs) was introduced by Rosenfeld [1] in 1975. He extended classical graph theory by incorporating fuzzy set principles, allowing for the representation of uncertainty in graph structures. Rosenfeld's work laid the foundation for various applications of FGs in areas such as pattern recognition, network analysis, and decision-making.

In a FG, edges and vertices are assigned membership values between 0 and 1, representing the degree of their presence or strength of connection. This framework is useful for modeling real-world problems where relationships are not strictly binary, such as social networks, transportation systems, and decision-making processes. By blending graph theory with fuzzy logic, FGs provide a more flexible and realistic representation of complex systems. Recent advancements in fuzzy graph theory are discussed in ([2]-[9]).

In 2020, Borzooei et al. [10] introduced the concept of the inverse FG (IFG), which extends traditional FG theory. In this framework, the membership value of an edge is always at least as large as the minimum membership value of its connected vertices, with all membership values ranging between 0 and 1. The approach proposed by Borzooei et al. [10] has been widely explored in various applications, as referenced in [11, 12].

Mixed Fuzzy Graphs (Mixed FGs), introduced by Das et al. in 2020 [13], are an advancement of traditional Fuzzy Graphs that combine both directed and undirected fuzzy arcs within a single framework. In this model, the membership values (MVs) of vertices, edges, and arcs range between 0 and 1, allowing for the representation of both uncertain and directional relationships. Mixed FGs are useful in real-world applications

*Corresponding Author: Ganesh Ghorai, Email: math.ganesh@mail.vidyasagar.ac.in, ORCID: 0000-0002-3877-8059
Received: 7 June 2025; **Revised**: 15 August 2025; **Accepted**: 24 August 2025; **Available Online**: 9 September 2025; **Published Online**: 7 May 2026.

How to cite: Mondal R, Ghorai G. Domination in inverse fuzzy mixed graphs with application. *Transactions on Fuzzy Sets and Systems.* 2026; 5(1): 54-71. DOI: https://doi.org/10.71602/tfss.2026.1209092

such as transportation networks, social influence analysis, decision-making systems, where both types of connections coexist and exhibit uncertainty.

Poulik and Ghorai [12] later expanded the ideas of inverse FGs and fuzzy mixed graphs by introducing inverse fuzzy mixed graphs (IFMGs). This advanced framework incorporates both directed and undirected edges (UDEs), offering a versatile approach to representing uncertainty and imprecision. Due to its flexibility, IFMGs have found applications across multiple domains including social influence analysis, decision-making systems, and real-world problem solving. Several recent studies on IFMGs are available in ([14] -[16]).

The Wiener index, introduced by Harold Wiener [17] in 1947, is a topological index used in graph theory to measure the structural properties of a graph. It is defined as the sum of the shortest path distances between all pairs of vertices in a graph. Originally developed for molecular chemistry to study boiling points of alkanes, the Wiener index has since found applications in various fields, including network analysis, biology, and communication systems. It serves as an important metric for evaluating the compactness and connectivity of a graph. Numerous extensions of H. Wiener's concept has been explored in the literature, as documented in [3, 18]. Mondal and Ghorai [16] recently introduced the inverse fuzzy mixed Wiener index (IFMWI) and the inverse fuzzy mixed connectivity index. Their work establishes a relationship between these two concepts and highlights the usefulness of the IFMWI as a quality measure for evaluating real-world phenomena.

In graph theory, domination refers to a set of vertices in a graph such that every vertex is either in this set or adjacent to at least one vertex in the set. Formally, a dominating set D of a graph $G = (V, E)$ is a subset of V where every $v \in V \setminus D$ has at least one neighbor in D, that is, v is adjacent to a vertex in D. The domination number is the minimum size of such a set. Crisp graphs (CGs) are traditional graphs where edges are either present or absent, without fuzzy or probabilistic associations. Domination in CGs has applications in network security, social networks, facility location problems, and wireless sensor networks. Variants include total domination, independent domination, and connected domination, each modifying the conditions for vertex selection to suit specific applications.

Domination in FGs extends the concept of domination in CGs by incorporating degrees of membership for vertices and edges. In a crisp graph, edges are either present or absent, whereas in a fuzzy graph, each edge and vertex has an associated membership value in the interval $[0, 1]$, representing the strength of connectivity and presence. A dominating set in an FG is a subset of vertices such that every other vertex in the graph is either in this set or has a strong enough connection (above a certain threshold) to at least one vertex in the set. The fuzzy domination number, is the minimum weight (sum of membership values) of such a set. Fuzzy domination generalizes to crisp domination. This generalization is useful in applications where relationships are uncertain or varying in strength, such as social networks, transportation systems, and biological networks, etc. Various extensions exist, such as strong domination, total domination, and independent domination in fuzzy graphs, which further refine the concept for different real-world scenarios. Some related important works that offer new perspectives in the literature can be found in [6, 7, 19]. In our current work, the idea of domination in an IFMG is introduced. Here, we have discussed the concepts of IFMDS and strong inverse fuzzy mixed dominating set (SIFMDS). It has been shown that the concept of minimal SIFMDS coincides with the concept of an IFM-irredundant set under certain conditions. The relation between (g, h)-IFMDS and IFMDF has been established. Also under certain conditions, the relation between (g, h)-IFMDS and (g, h)-IFMIS has been established. Most importantly, the usefulness of (g, h)-IFMDS has been demonstrated through a real-life application. This article explores a novel construct IFMGs and investigates the theory of domination within this framework. We develop fundamental definitions, establish new theoretical results, and demonstrate how these ideas can be applied to solve practical problems characterized by incomplete, imprecise, or asymmetric information.

2 Preliminaries

Definition 2.1. [10, 11] *We define two functions, f and g, associated with a simple finite graph $\mathcal{B} = (M, N)$, where $f : M \to [0, 1]$ and $g : N \to [0, 1]$. The structure $\mathcal{B}^I = (f, g)$ is referred to as an inverse FG (IFG) if the MV of any edge is at least as large as the smallest MV of its end vertices. Here, f and g represent the membership functions (MFs) of vertices and edges, respectively.*

An IFG can also be represented as (M, N, f, g). The corresponding underlying CG, denoted as $\mathcal{B}^* = (M^*, N^*)$, is derived from $\mathcal{B}^I$ based on MVs. The vertex set M^* consists of all vertices that have a positive MV and are connected to at least one other vertex by an edge with a positive MV in $\mathcal{B}^I$. Similarly, the edge set N^* includes all edges with positive MVs in $\mathcal{B}^I$. For more, refer to [10, 11]. The ideas of partial IF (PIF) subgraph and IF subgraph have been extensively explored in [10, 11]. It has been established in [10, 11] that every IF subgraph is necessarily a PIF subgraph. However, the converse does not always hold in general.

Definition 2.2. [13] *A fuzzy mixed graph is a tuple $G = (V, E_1, E_2, \mu_1, \mu_2, \sigma, \delta)$ where:*
** V is a non-empty set of vertices,*
** $E_1 \subseteq V \times V$ is the set of undirected edges,*
** $E_2 \subseteq V \times V$ is the set of directed edges,*
** $\sigma : V \to [0, 1]$ is the vertex membership function,*
** $\mu_1 : E_1 \to [0, 1]$ is the fuzzy membership of undirected edges,*
** $\mu_2 : E_2 \to [0, 1]$ is the fuzzy membership of directed edges,*
** $\delta : E_2 \to [0, 1]$ is the directedness measure of directed edges,*
satisfying :
1. $\mu_1(x, y) \leq \sigma(x) \wedge \sigma(y)$, for all $(x, y) \in E_1$,
2. $\mu_2(\overrightarrow{xy}) \leq \sigma(x) \wedge \sigma(y)$, for all $\overrightarrow{xy} \in E_2$,
3. $\delta(\overrightarrow{xy}) \leq |\sigma(x) - \sigma(y)|$, for all $\overrightarrow{xy} \in E_2$.

Definition 2.3. [10, 11] *In an IFG $\mathcal{R}$, with the edge set $\mathcal{F}$, the degree of a node p is determined using the following formula, $d(p) = \sum_{y \in N(p)} g(xy)$. Where $N(p)$ represents the set of neighbors of p, and $g(py)$ denotes the MV of the edge that connects x and y. This degree measure accounts for the total strength of the connections that involve p in the IFG.*

Furthermore, the total degree of a node p, is calculated as the sum of its degree and its MV. The order of $\mathcal{R}$ refers to the total sum of the MVs of all nodes, while the size of $\mathcal{R}$ is defined as the total sum of the MVs of all arcs.

$\mathcal{R}$ is classified as a complete IFG (CIFG) if the MV of each edge is exactly equal to the minimum of the MVs of its two end vertices.

Also $\mathcal{R}$ is termed vertex-stable (VS) if all its vertices share the same MV. Similarly, it is called edge-stable (ES) if all its edges have identical MVs. When $\mathcal{R}$ satisfies both conditions, it is referred to as a stable IFG (SIFMG). Various examples of SIFGs are available in the literature [10, 11].

