



## Callebaut and Hölder type inequalities for positive linear maps of selfadjoint operators via a Kittaneh-Manasrah result

S. S. Dragomir<sup>a,b,\*</sup>

<sup>a</sup>*Applied Mathematics Research Group, ISILC, Victoria University,  
PO Box 14428, Melbourne City, MC 8001, Australia.*

<sup>b</sup>*School of Computer Science & Applied Mathematics, University of the Witwatersrand,  
Private Bag 3, Johannesburg 2050, South Africa.*

Received 6 January 2025; Accepted 24 April 2025.

Communicated by Ghasem Soleimani Rad

---

**Abstract.** Some inequalities of Callebaut and Hölder type for positive linear maps of continuous functions of selfadjoint linear operators in Hilbert spaces, are given. Applications for power function are provided as well.

---

**Keywords:** Positive linear maps, selfadjoint operators, functions of selfadjoint operators, Callebaut inequality, Hölder inequality.

**2010 AMS Subject Classification:** 47A63, 47A99.

### 1. Introduction

As is well-known, the famous *Young inequality* for scalars says that if  $a, b > 0$  and  $\nu \in [0, 1]$ , then

$$a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b \quad (1)$$

with equality if and only if  $a = b$ . The inequality (1) is also called as  *$\nu$ -weighted arithmetic-geometric mean inequality*.

Kittaneh and Manasrah [9, 10] provided a refinement and a reverse for Young inequality

---

\*Corresponding author.

E-mail address: sever.dragomir@vu.edu.au (S. S. Dragomir).

as follows:

$$r \left( \sqrt{a} - \sqrt{b} \right)^2 \leq (1 - \nu) a + \nu b - a^{1-\nu} b^\nu \leq R \left( \sqrt{a} - \sqrt{b} \right)^2, \quad (2)$$

where  $a, b > 0$ ,  $\nu \in [0, 1]$ ,  $r = \min \{1 - \nu, \nu\}$  and  $R = \max \{1 - \nu, \nu\}$ . The case  $\nu = \frac{1}{2}$  reduces (2) to an identity and is of no interest. We observe that if  $a, b \in [m, M] \subset (0, \infty)$ , then  $\left| \sqrt{a} - \sqrt{b} \right| \leq \sqrt{M} - \sqrt{m}$  and by (2) we obtain the following simple reverse of Young inequality

$$(1 - \nu) a + \nu b - a^{1-\nu} b^\nu \leq R \left( \sqrt{M} - \sqrt{m} \right)^2. \quad (3)$$

We can give a direct proof for (2) as follows. Recall the following result obtained by Dragomir in 2006 [3] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} n \min_{j \in \{1, 2, \dots, n\}} \{p_j\} \left[ \frac{1}{n} \sum_{j=1}^n h(x_j) - h \left( \frac{1}{n} \sum_{j=1}^n x_j \right) \right] &\leq \frac{1}{P_n} \sum_{j=1}^n p_j h(x_j) - h \left( \frac{1}{P_n} \sum_{j=1}^n p_j x_j \right) \\ &\leq n \max_{j \in \{1, 2, \dots, n\}} \{p_j\} \left[ \frac{1}{n} \sum_{j=1}^n h(x_j) - h \left( \frac{1}{n} \sum_{j=1}^n x_j \right) \right], \end{aligned} \quad (4)$$

where  $h : C \rightarrow \mathbb{R}$  is a convex function defined on convex subset  $C$  of the linear space  $X$ ,  $\{x_j\}_{j \in \{1, 2, \dots, n\}}$  are vectors in  $C$  and  $\{p_j\}_{j \in \{1, 2, \dots, n\}}$  are nonnegative numbers with  $P_n = \sum_{j=1}^n p_j > 0$ . For  $n = 2$ , we deduce from (4) that

$$\begin{aligned} 2 \min \{\nu, 1 - \nu\} \left[ \frac{h(x) + h(y)}{2} - h \left( \frac{x + y}{2} \right) \right] &\leq \nu h(x) + (1 - \nu) h(y) - h[\nu x + (1 - \nu) y] \\ &\leq 2 \max \{\nu, 1 - \nu\} \left[ \frac{h(x) + h(y)}{2} - h \left( \frac{x + y}{2} \right) \right] \end{aligned} \quad (5)$$

for any  $x, y \in \mathbb{R}$  and  $\nu \in [0, 1]$ . If we take  $h(x) = \exp(x)$ , then we get from (5) that

$$\begin{aligned} 2 \min \{\nu, 1 - \nu\} \left[ \frac{\exp(x) + \exp(y)}{2} - \exp \left( \frac{x + y}{2} \right) \right] & \\ &\leq \nu \exp(x) + (1 - \nu) \exp(y) - \exp[\nu x + (1 - \nu) y] \\ &\leq 2 \max \{\nu, 1 - \nu\} \left[ \frac{\exp(x) + \exp(y)}{2} - \exp \left( \frac{x + y}{2} \right) \right] \end{aligned} \quad (6)$$

for any  $x, y \in \mathbb{R}$  and  $\nu \in [0, 1]$ . Further, denote  $\exp(x) = a$ ,  $\exp(y) = b$  with  $a, b > 0$ , then from (6) we obtain the inequality (2).

Let  $H$  be a complex Hilbert space and  $\mathcal{B}(H)$ , the Banach algebra of bounded linear operators acting on  $H$ . We denote by  $\mathcal{B}^+(H)$  the convex cone of all positive operators on  $H$  and by  $\mathcal{B}^{++}(H)$  the convex cone of all positive definite operators on  $H$ . Let  $H$  and  $K$  be complex Hilbert spaces. Following [2] (see also [15, p. 18]) we can introduce:

**Definition 1.1** A map  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is linear if it is additive and homogeneous, namely  $\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$  for any  $\lambda, \mu \in \mathbb{C}$  and  $A, B \in \mathcal{B}(H)$ . The linear map  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is positive if it preserves the operator order, i.e. if  $A \in \mathcal{B}^+(H)$  then  $\Phi(A) \in \mathcal{B}^+(K)$ . We write  $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ . The linear map  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is normalized if it preserves the identity operator, i.e.  $\Phi(1_H) = 1_K$ . We write  $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ .

We observe that a positive linear map  $\Phi$  preserves the *order relation*, namely  $A \leq B$  implies  $\Phi(A) \leq \Phi(B)$  and preserves the adjoint operation  $\Phi(A^*) = \Phi(A)^*$ . If  $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$  and  $\alpha 1_H \leq A \leq \beta 1_H$ , then  $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$ . If the map  $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is linear, positive and  $\Psi(1_H) \in \mathcal{B}^{++}(K)$  then by putting  $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$  we get that  $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ , namely it is also normalized. For some recent inequalities for positive maps of Hilbert space operators see [4–7], [8, 11–14], [16–18] and the references therein.

In this paper, we establish some inequalities of Callebaut and Hölder type for positive linear maps of continuous functions of selfadjoint linear operators in Hilbert spaces. Applications for power function are provided as well.

## 2. Callebaut's type inequalities

The following refinement of Cauchy-Bunyakowsky-Schwarz inequality for  $n$ -tuples of nonnegative real numbers  $(a_1, \dots, a_n)$ ,  $(b_1, \dots, b_n)$  was established by Callebaut [1, 1965] and can be stated as follows:

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^{2(1-\lambda)} b_i^{2\lambda} \sum_{i=1}^n a_i^{2\lambda} b_i^{2(1-\lambda)} \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2, \quad (7)$$

for any  $\lambda \in [0, 1]$ . In this section we obtain, as a main result, the following refinement and reverse of the second part of Callebaut inequality (7) in the case of positive maps:

$$\begin{aligned} 0 &\leq 2r \left( \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(A))x, x \rangle - \langle \Phi(f(A)g(A))x, x \rangle^2 \right) \\ &\leq \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(A))x, x \rangle \\ &\quad - \langle \Phi(f^{2\nu}(A)g^{2(1-\nu)}(A))x, x \rangle \langle \Phi(f^{2(1-\nu)}(A)g^{2\nu}(A))x, x \rangle \\ &\leq R \left( \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(A))x, x \rangle - \langle \Phi(f(A)g(A))x, x \rangle^2 \right), \end{aligned} \quad (8)$$

for any  $x \in K$ , where  $f, g : I \rightarrow \mathbb{R}$  are continuous functions on the interval  $I$ ,  $A$  is a selfadjoint operator with  $Sp(A) \subset I$ ,  $\nu \in [0, 1]$ ,  $r = \min\{1 - \nu, \nu\}$ ,  $R = \max\{1 - \nu, \nu\}$  and  $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ . This result can be obtained as a particular case of the more general result incorporated in

**Theorem 2.1** Let  $f, g : I \rightarrow \mathbb{R}$  be continuous functions on the interval  $I$  and  $A, B$  be two selfadjoint operators with  $Sp(A), Sp(B) \subset I$ . If  $\nu \in [0, 1]$  and  $\Phi, \Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ ,

then

$$\begin{aligned}
0 &\leq r (\langle \Phi (g^2 (A)) x, x \rangle \langle \Psi (f^2 (B)) x, x \rangle - 2 \langle \Phi (f (A) g (A)) x, x \rangle) \\
&\quad \times \langle \Psi (f (B) g (B)) x, x \rangle + \langle \Phi (f^2 (A)) x, x \rangle \langle \Psi (g^2 (B)) x, x \rangle) \\
&\leq (1 - \nu) \langle \Phi (g^2 (A)) x, x \rangle \langle \Psi (f^2 (B)) x, x \rangle + \nu \langle \Phi (f^2 (A)) x, x \rangle \langle \Psi (g^2 (B)) x, x \rangle \\
&\quad - \langle \Phi (f^{2\nu} (A) g^{2(1-\nu)} (A)) x, x \rangle \langle \Psi (f^{2(1-\nu)} (B) g^{2\nu} (B)) x, x \rangle \\
&\leq R (\langle \Phi (g^2 (A)) x, x \rangle \langle \Psi (f^2 (B)) x, x \rangle - 2 \langle \Phi (f (A) g (A)) x, x \rangle \\
&\quad \times \langle \Psi (f (B) g (B)) x, x \rangle + \langle \Phi (f^2 (A)) x, x \rangle \langle \Psi (g^2 (B)) x, x \rangle)
\end{aligned} \tag{9}$$

for any  $x \in K$ , where  $r = \min \{1 - \nu, \nu\}$  and  $R = \max \{1 - \nu, \nu\}$ .

**Proof.** Let  $t, s \in I$  such that  $g(t), g(s) \neq 0$ . If we use the inequalities (2) for  $a = \frac{f^2(t)}{g^2(t)}$  and  $b = \frac{f^2(s)}{g^2(s)}$ , then we get

$$\begin{aligned}
r \left( \frac{f(t)}{g(t)} - \frac{f(s)}{g(s)} \right)^2 &\leq (1 - \nu) \frac{f^2(t)}{g^2(t)} + \nu \frac{f^2(s)}{g^2(s)} - \left( \frac{f^2(t)}{g^2(t)} \right)^{1-\nu} \left( \frac{f^2(s)}{g^2(s)} \right)^\nu \\
&\leq R \left( \frac{f(t)}{g(t)} - \frac{f(s)}{g(s)} \right)^2,
\end{aligned} \tag{10}$$

where  $\nu \in [0, 1]$ ,  $r = \min \{1 - \nu, \nu\}$  and  $R = \max \{1 - \nu, \nu\}$ . Therefore,