Definition 2.4. [20] *A fuzzy digraph (FDG) is an extension of a classical directed graph (digraph / DG) that incorporates fuzzy set theory to handle uncertainty in relationships between nodes. It is defined as a triple $D^f = (V, \mu_V, \mu_E)$, where:*

- *V is a non-empty finite set of vertices.*
- *$\mu_V : V \to [0, 1]$ is a vertex MF, assigning each vertex a degree of belonging.*
- *$\mu_E : V \times V \to [0, 1]$ is an edge MF, where $\mu_E(u, v)$ represents the strength or existence of a directed edge (DE) from vertex u to vertex v.*

A FDG generalizes classical digraphs by allowing edges and vertices to have varying degrees of presence rather than being strictly binary (0 or 1). It is widely used in applications like decision-making, communication networks, and social influence modeling. A fuzzy sub-DG is a smaller part of an FDG that retains its fundamental structure while possibly reducing the number of vertices and edges. It is formed by selecting a subset of vertices from the original FDG, ensuring that their MVs do not exceed those in the original graph. Similarly, the edges in the fuzzy sub-DG must also have MVs that are equal to or lower than their corresponding values in the original DG. This concept allows for analyzing specific portions of an FDG while maintaining its fuzzy relationships.

An IFG is a mathematical structure that consists of a set of vertices and two types of edges: directed and undirected. Each vertex is assigned an MV that lies between 0 and 1, representing its degree of presence on the graph. Similarly, each edge, whether directed or undirected, is also assigned an MV.

Definition 2.5. [12] *A structure* $\mathcal{R} = (P, D_1, D_2, \phi_1, \phi_2, \xi, \tau)$ *is defined as an IFMG, if*

- P *is a non-empty set of vertices.*
- D_1 *and* D_2 *are subsets of* $P \times P$*, where* D_1 *represents the set of UDEs and* D_2 *represents the set of DEs.*
- $\xi : P \to [0, 1]$ *assigns an MV to each vertex.*
- $\phi_1 : D_1 \to [0, 1]$ *and* $\phi_2, \tau : D_2 \to [0, 1]$ *assign MVs to UDEs and DEs, respectively.*

These functions satisfy the following conditions:
1. $\phi_1(a_1, b_1) \geq \xi(a_1) \wedge \xi(b_1)\ \forall (a_1, b_1) \in D_1$.
2. $\phi_2(a_1, b_1) \geq \xi(a_1) \wedge \xi(b_1)\ \forall (a_1, b_1) \in D_2$.
3. $\tau(a_1, b_1) \geq |\xi(a_1) - \xi(b_1)|\ \forall (a_1, b_1) \in D_2$.

Here, ϕ_1 and ϕ_2 represent the MV of UDE and DE, respectively, while τ denotes the measure of directedness for DEs.

In an IFMG, the MV of an UDE must be at least as large as the lowest MV of its two connecting vertices. A similar condition applies to DEs, ensuring that their MVs are not smaller than the minimum MV of their connected vertices. Additionally, for a DE, there is a measure of how strongly it is directed, which must be at least the absolute difference between the MVs of its two end vertices. Some recent studies on IFMG can be found in [12, 14, 15, 16]. In Figure 1, a detailed example of an IFMG is presented. There are three vertices A, B, C in the graph with associated weights of 0.25, 0.2, and 0.1 respectively. There are two UDEs (A,B) and (B, C) in the graph with associated weights of 0.6 and 0.5 respectively. Also there is a DE (A, C) in the graph with an associated weight of 0.65 and a measure of directedness of 0.68, respectively.

Definition 2.6. [15] *An IFMG is considered an ES when all edges, directed and undirected, have identical MVs. Additionally, the DEs in the graph must have a measure of directedness equal to this common value. This ensures uniformity in edge strength across the entire graph.*

On the other hand, an IFMG is classified as VS if, for every edge in the graph, the two connected vertices possess the same MV. This means that all directly linked nodes exhibit an equal degree of presence in the network.

Definition 2.7. [15] *In an IFMG, an edge is classified as effective if its MV is at least as large as the highest MV among its two end vertices. This means that the edge maintains a strong presence in relation to the nodes it connects.*

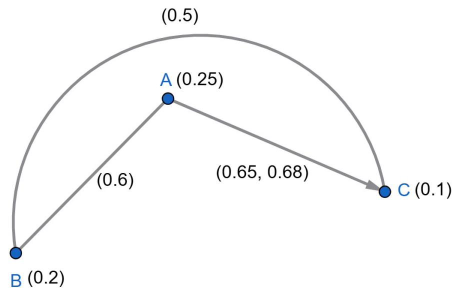

Figure 1: Example of an IFMG

Conversely, if the MV of an edge is lower than the maximum MV of its end vertices, it is considered a non-effective edge. In this case, the edge does not fully reflect the influence of its connected vertices.

As discussed in [14], if an IFMG is complete, then the MV of each UDE is determined by the minimum MV of its end vertices. Similarly, the MV of each DE follows the same rule, meaning that for any pair of vertices in the graph, their connecting edge whether directed or undirected, takes on the lowest membership value between the two vertices.

Let S be a fuzzy set consisting of a collection of elements, where each element has an MV assigned by a function. The smallest MV among all the elements in this fuzzy set is called its minimum MV.

Now, consider two fuzzy sets (FSs), each with its own MF that assigns values to their elements. The MF of the intersection of these two FSs is determined by comparing the MVs of each element that appears in both sets. For every shared element, the MV in the intersection is the smaller of the two values assigned by the original FSs. This method ensures that the intersection reflects the least degree of membership for common elements.

Definition 2.8. [16] *An IFMG is considered a partial IFM subgraph of another IFMG, if*

- *The MV of each vertex in the subgraph is less than or equal to its corresponding value in the original graph.*
- *The MV of each UDE in the subgraph does not exceed its value in the original graph.*
- *The MV of each DE in the subgraph is at most the same as in the original graph.*
- *The measure of directedness for each DE in the subgraph is not greater than its corresponding value in the original graph.*

These conditions ensure that the subgraph maintains a structure similar to the original graph while preserving a hierarchy of membership values and directedness.

Definition 2.9. [16] *An IFMG is considered an IFM subgraph of another IFMG, if*

- *The MV of each vertex in the subgraph is exactly the same as its corresponding value in the original graph.*
- *The MV of each UDE in the subgraph is equal to its corresponding value in the original graph.*

- *The MV of each DE in the subgraph matches its value in the original graph.*
- *The measure of directedness for each DE in the subgraph is identical to its corresponding value in the original graph.*

These conditions ensure that the subgraph retains the same MVs and structure as the original graph while being a subset of it. An IFMG is called an IFM spanning subgraph of another IFMG if the MVs of all vertices in the subgraph are exactly the same as in the original graph. This means that while the edges in the subgraph may be a subset of those in the original graph, the MVs of the vertices remain unchanged.

In a FG, the strength of a path is determined by the smallest weight among all the edges that make up the path. This means that the weakest link in the sequence of edges defines the overall strength of the path.

Definition 2.10. [16] *A connected IFMG is considered an IFM tree if it contains a spanning IFM subgraph that forms a tree.*

Additionally, for any edge that is not part of this spanning tree, there must exist an alternative path within the tree where the overall strength exceeds the weight of the missing edge (if UDE) or the directed weight of the missing edge (if DE). This ensures that the structure maintains tree-like properties while preserving the fuzzy and mixed characteristics of the graph.

Definition 2.11. [16] *Two IFMGs are considered isomorphic if there exists a one-to-one and onto mapping between their vertex sets that preserves the structure of the graph. This means that the MVs of corresponding vertices remain unchanged, the weights of corresponding edges (both directed and undirected) are identical, and the measure of directedness for DEs is maintained.*

3 Inverse Fuzzy Mixed Domination Number

Throughout $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ or simply $\mathcal{R}$ stands for an IFMG, which is associated with D_1, D_2, Φ_1, Φ_2, ξ, τ as defined above, unless otherwise stated. A path between two vertices from p to q in $\mathcal{R}$ is said to be a directed inverse fuzzy mixed path or directed if all the edges associated with this path are directed towards q. Otherwise, (if some of them are undirected edges) is considered an undirected inverse fuzzy mixed path or simply an undirected path. Also if it contains both directed (towards q) and undirected edges then it is said to be an inverse fuzzy mixed path. An inverse fuzzy mixed path is an undirected inverse fuzzy mixed path. Generally, a directed or undirected path between two vertices will be called a path between these two vertices. In $\mathcal{R}$ the IFM open open neighborhood of a vertex p is the collection of vertices adjacent to p and is denoted by IFM-N(p) and IFM-N(p) $\cup \{p\}$ is the closed neighborhood of p, denoted by IFM-N[p]. For a subset A of the vertex set, IFM-N(A) $= \bigcup_{p\in A} IFM\text{-}N(p)$ and IFM-N[A] $= \bigcup_{p\in A} IFM\text{-}N[p]$.

Definition 3.1. *In $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ a subset P_1 of P is said to be inverse fuzzy mixed dominating set (IFMDS) in $\mathcal{R}$ if for any $u \in P \setminus P_1$ there exists a vertex in P_1, which is adjacent to u.*

P_1 is said to be minimal if there is no proper proper subset of it which is IFMDS of $\mathcal{R}$. The inverse fuzzy mixed domination number (IFMDN) is the minimal cardinality among all the minimal IFMDSs of $\mathcal{R}$, will be denoted as $\mathcal{R}^{\gamma}$. The upper inverse fuzzy mixed domination number (UIFMDN) is the maximum cardinality taken over all minimal IFMDSs in $\mathcal{R}$, will be denoted as $\mathcal{R}^{\Gamma}$.

Definition 3.2. *A function $g : P \longrightarrow [0,1]$ is said to be a IFM dominating function or IFMDF of $\mathcal{R}$ if $g($ IFM-N[p] $) \geq 1$ for all vertex $p \in P$. Where for a subset S of the vertex set $f(S) = \sum_{x\in S} f(x)$.*

Definition 3.3. *Let K be a subset of the vertex set of $\mathcal{R}$. A vertex x is said to be an inverse fuzzy mixed private neighbor (IFMPN) of a vertex y in K with respect to K if the closed neighborhood of x intersects with K as the singleton set $\{y\}$. K is said to be an IFM-irredundant set or IFMIS if every vertex contained in it must have an IFMPN.*

Also K is said to be maximal IFMIS if no proper super set of it is IFMIS and minimal if no proper subset of it is IFMIS. The minimum cardinality of a maximal IFMIS in $\mathcal{R}$ is called the IFM-irredundance number of $\mathcal{R}$, denoted by $\mathcal{R}^{ir}$. The maximum cardinality of an IFMIS in $\mathcal{R}$ is called the IFM-upper irredundance number of $\mathcal{R}$, denoted by $\mathcal{R}^{IR}$.

Theorem 3.4. *A IFMDS K in $\mathcal{R}$ is a minimal IFMDS if and only if it is IFMDS and IFMIS.*

Proof. Let K be a minimal IFMDS in $\mathcal{R}$. Now if possible let K be not IFMIS. Then there exists a vertex a which does not have any IFMPN with respect to K. It is given that K is an IFMDS. Hence there exists a vertex b in $P \setminus K$ which is adjacent with a and also a vertex other then a in K. Now it can be clearly observed that $K \setminus \{a\}$ is also an IFMDS. This is a contradiction to the fact that K is a minimal IFMDS. Therefore K is IFMDS and IFMIS.

Conversely let K be IFMDS and IFMIS. If K is not minimal IFMDS then there exists a proper subset K_1 of K, which is also a IFMDS. Hence there exists a vertex b in $K \setminus K_1$ with its IFMPN b_1 with respect to K in $P \setminus K$. Also since K_1 is a IFMDS b_1 must be connected to a vertex b_2 in K_1 through an edge. This is a contradiction to the fact that b_1 is IFMPN of b with respect to K. This contradiction ensures that K is a minimal IFMDS. □

Definition 3.5. *Let K be a subset of the vertex set of $\mathcal{R}$. We say that K is an IFM-independent set if no two vertices within it are adjacent. Furthermore, K is referred to as a maximal IFM-independent set if there is no proper superset of K that is also independent.*

Theorem 3.6. *An IFM-independent set K in a graph G is maximal if and only if K is IFM-independent and is a minimal IFMDS.*

Proof. First let us suppose that K is maximal IFM-independent set. We can can clarify that K must be an IFMDS. Because if K is not an IFMDS then there exists a node a in $P \setminus K$ which is not adjacent to any vertex in K. Hence $K \cup \{a\}$ becomes an IFM-independent set. This is a contradiction to the fact that K is maximal IFM-independent set. Therefore K must be a IFMDS. Now we will show that K is minimal IFMDS. If K is not minimal IFMDS then there exists a proper sub set J of K, which is IFMDS. Hence there exis a vertex p in $K \setminus J$. Now since J is a IFMDS, p must be adjacent to some vertex in J. This will contradicts our assumption that K is maximal IFM-independent set. Therefore K is minimal IFMDS.

Conversely let K be IFM-independent and a minimal IFMDS. Now if K is not maximal IFM-independent set then there exists a proper super set of K which is also a IFM-independent set. So we can find a vertex outside of K which is not adjacent to any vertex in K, which is a contradiction to the fact that K is IFMDS. Hence K must be maximal IFM-independent set. □

The IFM independent domination number of $\mathcal{R}$ is denoted as $\mathcal{R}^i$ and is defined as the minimum cardinality of a maximal IFM-independent set in $\mathcal{R}$. Also the IFM-independence number of $\mathcal{R}$ is the maximum cardinality of an IFM-independent set in $\mathcal{R}$ and is denoted as $\mathcal{R}^{\beta_0}$.

Theorem 3.7. *A minimal IFMDS in $\mathcal{R}$ must be a maximal IFMIS.*

Proof. Let S be a minimal IFMDS in $\mathcal{R}$. Now it is clear to us that S must be a IFMIS. Because if S is not IFMIS then there exists a node e in S, which has no IFMPN with respect to S. Now since S is a minimal IFMDS in $\mathcal{R}$, e must be connected to a vertex h in $S \setminus P$. Otherwise $S \setminus \{e\}$ would be a IFMDS. Since e does not have any IFMPN with respect to S, h must be adjacent to another node e_1 in S. Hence we can say

$S \setminus \{e\}$ and $S \setminus \{e_1\}$ must be a IFMDS, whch is a contradiction to the fact that S be a minimal IFMDS in $\mathcal{R}$. Therefore S must be a IFMIS.

Now we will show that S is a maximal IFMIS. Now every member of S has a IFMPN with respect to S. Let us suppose that there exists a IFMIS B containing S, which contains a node p outside of S. Then it can be observed that p can not have an IFMPN with respect to B. If p has a IFMPN t with respect to B then t would be adjacent to another member of B other than p because S is a IFMDS and hence t must be adjacent to a vertex in S. This is a contradiction to the fact that t is a IFMPN of p with respect to B. Hence the result follows. □

Theorem 3.8. *For any IFMG $\mathcal{R}$, the following inequality holds:*

$$\mathcal{R}^{ir} \leq \mathcal{R}^{\gamma} \leq \mathcal{R}^{i} \leq \mathcal{R}^{\beta_0} \leq \mathcal{R}^{\Gamma} \leq \mathcal{R}^{IR}$$

Proof. A minimal IFMDS in $\mathcal{R}$ must be IFMDS and IFMIS. Let S be a minimal IFMDS with the cardinality $\mathcal{R}^{\gamma}$. Now by previous theorem we can say that S is a maximal IFMIS. Hence the cardinality of S must be grater or equal to $\mathcal{R}^{ir}$. Hence it can be easy observed that $\mathcal{R}^{ir} \leq \mathcal{R}^{\gamma}$, because $\mathcal{R}^{ir}$ is the minimum cardinality of a maximal IFMIS in $\mathcal{R}$.

Let T be a maximal IFM independent set with the cardinality $\mathcal{R}^{i}$. From the previous theorem we have seen that a maximal IFM-independent set must be a minimal IFMDS. Hence the minimum cardinality of a minimal IFMDS must be less or equal to $\mathcal{R}^{\gamma}$. Therefore we can write $\mathcal{R}^{\gamma} \leq \mathcal{R}^{i}$.

The maximum cardinality of an IFM-independent set in $\mathcal{R}$ is $\mathcal{R}^{\beta_0}$. Hence from definition it can be observed that $\mathcal{R}^{i} \leq \mathcal{R}^{\beta_0}$.

Let V be an maximal IFM-independent set with the cardinality $\mathcal{R}^{\beta_0}$. Then V is IFM-independent and is a minimal IFMDS. Therefore the minimal IFMDS with the maximum cardinality must be with the cardinality greater or equal to $\mathcal{R}^{\Gamma}$. Hence we can write $\mathcal{R}^{\beta_0} \leq \mathcal{R}^{\Gamma}$.

Let E be the minimal IFMDS with the maximum cardinality $\mathcal{R}^{\Gamma}$. Now E must be a IFMIS and hence the IFMIS with the maximum cardinality must be with the cardinality greater or equal to $\mathcal{R}^{\Gamma}$. Therefore we can write $\mathcal{R}^{\Gamma} \leq \mathcal{R}^{IR}$.