$$\begin{aligned}
r \left( \frac{f^2(t)}{g^2(t)} - 2 \frac{f(t)f(s)}{g(t)g(s)} + \frac{f^2(s)}{g^2(s)} \right) &\leq (1 - \nu) \frac{f^2(t)}{g^2(t)} + \nu \frac{f^2(s)}{g^2(s)} - \left( \frac{f^2(t)}{g^2(t)} \right)^{1-\nu} \left( \frac{f^2(s)}{g^2(s)} \right)^\nu \\
&\leq R \left( \frac{f^2(t)}{g^2(t)} - 2 \frac{f(t)f(s)}{g(t)g(s)} + \frac{f^2(s)}{g^2(s)} \right).
\end{aligned} \tag{11}$$

If we multiply (11) by  $g^2(t)g^2(s)$ , then we get

$$\begin{aligned}
0 &\leq r (f^2(t)g^2(s) - 2f(t)g(t)f(s)g(s) + f^2(s)g^2(t)) \\
&\leq (1 - \nu) f^2(t)g^2(s) + \nu g^2(t)f^2(s) - f^{2(1-\nu)}(t)g^{2\nu}(t)f^{2\nu}(s)g^{2(1-\nu)}(s) \\
&\leq R (f^2(t)g^2(s) - 2f(t)g(t)f(s)g(s) + f^2(s)g^2(t)),
\end{aligned} \tag{12}$$

which holds for any  $t, s \in I$ . Fix  $t \in I$  and use the continuous functional calculus for  $A$  to get

$$\begin{aligned}
0 &\leq r (f^2(t)g^2(A) - 2f(t)g(t)f(A)g(A) + g^2(t)f^2(A)) \\
&\leq (1 - \nu) f^2(t)g^2(A) + \nu g^2(t)f^2(A) - f^{2(1-\nu)}(t)g^{2\nu}(t)f^{2\nu}(A)g^{2(1-\nu)}(A) \\
&\leq R (f^2(t)g^2(A) - 2f(t)g(t)f(A)g(A) + g^2(t)f^2(A)).
\end{aligned} \tag{13}$$

If we take in the inequality (13) the map  $\Phi$ , then we get

$$\begin{aligned} 0 &\leq r (f^2(t) \Phi(g^2(A)) - 2f(t)g(t) \Phi(f(A)g(A)) + g^2(t) \Phi(f^2(A))) & (14) \\ &\leq (1-\nu) f^2(t) \Phi(g^2(A)) + \nu g^2(t) \Phi(f^2(A)) \\ &\quad - f^{2(1-\nu)}(t) g^{2\nu}(t) \Phi(f^{2\nu}(A) g^{2(1-\nu)}(A)) \\ &\leq R (f^2(t) \Phi(g^2(A)) - 2f(t)g(t) \Phi(f(A)g(A)) + g^2(t) \Phi(f^2(A))) \end{aligned}$$

for any  $t \in I$ . If we take the inner product in (14), then we get

$$\begin{aligned} 0 &\leq r (f^2(t) \langle \Phi(g^2(A)) x, x \rangle - 2f(t)g(t) \langle \Phi(f(A)g(A)) x, x \rangle & (15) \\ &\quad + g^2(t) \langle \Phi(f^2(A)) x, x \rangle) \\ &\leq (1-\nu) f^2(t) \langle \Phi(g^2(A)) x, x \rangle + \nu g^2(t) \langle \Phi(f^2(A)) x, x \rangle \\ &\quad - f^{2(1-\nu)}(t) g^{2\nu}(t) \langle \Phi(f^{2\nu}(A) g^{2(1-\nu)}(A)) x, x \rangle \\ &\leq R (f^2(t) \langle \Phi(g^2(A)) x, x \rangle - 2f(t)g(t) \langle \Phi(f(A)g(A)) x, x \rangle \\ &\quad + g^2(t) \langle \Phi(f^2(A)) x, x \rangle) \end{aligned}$$