Therefore the inequality $\mathcal{R}^{ir} \leq \mathcal{R}^{\gamma} \leq \mathcal{R}^{i} \leq \mathcal{R}^{\beta_0} \leq \mathcal{R}^{\Gamma} \leq \mathcal{R}^{IR}$ holds for any IFMG $\mathcal{R}$. □

Definition 3.9. *In $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$, a subset P_1 of P is said to be a strong inverse fuzzy mixed dominating set (SIFMDS) in $\mathcal{R}$ if for any $u \in P \setminus P_1$ there exists a vertex in P_1, which is adjacent to u, with the condition that we can find at least one $u \in P \setminus P_1$ which is connected to a vertex in P_1 through a directed edge and also we can find at least one $u_1 \in P \setminus P_1$ which is connected to a vertex in P_1 through an undirected edge.*

P_1 is said to be minimal SIFMDS if there is no proper subset of it which is SIFMDS of $\mathcal{R}$. The strong inverse fuzzy mixed domination number (SIFMDN) is the minimal cardinality among all minimal SIFMDSs of $\mathcal{R}$ and will be denoted as $\mathcal{R}^{S\gamma}$. The upper strong inverse fuzzy mixed domination number (USIFMDN) is the maximum cardinality taken over all minimal SIFMDSs in $\mathcal{R}$ and will be denoted as $\mathcal{R}^{S\Gamma}$.

Theorem 3.10. *The complement of a minimal SIFMDS must have at least two distinct elements if it is also a minimal IFMDS.*

Proof. Let a minimal SIFMDS K in $\mathcal{R}$ be also a minimal SIFMDS, containing only element p in its complement. Suppose p is connected with a vertex p_1 in K through a directed edge and also connected with a vertex p_2 in K through an undirected edge. Hence $K \setminus p_1$ is also an IFMDS, which is a contradiction to the fact that K is minimal IFMDS. Hence the result follows. □

Lemma 3.11. *If K is a minimal SIFMDS in $\mathcal{R}$ then every member in K must be adjacent to a vertex in $P \setminus K$.*

Proof. Let K be a minimal SIFMDS in $\mathcal{R}$. If possible, let $\exists$ a node j in K which is not adjacent to any vertex in $P \setminus K$. Then it obvious that $K \setminus \{j\}$ is also an SIFMDS. This is a contradiction to the fact that K be a minimal SIFMDS in $\mathcal{R}$. Therefore, every member in K must be adjacent to a vertex in $P \setminus K$. □

Similarly with the same argument we can say that if K is a minimal IFMDS in $\mathcal{R}$ then every member in K must be adjacent to a vertex in $P \setminus K$.

Theorem 3.12. *A SIFMDS K in $\mathcal{R}$, where a vertex can only be associated with either a directed edge or an undirected edge is a minimal SIFMDS if and only if it is SIFMDS and IFMIS.*

Proof. Let K be a minimal SIFMDS in $\mathcal{R}$. Now if possible, let K be not IFMIS. Then $\exists$ a vertex a which does not have any IFMPN with respect to K. It is given that K is a SIFMDS. Hence there exists a vertex b in $P \setminus K$ which is adjacent with a and also a vertex other than a in K and the connecting edges are either both directed or both undirected. Now it can be clearly observed that $K \setminus \{a\}$ is also an SIFMDS. This is a contradiction to the fact that K is a minimal SIFMDS. Therefore, K is an SIFMDS and it is also an IFMIS.

Conversely, let K be an SIFMDS and an IFMIS. If K is not minimal SIFMDS then $\exists$ a proper subset K_1 of K, which is also a SIFMDS. Hence there exists a vertex b in $K \setminus K_1$ with its IFMPN b_1 with respect to K in $P \setminus K$. Also since K_1 is a SIFMDS b_1 must be connected to a vertex b_2 in K_1 through an edge. This is a contradiction to the fact that b_1 is IFMPN of b with respect to K. This contradiction ensures that K is a minimal SIFMDS. □

Theorem 3.13. *For any IFMG $\mathcal{R}$, with the condition ($\boldsymbol{E}$) as given below the following inequality holds:*

$$\mathcal{R}^{\beta_0} \leq \mathcal{R}^{S\Gamma} \leq \mathcal{R}^{IR}.$$

E: *A minimal IFMDS in $\mathcal{R}$ must be a minimal SIFMDS in $\mathcal{R}$ and vice versa.*

Proof. Let V be an maximal IFM-independent set with cardinality $\mathcal{R}^{\beta_0}$ then V is IFM-independent and is a minimal IFMDS and hence it is a minimal SIFMDS. Therefore, the minimal SIFMDS with the maximum cardinality must be with cardinality greater or equal to $\mathcal{R}^{\Gamma}$. Hence, we can write $\mathcal{R}^{\beta_0} \leq \mathcal{R}^{S\Gamma}$.

Let E be the minimal SIFMDS with the maximum cardinality $\mathcal{R}^{S\Gamma}$. So E must be a minimal IFMDS with the maximum cardinality $\mathcal{R}^{S\Gamma}$. Now E must be a IFMIS and hence the IFMIS with the maximum cardinality must be with the cardinalty greater or equal to $\mathcal{R}^{S\Gamma}$. Therefore we can write $\mathcal{R}^{S\Gamma} \leq \mathcal{R}^{IR}$.

Therefore, the inequality $\mathcal{R}^{\beta_0} \leq \mathcal{R}^{S\Gamma} \leq \mathcal{R}^{IR}$ holds. □

4 Domination in Inverse Fuzzy Mixed Graphs

Definition 4.1. *In an IFMG, the directed weight or d-weight (DW) of a directed edge (DE) is calculated by taking the smaller value between its MV and the measure of directedness. Also, by effective weight (EW) of a directed edge (DE) is determined by taking the maximum between its MV and the measure of directedness.*

It should be noted that for an UDE, the concepts of DW and EW will coincide with the concept of weight. If f_1 and f be two fuzzy sets on P, then f_1 is said to be a fuzzy subset or inverse fuzzy mixed subset (IFMS) of f if $f_1(x) \leq f(x)$ for all $x \in P$. If f_1 is a IFMS of f then we will write $f_1 \leq f$. Also by $f_1 < f$ we will mean $f_1 \leq f$ and $f_1(x) < f(x)$ for at least one vertex x.

The weight of f is denoted by $f(P)$ and is defined by $|f| = f(P) = \sum_{a \in P} f(a)$.

Definition 4.2. *Let $\mathcal{R}$ be an IFMG for a vertex s the μ_t- weight with respect to a fuzzy set μ on P is defined as $\sum_{EW(a,s) \geq t} \mu(a) + \mu(s)$ and is denoted as $W_t^{\mu}(s)$.*

Definition 4.3. *Let $\mathcal{R}$ be an IFMG and μ be a fuzzy subset of ξ. Then μ is said to be a (g,h)-inverse fuzzy mixed dominating set ((g,h)-IFMDS) of $\mathcal{R}$, where $0 \leq g < h \leq 1$ if $W_g^{\mu}(s) \geq h$ for all $s \in P$.*

$min_\mu \sum_{p\in P}\mu(p)$, the minimum value of $\sum_{p\in P}\mu(p)$, considering all (g,h)-IFMDSs of $\mathcal{R}$ is called (g,h)-inverse fuzzy domination number ((g,h)-DN) of $\mathcal{R}$. Throughout this discussion we will consider g and h such that $0\leq g\leq 1$, $0\leq h\leq 1$ unless otherwise stated. In the case when no (g,h)-IFMDS exists for $\mathcal{R}$, the (g,h)-DN is defined as $\sum_{p\in P}\xi(p)$.

Also $max_\mu \sum_{p\in P}\mu(p)$, the maximum value of $\sum_{p\in P}\mu(p)$, considering all minimal ((g,h)-IFMDSs of $\mathcal{R}$ is called (g,h)-inverse fuzzy upper domination number ((g,h)-UDN) of $\mathcal{R}$.

Definition 4.4. *Let $\mathcal{R}$ be an IFMG and μ be a (g,h)-IFMDS of $\mathcal{R}$. Then μ is said to be a minimal (g,h)-IFMDS of $\mathcal{R}$ if for any fuzzy subset ζ of μ with $\zeta(x)<\mu(x)$ for some $x\in P$, ζ is not a (g,h)-IFMDS of $\mathcal{R}$.*

Example 4.5. In the Figure 1, we can see an IFMG with three vertices A, B and C. Also we can see that the two edges in the IFMG, one directed edge (B, A) and one undirected edge (C, A). Now let us consider a fuzzy subset f of the fuzzy set representing the MVs of the vertices, where $f(A)=0.2$, $f(B)=0.1$ and $f(C)=0.1$. Also EW(B, A) = 0.5, EW(A, C) = 0.6, as (A, C) is undirected its EW equals to its weight.

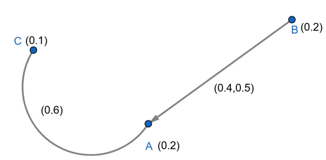

Figure 2: Example of an IFMDS

If we consider $g=0.2$ then we will find $W_g^f(C)=0.1+0.2=0.3$, $W_g^f(A)=0.1+0.2+0.1=0.4$ and $W_g^f(B)=0.1+0.2=0.3$. Hence, for any $h\leq 0.3$, f is a ((g,h)-IFMDS for the IFMG as shown in the Figure 2. It can be observed that for any $g\leq 0.5$, the IFMG will be a (g,h)-IFMDS.

Theorem 4.6. *A (g,h)-IFMDS f is a minimal (g,h)-IFMDS of $\mathcal{R}$ if and only if for any vertex x (with $f(x)>0$) we can find a vertex s satisfying the conditions $EW(x,s)\geq g$ and $\sum_{EW(a,s)\geq g} f(a)+f(s)=h$.*

Proof. First, let us suppose that the conditions hold, that is, for any vertex x with positive MV $\exists$ a vertex s with the conditions $EW(x,s)\geq g$ and $\sum_{EW(a,s)\geq g} f(a)+f(s)=h$. Now if f_1 be a IFMS of f with $f_1(x)<f(x)$ for some vertex x in $\mathcal{R}$, then $\exists$ a vertex s with the conditions $EW(x,s)\geq g$ and $\sum_{EW(a,s)\geq g} f(a)+f(s)=h$. Now since $f_1(x)<f(x)$ and $EW(x,s)\geq g$ will give us $\sum_{EW(a,s)\geq g} f_1(a)+f_1(s)<h$, which ensures that f_1 can not be a (g,h)-IFMDS of $\mathcal{R}$. Hence f is a minimal (g,h)-IFMDS of $\mathcal{R}$.

Conversely, let f be a minimal (g,h)-IFMDS of $\mathcal{R}$. If possible let for every vertex s satisfying the condition $EW(x,s)\geq g$, the inequality $\sum_{EW(a,s)\geq g} f(a)+f(s)>h$ holds, where $x\in P$. Let us consider $min_s\{\sum_{EW(a,s)\geq g} f(a)+f(s)-h\}=\beta^*$, where the minimum is evaluated considering all such s satisfying $EW(x,s)\geq g$. Now, let us denote $min\{(\sum_{EW(a,x)\geq g} f(a)+f(x)-h),\beta^*\}$ by β.

Now we will consider the following function as defined below:

$$f_1(m)=max\{0,f(m)-\tfrac{\beta}{4}\},\text{ if } m=x$$
$$f_1(m)=f(m),\text{ otherwise.}$$

It can be easily clarified that f_1 is a (g,h)-IFMDS of $\mathcal{R}$ and clearly from the construction we can say $f_1(x)<f(x)$. This is a contradiction to the fact that f is a minimal (g,h)-IFMDS of $\mathcal{R}$. This contradiction ensures that for any vertex x $\exists$ a vertex s with the conditions $EW(x,s)\geq g$ and $\sum_{EW(a,s)\geq g} f(a)+f(s)=h$ and hence the result follows. □

Theorem 4.7. *In $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ a subset P_1 of P be IFMDS if and only if $\exists$ a fuzzy subset f(as defined below) of ξ, which is $(g, 1)$-IFMDS of $\mathcal{R}$ for any $g > 0$ as defined above, where $f : P \longrightarrow [0, 1]$ is defined below:*

$$f(a) = 1, \text{ if } a \in P_1; \\ = 0, \text{ otherwise}$$

Proof. First let us suppose that P_1 be IFMDS of $\mathcal{R}$. Hence for any $u \in P \setminus P_1$ $\exists$ a vertex in P_1, which is adjacent to u. Therefore, if $s \in P_1$ then, $\sum_{EW(a,s)\geq g} f(a) + f(s) \geq 1$, because $f(s) = 1$ for any $s \in P_1$ and if s does not belong to P_1 then the sum $\sum_{EW(a,s)\geq g} f(a) + f(s)$ must be greater or equal to 1, because in this case $s \in P \setminus P_1$ and hence $\exists$ a $x \in P_1$ such that $EW(s, x) = 1$ as $f(x) = 1$. Hence, it is obvious that f is $(g, 1)$-IFMDS of $\mathcal{R}$ for any $g > 0$.

Conversely let us suppose that f be $(g, 1)$-IFMDS of $\mathcal{R}$ for any $g > 0$. Hence if s does not belong to P_1 then the sum $\sum_{EW(a,s)\geq g} f(a) + f(s)$ must be greater or equal to 1. It is only possible when $\exists$ a $x \in P_1$ with $EW(x, s) \geq g$. Hence the result follows. □

Theorem 4.8. *If a function f is a $(g, 1)$-IFMDS of $\mathcal{R}$ for any $g > 0$ then it is an IFMDF of $\mathcal{R}$.*

Proof. Let f be a $(g, 1)$-IFMDS of $\mathcal{R}$ for any $g > 0$. Then the sum $\sum_{EW(a,s)\geq g} f(a) + f(s) \geq 1$ for all $s \in P$. Now it is obvious that $f(\text{IFM-}N[s]) \geq \sum_{EW(a,s)\geq g} f(a) + f(s)$ for all $s \in P$ and for any g. Therefore, for any vertex s, $f(\text{IFM-}N[s]) \geq 1$. Hence, the result follows. □

It can be easily clarified that the converse part of the above result is true for the vertices which takes the unit value with respect to f.

Theorem 4.9. *In $\mathcal{R}$, for $g^{'} \leq g$ and $g \leq s$, we must have $(g^{'}, s)$-DN $\leq (g, s)$-DN.*

Proof. If a function f is a (g, s)-IFMDS of $\mathcal{R}$. Then it is obvious that f is a $(g^{'}, s)$-IFMDS of $\mathcal{R}$. But the converse may not holds, that is, if f is a $(g^{'}, s)$-IFMDS of $\mathcal{R}$ then it may not be a (g, s)-IFMDS of $\mathcal{R}$.

Because for a particular vertex b, the set $\{x : EW(b, x) \geq g\} \subset \{x : EW(b, x) \geq g^{'}\}$. So $\sum_{EW(b,x)\geq g^{'}} f(x) + f(b) \geq \sum_{EW(b,x)\geq g} f(x) + f(b)$. Therefore, $\sum_{EW(b,x)\geq g^{'}} f(x) + f(b) \geq s$ may not imply $\sum_{EW(b,x)\geq g} f(x) + f(b) \geq s$. Hence we can say $(g^{'}, s)$-DN $\leq (g, s)$-DN and hence the result follows. □

Theorem 4.10. *In $\mathcal{R}$, for $s^{'} \leq s$ and $g \leq s^{'} \leq s$, we must have $(g, s^{'})$-DN $\leq (g, s)$-DN.*

Proof. If f is a (g, s)-IFMDS of $\mathcal{R}$, then it is evident that f is also a (g, s')-IFMDS of $\mathcal{R}$. However, the converse of this statement may not hold in general.

Because for a particular vertex b the inequality $\sum_{EW(b,x)\geq g} f(x) + f(b) \geq s^{'}$ may not imply $\sum_{EW(b,x)\geq g} f(x) + f(b) \geq s$. However, if $\sum_{EW(b,x)\geq g} f(x) + f(b) \geq s$ holds then it will imply $\sum_{EW(b,x)\geq g} f(x) + f(b) \geq s^{'}$. Hence, a (g, s)-IFMDS must be a $(g, s^{'})$-IFMDS. Therefore, $(g, s^{'})$-DN $\leq (g, s)$-DN. □

Theorem 4.11. *Let $\mathcal{R}^*$ be a IFM subgraph of an IFMG $\mathcal{R}$, Then (g, s)-DN$(\mathcal{R}) \leq (g, s)$-DN$(\mathcal{R}^*)$.*

Proof. Since $\mathcal{R}^*$ is an IFM subgraph of an IFMG $\mathcal{R}$ for any edge (a, b) in $\mathcal{R}^*$, the EW will be preserved in $\mathcal{R}$. Also, for any vertex in $\mathcal{R}^*$, the MV will be preserved in $\mathcal{R}$. Hence any (g, s)-IFMDS of $\mathcal{R}^*$ must be a (g, s)-IFMDS of $\mathcal{R}$, but the converse may not hold. Hence from the definition of (g, s)-DN result directly holds. □

Definition 4.12. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG and consider a f-subset ξ_1 of ξ that maps from P to $[0, 1)$. We will define the following two sets :*

$PO_{\xi_1} = \{t \in P : \xi_1(t) > 0\}$, the IFM positive set related to ξ_1,

and $BO^t_{(\xi_1,s)} = \{a \in P : W^{\xi_1}_t(a) = s\}$, called the IFM s-boundary set depending on t.

Definition 4.13. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG and U and V be two subsets of P. Then U is said to be t-dominates V if for any $v \in V$, either $v \in U$ or $\exists$ a $u \in U$ such that $EW(v, u) \geq t$ and is denoted by $U \longrightarrow^t V$.*

Theorem 4.14. *A (g, h)-IFMDS f is a minimal (g, h)-IFMDS of $\mathcal{R}$ if and only if $BO^g_{(f,h)} \longrightarrow^g PO_f$, provided $BO^g_{(f,h)} \cap PO_f = \emptyset$.*

Proof. First, suppose that f is a minimal (g, h)-IFMDS of $\mathcal{R}$. Hence, by previous theorem, we have for any vertex x (with $f(x) > 0$) $\exists$ a vertex s satisfying the conditions $EW(x, s) \geq g$ and $\sum_{EW(a,s)\geq g} f(a)+f(s) = h$. Now, let us suppose that $x \in PO_f$. Therefore, $f(x) > 0$ and hence we can find a vertex s satisfying the conditions $EW(x, s) \geq g$ and $\sum_{EW(a,s)\geq g} f(a) + f(s) = h$. So $s \in BO^g_{(f,h)}$ and clearly we can see that $EW(x, s) \geq g$. Therefore, $BO^g_{(f,h)} \longrightarrow^g PO_f$.

Conversely, let $BO^g_{(f,h)} \longrightarrow^g PO_f$ holds and $f(x) > 0$ for some vertex x. So $x \in PO_f$ and hence $\exists$ a vertex s in $BO^g_{(f,h)}$ satisfying $EW(x, s) \geq g$, because from the given condition we can say $x \notin BO^g_{(f,h)}$. Therefore, in this case for a vertex x (with $f(x) > 0$) we can find a vertex s satisfying the conditions $EW(x, s) \geq g$ and $\sum_{EW(a,s)\geq g} f(a) + f(s) = h$. So using the previous Theorem, we can say f is a minimal (g, h)-IFMDS of $\mathcal{R}$. □

Definition 4.15. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG and μ be a fuzzy subset of ξ. Then μ is said to be a (g, h)-IFM irredundant set ((g,h)-IFMIS) of $\mathcal{R}$ if for any $x \in P$ with $\mu(x) > 0$, $\exists$ a $y \in P$ such that $EW(x, y) \geq g$ and $W^\mu_g(y) = h$.*

Example 4.16. In the Figure 2, we can see an IFMG with three vertices A, B and C. Also we can see that there are two edges in the IFMG, one directed edge (A, C) and one undirected edge (A, B). Now let us consider a fuzzy subset f of the fuzzy set representing the MVs of the vertices, where $f(A) = 0.1$, $f(B) = 0$ and $f(C) = 0.1$. Also EW(A,C) = 0.5, EW(A, B) = 0.6, as (A, B) is undirected its EW equals to its weight. If we consider $g = 0.2$ then we will find $W^f_g(C) = 0.1 + 0.1 = 0.2$, $W^f_g(A) = 0.1 + 0 + 0.1 = 0.2$ and

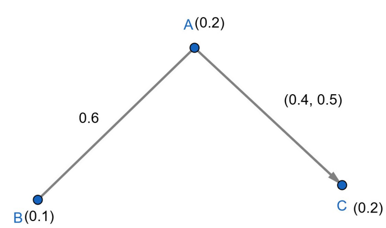

Figure 3: Example of an IFMIS

$W^f_g(B) = 0.1 + 0 = 0.1$.
It can be easily verified that for any $x \in P$ with $f(x) > 0$, $\exists$ a $y \in P$ such that $EW(x, y) \geq g$ and $W^f_g(y) = h$, where P is the set of all vertices and $h = 0.2$. It can be observed that for any $g \leq 0.5$, the IFMG will be a $((g, h)$-IFMIS.

Theorem 4.17. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG. A (g, h)-IFMDS μ of $\mathcal{R}$ is a minimal (g, h)-IFMDS of $\mathcal{R}$ if and only if μ is (g, h)-IFMDS and (g,h)-IFMIS of $\mathcal{R}$.*

Proof. First, let us suppose that μ be a (g, h)-IFMDS and (g,h)-IFMIS of $\mathcal{R}$. If f is a f-subset of μ with $f(l) < \mu(l)$ for some $l \in P$, then it is obvious that $\mu(l) > 0$. Now since μ is (g,h)-IFMIS $\exists$ a $y \in P$ with $EW(l, y) \geq g$ such that $W^\mu_g(y) = h$. Now, since $f(l) < \mu(l)$ and $EW(l, y) \geq g$, we can write $W^f_g(y) < h$.

Therefore, it is clear to us that f can not be a (g,h)-IFMDS of $\mathcal{R}$. This is a contradiction to our assumption. Hence μ is a minimal (g,h)-IFMDS of $\mathcal{R}$.

Conversely let μ be a minimal (g,h)-IFMDS of $\mathcal{R}$. Hence from the previous theorem we can say that for any $x \in P$ with $\mu(x) > 0$, $\exists$ a $y \in P$ such that $EW(x,y) \geq g$ and $W_g^{\mu}(y) = h$. So μ is (g,h)-IFMDS and (g,h)-IFMIS of $\mathcal{R}$. □

Definition 4.18. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG and f be a fuzzy subset of ξ. Let x be a vertex in $\mathcal{R}$ with positive MV. Then a vertex s is said to be a (g,h)-private neighbor ((g,h)-PN) of x with respect to f if $EW(x,s) \geq g$ and $W_g^f(s) = h$.*

Theorem 4.19. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG and f be a fuzzy subset of ξ. Then f is (g,h)-IFMIS of $\mathcal{R}$ if and only if for any vertex with positive MV there is a (g,h)-PN of it.*

Proof. First suppose that f is (g,h)-IFMIS of $\mathcal{R}$. Let x be a vertex in $\mathcal{R}$ with positive MV. Hence from definition we can say $\exists$ a s such that $EW(x,s) \geq g$ and $W_g^f(s) = h$. Therefore s is a (g,h)-PN of x.

Conversely let for any vertex with positive MV there is a (g,h)-PN of it. So if x is a vertex in $\mathcal{R}$ with positive MV, then $\exists$ a (g,h)-PN s of it. Therefore, we can say that $EW(x,s) \geq g$ and $W_g^f(s) = h$. Hence, the result follows immediately. □

Theorem 4.20. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG. A minimal (g,h)-IFMDS μ of $\mathcal{R}$ is a maximal (g,h)-IFMIS of $\mathcal{R}$.*

Proof. First let us suppose that μ be a minimal (g,h)-IFMDS of $\mathcal{R}$. So using the previous theorem we can say that μ is a (g,h)-IFMIS. We have to show that μ be a maximal (g,h)-IFMIS. If possible let μ is not maximal (g,h)-IFMIS. Hence $\exists$ f such that $\mu \leq f$ with $\mu(x) < f(x)$ for atleast one vertex x, where f is a (g,h)-IFMIS. Now clearly $f(x) > 0$ and hence for $x \in P$ we can find a, $y \in P$ with $EW(x,y) \geq g$ such that $W_g^f(y) = h$. Now this will imply that $W_g^{\mu}(y) < h$, which is a contradiction to the fact that μ is a (g,h)-IFMDS. Hence the result follows. □

$min_{\mu} \sum_{p \in P} \mu(p)$, the minimum value of $\sum_{p \in P} \mu(p)$ considering all maximal (g,h)-IFMISs, is called (g,h)-inverse fuzzy mixed irredundance number ((g,h)-IN) of $\mathcal{R}$. It has been already mentioned that throughout this discussion we will consider g and h such that $0 \leq g \leq 1$, $0 \leq h \leq 1$ unless otherwise stated.

Also $max_{\mu} \sum_{p \in P} \mu(p)$, the maximum value of $\sum_{p \in P} \mu(p)$ considering all ((g,h)-IFMISs, is called (g,h)-inverse fuzzy upper irredundance number ((g,h)-UIN) of $\mathcal{R}$.

If there does not exist any (g,h)-inverse fuzzy mixed irredundance set of $\mathcal{R}$ then we write (g,h)-IN($\mathcal{R}$)= (g,h)-UIN($\mathcal{R}$)= $\xi(P)$.

Theorem 4.21. *For any IFMG $\mathcal{R}$ we will have the following relation: $(g,h) - IN(\mathcal{R}) \leq (g,h) - DN(\mathcal{R}) \leq (g,h) - UDN(\mathcal{R}) \leq (g,h) - UIN(\mathcal{R})$.*

Proof. It has been discussed previously that for an IFMF $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$, a minimal (g,h)-IFMDS μ of $\mathcal{R}$ is a maximal (g,h)-IFMIS of $\mathcal{R}$. Hence we can write (g,h)-IN($\mathcal{R}$) $\leq$ (g,h)-DN($\mathcal{R}$).

It follows directly from the definitions of (g,h)-UDN($\mathcal{R}$) and (g,h)-DN($\mathcal{R}$) that (g,h)-DN($\mathcal{R}$) $\leq$ (g,h)-UDN($\mathcal{R}$).

It has been clarified before that a (g,h)-IFMDS μ of $\mathcal{R}$ is a minimal (g,h)-IFMDS of $\mathcal{R}$ if and only if μ is (g,h)-IFMDS and (g,h)-IFMIS of $\mathcal{R}$. Hence from the definitions of (g,h)-UDN($\mathcal{R}$) and (g,h)-UIN($\mathcal{R}$) we can write that (g,h)-UDN($\mathcal{R}$) $\leq$ (g,h)-UIN($\mathcal{R}$). Hence the result follows. □

Definition 4.22. *Let $\mathcal{R}$ be an IFMG and μ be a fuzzy subset of ξ. Then μ is said to be a (g,h)-inverse fuzzy mixed independent set ((g,h)-IFM independent set, where $g, h \in [0,1]$) of $\mathcal{R}$, if for every vertex s satisfying $\mu(a) \geq g > 0$, we can write $W_g^{\mu}(s) = h$.*

Let $\mathcal{R}$ be an IFMG and μ be a (g,h)-IFM independent set of $\mathcal{R}$. Then μ is said to be a maximal (g,h)-IFM independent set of $\mathcal{R}$ if for any fuzzy super-set ζ of μ satisfying $\mu(x) < \zeta(x)$ for some $x \in P$, will imply ζ is not a (g,h)-IFM independent set of $\mathcal{R}$.

It is discussed previously that in $\mathcal{R}$ be an IFMG for a vertex s the μ_t- weight with respect to a fuzzy set μ on P is defined as $\sum_{EW(a,s)\geq t} \mu(a) + \mu(s)$ and is denoted as $W_t^{\mu}(s)$.

A (g,h)-IFM independent set μ is said to be a maximal (g,h)-IFM independent set of $\mathcal{R}$ if we can not find any (g,h)-IFM independent set μ_1 of $\mathcal{R}$ such that $\mu \leq \mu_1$ with $\mu(f) \leq \mu_1(f)$ for some vertex f.

Theorem 4.23. *Let $\mathcal{R} = (P, D_1, D_2, \Phi_1, \Phi_2, \xi, \tau)$ be an IFMG in which $W_g^{\mu}(a) = W_g^{\mu}(b)$ every pair of vertex a,b satisfying $EW(a,b) \geq g$. Then a (g,h)-IFM independent set μ of $\mathcal{R}$ is a maximal (g,h)-IFM independent set of $\mathcal{R}$ if and only if μ is (g,h)-IFMDS and (g,h)-IFM independent set of $\mathcal{R}$, where $0 < g < 1$ and $0 < h < 1$.*

Proof. First suppose that μ is (g,h)-IFMDS and (g,h)-IFM independent set of $\mathcal{R}$. We will prove that μ is a maximal (g,h)-IFM independent set of $\mathcal{R}$. If possible let $\mu \leq \mu_1$ with $\mu(f) \leq \mu_1(f)$ for some vertex f, where μ_1 is a fuzzy subset of ξ. Therefore we can say that $\mu_1(f) > 0$. It is given that μ is (g,h)-IFMDS and hence $W_g^{\mu}(f) \geq h$. Now since $\mu \leq \mu_1$ with $\mu(f) \leq \mu_1(f)$ for some vertex f, we can write $W_g^{\mu_1}(f) > h$. This is a contradiction to the fact that μ is (g,h)-IFM independent set. Hence μ is a maximal (g,h)-IFM independent set of $\mathcal{R}$.

Conversely let μ is a maximal (g,h)-IFM independent set of $\mathcal{R}$. So μ must be is a (g,h)-IFM independent set of $\mathcal{R}$. We will so that μ is a (g,h)-IFMDS of $\mathcal{R}$. If μ is not a (g,h)-IFMDS of $\mathcal{R}$ then $\exists$ a $s \in P$ such that $W_g^{\mu}(s) < h \longrightarrow (i)$. Let us consider $\eta = h - W_g^{\mu}(s)$ and hence $0 < \eta < 1$. It can be easily clarified that (i) is only possible when $\mu(s) = 0$.

Now let us define a function $\mu^* : P \longrightarrow [0,1]$ such that $\mu^*(x) = \eta$ if $x = s$ and $\mu^*(x) = \mu(x)$, otherwise. It can be observed that $\mu < \mu^*$ and $W_g^{\mu^*}(s) = \sum_{EW(a,s)\geq g} \mu^*(a) + \mu^*(s) = \sum_{EW(a,s)\geq g} \mu(a) + \eta = \sum_{EW(a,s)\geq g} \mu(a) + h - \sum_{EW(a,s)\geq g} \mu(a) = h$, since $\mu(s) = 0$ it is obvious that $W_g^{\mu}(s) = \sum_{EW(a,s)\geq g} \mu(a)$. Also for $p \neq s$ we can write $W_g^{\mu^*}(p) = \sum_{EW(a,p)\geq g} \mu^*(a) + \mu^*(p) = \sum_{EW(a,p)\geq g} \mu(a) + \mu(p) = W_g^{\mu}(p)$ if $EW(s,p) < g$. In the case when $EW(s,p) \geq g$ using the given condition we can also write $W_g^{\mu^*}(p) = W_g^{\mu}(p)$. It follows that μ^* is also a (g,h)-IFM independent set of $\mathcal{R}$. Thich is a contradiction to the fact that μ is a maximal (g,h)-IFM independent set of $\mathcal{R}$. Hence the result follows. □

5 Application of (g,h)-IFMDS in ATM network

The ATM (Automated Teller Machine) network in India is an extensive and robust system that facilitates cash withdrawals, balance inquiries, and other banking services. Managed by banks and third-party service providers, ATMs are strategically placed in both urban and rural areas to enhance financial inclusion.

The National Financial Switch (NFS), managed by the National Payments Corporation of India (NPCI), serves as the backbone of the interbank ATM network, connecting various banks and enabling seamless transactions. NFS supports features like cash withdrawals, fund transfers, and mini statements across different bank ATMs. Various private and government sector banks maintain an extensive ATM network in India.

The concept of (g,h)-IFMDS can be applied to determine the importance index of ATM networks in a specific region of India. Let us represent all block towns and district towns in a specific region as nodes, where the MV of each node is given by $(1-\frac{1}{m})$, where m being the population of the respective town. We will represent the connecting roads (Highways with the shortest distance) between these towns as edges between the corresponding vertices. If the localities along these roads have a population of fewer than ten thousand people, we will represent them with an undirected edge. Otherwise, we will use a directed edge toward towns with higher populations. The MV of edges is given by $\frac{1}{1000}(1-\frac{1}{p}) + k$, where p being the number of ATMs along the respective road and k being the minimum MV of its end nodes. Additionally, the measure of directedness for a directed edge is given by $\frac{q}{1000} + r$, where q represents the percentage of people dependent on ATMs in the localities along the road and in the end block towns (nodes) and r being the minimum MV of its end nodes.

We can consider the graphical representation (in the form of IFMGs) of a specific region in the above format as discussed above, noting that the function representing the MVs of nodes is a fuzzy subset of itself.

Let P and Q be two specific regions of India containing a large number of block towns. Also suppose that the membership function (MF) representing the MVs of nodes of the region P of India is a (g, h)-IFMDS. Based on the above discussion, we can conclude that P will have a higher ATM demand index compared to another region Q of the same area, whose MF representing the MVs of nodes in Q is not an (x, y)-IFMDS for $x \geq g$ and $y \geq h$.

A zone can be represented using the IFMG format as discussed above, where the membership function (MF) of the nodes constitutes a (g, h)-IFMDS. Here, g correlates with the number of ATMs in the region and the percentage of people dependent on ATMs in localities outside block towns. On the other hand, h is associated with the main population mass. Therefore, higher values of g and h indicate greater overall demand and significance of the region concerning the ATM network.

We have illustrated two different regions, A and B, of the same area in the following Figures 3 and 4, represented as IFMGs. The MVs of nodes and edges are presented in the format discussed earlier. Additionally, the MDs of directed edges are depicted in the format discussed earlier and are labelled as MD.

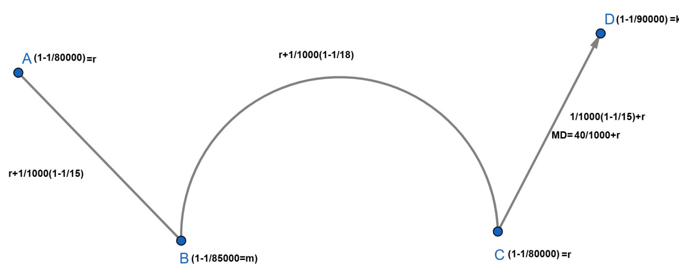

Figure 4: Region A

In region-A, we have depicted an area with four towns (nodes) A, B, C, and D, having populations of $80,000, 85,000, 80,000$, and $90,000$, respectively. Also in the road AB (edge) there is 15 ATMs , in the road BC (edge) there is 18 ATMs and in the road CD (directed edge) there is 15 ATMs and 40 percent people people dependent on ATMs in the localities along the road (edge CD) and in the end block towns (nodes).

In region-B, we have depicted an area with four towns (nodes) A, B, C, and D, having populations of $12,000, 22,000, 18,000$, and $18,000$, respectively. Also in the road AB (edge) there is 5 ATMs, in the road CB (directed edge) there is 8 ATMs, where 25 percent people dependent on ATMs in the localities along the road (edge CB) and in the end block towns (nodes) and also in the road CD (edge) there is 4 ATMs. It can be easily clarified that the region-A will be a (x, y)-IFMDS with higher values of x and y both compared to region-B. Hence, we can conclude that the overall demand and significance of region A is higher than that of region B concerning the ATM network. This application is initially associated with a small area on a small scale but can be implemented on a larger scale across wider regions. Similarly many applications can be explored to address various real-life problems using the concept of (x, y)-IFMDS.

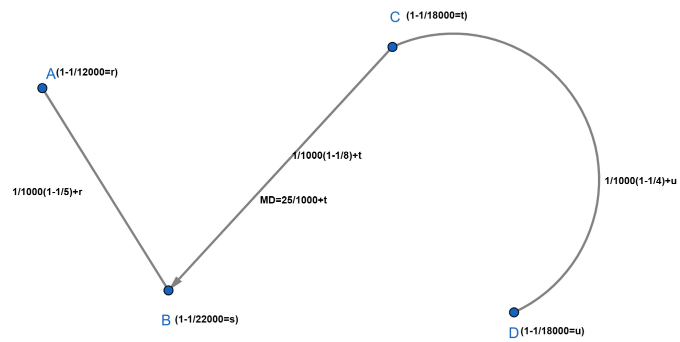

Figure 5: Region B

6 Conclusion and future direction

In this study, we introduced and rigorously explored the concept of domination in inverse fuzzy mixed graphs, a novel and significant extension of classical graph theoretic models. By integrating the characteristics of fuzziness, mixed edge orientation, and inverse graph structures, we established a more comprehensive framework capable of modeling real-world systems that are inherently uncertain, asymmetric, and complex. The concepts of IFMDS and IFMDN have been explored for IFMGs. The inequality $\mathcal{R}^{ir} \leq \mathcal{R}^{\gamma} \leq \mathcal{R}^{i} \leq \mathcal{R}^{\beta_0} \leq \mathcal{R}^{\Gamma} \leq \mathcal{R}^{IR}$ has been established for an IFMG $\mathcal{R}$. Also the concept of (g, h)-IFMDS have explored. The relation (g, h)-IN$(\mathcal{R}) \leq (g, h)$-DN$(\mathcal{R}) \leq (g, h)$-UDN$(\mathcal{R}) \leq (g, h)$-UIN$(\mathcal{R})$ has been established for an IFMG $\mathcal{R}$. Finally a real life application of the concept of (g, h)-IFMDS is presented. This work lays a solid foundation for further research in several promising directions. Future work may include the investigation of weighted domination, probabilistic fuzzy domination, dynamic inverse fuzzy mixed graphs, and algorithmic approaches for efficiently computing domination parameters in large-scale systems. Moreover, integrating machine learning techniques for learning fuzzy relationships directly from data could further enhance the practical utility of the proposed model. We hope that further research may explore new applications in science and technology.

Acknowledgements: "The authors are highly grateful to the anonymous referees for a careful checking of the details and for helpful comments that improved the overall presentation of this paper."

Conflict of Interest: "The authors declare no conflict of interest."

References

[1] Rosenfeld A. *Fuzzy graphs. In: Zadeh LA, Fu K-S, Tanaka K, Shimura M. (eds) Fuzzy sets and their applications*. New York: Academic Press; 1975. p.77-95. DOI: https://doi.org/10.1016/B978-0-12-775260-0.50008-6

[2] Akram M, Sattar A, Saeid AB. Competition graphs with complex intuitionistic fuzzy information. *Granular Computing*. 2022; 7: 25-47. DOI: https://doi.org/10.1007/s41066-020-00250-2

[3] Akram M, Siddique S, Alcantud JCR. Connectivity indices of m-polar fuzzy network model, with an application to a product manufacturing problem. *Artificial Intelligence Review*. 2023; 56: 7795-7838. DOI: https://doi.org/10.1007/s10462-022-10360-9

[4] Bhutani KR, Rosenfeld A. Geodesics in fuzzy graphs. *Electronic Notes in Discrete Mathematics*. 2003; 15: 49-52. DOI: https://doi.org/10.1016/S1571-0653(04)00526-8

[5] Brigham RC, McMorris FR, Vitray RP. Tolerance competition graphs. *Linear Algebra and its Application*. 1995; 217: 41-52. DOI: https://doi.org/10.1016/0024-3795(94)00059-M

[6] Cockayne EJ, Mynhardt CM. The sequence of upper and lower domination, independence and irredundance numbers of a graph. *Discrete Mathematics*. 1993; 122(1-3): 89-102. DOI: https://doi.org/10.1016/0012-365X(93)90288-5

[7] Haynes TW, Hedetniemi ST, Slater PJ. *Fundamentals of Domination in Graphs*. Cleveland: CRC Press; 1998.

[8] Mathew S, Sunitha, MS. Types of arcs in a fuzzy graph. *Information Sciences*. 2009; 179(11): 1760-1768. DOI: https://doi.org/10.1016/j.ins.2009.01.003

[9] Mordeson JN, Nair PS. *Fuzzy Graphs and Fuzzy Hypergraphs*. New York: Physica-Verlag; 2000. https://doi.org/10.1007/978-3-7908-1854-3

[10] Borzooei RA, Almallah R, Jun YB, Ghaznavi H. Inverse fuzzy graphs with applications. *New Mathematics and Natural Computation*. 2020; 16(02): 397-418. DOI: https://doi.org/10.1142/S1793005720500246

[11] Almallah R, Borzooei RA, Jun YB. Domination numbers of inverse fuzzy graphs with application in decision-making problems. *New Mathematics and Natural Computation*. 2022; 18(01): 19-42. DOI: https://doi.org/10.1142/S179300572250003X

[12] Poulik S, Ghorai G. New concepts of inverse fuzzy mixed graphs and its application. *Granular computing*. 2021; 7(3): 549-559. DOI: https://doi.org/10.1007/s41066-021-00284-0

[13] Das K, Naseem U, Samanta S, Khan SK, De K. Fuzzy mixed graphs and its application to identification of COVID19 affected central regions in India. *Journal of Intelligent & Fuzzy Systems*. 2020; 40(1):1051-1064. DOI: https://doi.org/10.3233/JIFS-201249

[14] Mondal R, Ghorai G. On the m-step inverse fuzzy mixed competition graphs with application. *International Journal of Mathematics for Industry*. 2024; 16(01): 2450006. DOI: https://doi.org/10.1142/S2661335224500060

[15] Mondal R, Ghorai G. Inverse fuzzy mixed planar graphs with application. *International Journal of Applied and Computational Mathematics*. 2024; 10(131). DOI: https://doi.org/10.1007/s40819-024-01764-y

[16] Mondal R, Ghorai G. Wiener index of inverse fuzzy mixed graphs with application in education system. *Journal of Applied Mathematics and Computing*. 2025; 71: 725-742. DOI: https://doi.org/10.1007/s12190-024-02263-5

[17] Wiener H. Structural determination of paraffin boiling points. *Journal of the American Chemical Society*. 1947; 69(1): 17-20. DOI: https://doi.org/10.1021/ja01193a005

[18] Selvakumar K, Gangaeswari P, Arunkumar G. The wiener index of the zero-divisor graph of a finite commutative ring with unity. *Discrete Applied Mathematics*. 2022; 311: 72-84. DOI: https://doi.org/10.1016/j.dam.2022.01.012

[19] Somasundaram A, Somasundaram S. Domination in fuzzy graphs-I. *Pattern Recognition Letters*. 1998; 19(9): 787-791. DOI: https://doi.org/10.1016/S0167-8655(98)00064-6

[20] Samanta S, Akram M, Pal M. m-Step fuzzy competition graphs. *Journal of Applied Mathematics and Computing*. 2015; 47: 461-472. DOI: https://doi.org/10.1007/s12190-014-0785-2

Rahul Mondal
Department of Mathematics
Vivekananda Satavarshiki Mahavidyalaya
Manikpara, Jhargram -721513, West Bengal, India.
E-mail: imondalrahul@gmail.com

Ganesh Ghorai
Department of Applied Mathematics
Vidyasagar University, Midnapore - 721 102
West Bengal, India
E-mail: math.ganesh@mail.vidyasagar.ac.in