for any  $t \in I$  and  $x \in K$ . If we use the functional calculus for the operator  $B$  we have by (15) that

$$\begin{aligned} 0 &\leq r (\langle \Phi(g^2(A)) x, x \rangle f^2(B) - 2 \langle \Phi(f(A)g(A)) x, x \rangle f(B)g(B) & (16) \\ &\quad + \langle \Phi(f^2(A)) x, x \rangle g^2(B)) \\ &\leq (1-\nu) \langle \Phi(g^2(A)) x, x \rangle f^2(B) + \nu \langle \Phi(f^2(A)) x, x \rangle g^2(B) \\ &\quad - \langle \Phi(f^{2\nu}(A) g^{2(1-\nu)}(A)) x, x \rangle f^{2(1-\nu)}(B) g^{2\nu}(B) \\ &\leq R (\langle \Phi(g^2(A)) x, x \rangle f^2(B) - 2 \langle \Phi(f(A)g(A)) x, x \rangle f(B)g(B) \\ &\quad + \langle \Phi(f^2(A)) x, x \rangle g^2(B)) \end{aligned}$$

for any  $x \in K$ . If we take in the inequality (16) the map  $\Psi$  and the inner product for  $y \in K$ , then we get

$$\begin{aligned} 0 &\leq r (\langle \Phi(g^2(A)) x, x \rangle \langle \Psi(f^2(B)) y, y \rangle - 2 \langle \Phi(f(A)g(A)) x, x \rangle & (17) \\ &\quad \times \langle \Psi(f(B)g(B)) y, y \rangle + \langle \Phi(f^2(A)) x, x \rangle \langle \Psi(g^2(B)) y, y \rangle) \\ &\leq (1-\nu) \langle \Phi(g^2(A)) x, x \rangle \langle \Psi(f^2(B)) y, y \rangle \\ &\quad + \nu \langle \Phi(f^2(A)) x, x \rangle \langle \Psi(g^2(B)) y, y \rangle \\ &\quad - \langle \Phi(f^{2\nu}(A) g^{2(1-\nu)}(A)) x, x \rangle \langle \Psi(f^{2(1-\nu)}(B) g^{2\nu}(B)) y, y \rangle \\ &\leq R (\langle \Phi(g^2(A)) x, x \rangle \langle \Psi(f^2(B)) y, y \rangle - 2 \langle \Phi(f(A)g(A)) x, x \rangle \\ &\quad \times \langle \Psi(f(B)g(B)) y, y \rangle + \langle \Phi(f^2(A)) x, x \rangle \langle \Psi(g^2(B)) y, y \rangle) \end{aligned}$$

for any  $x, y \in K$ . If we take in (17)  $y = x$ , then we get the desired result (9). ■

**Remark 1** In particular, under the assumptions of Theorem 2.1 we have

$$\begin{aligned}
0 &\leq r \left( \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(B))x, x \rangle - 2 \langle \Phi(f(A)g(A))x, x \rangle \right. \\
&\quad \times \langle \Phi(f(B)g(B))x, x \rangle + \langle \Phi(f^2(A))x, x \rangle \langle \Phi(g^2(B))x, x \rangle \left. \right) \\
&\leq (1 - \nu) \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(B))x, x \rangle + \nu \langle \Phi(f^2(A))x, x \rangle \langle \Phi(g^2(B))x, x \rangle \\
&\quad - \langle \Phi(f^{2\nu}(A)g^{2(1-\nu)}(A))x, x \rangle \langle \Phi(f^{2(1-\nu)}(B)g^{2\nu}(B))x, x \rangle \\
&\leq R \left( \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(B))x, x \rangle - 2 \langle \Phi(f(A)g(A))x, x \rangle \right. \\
&\quad \times \langle \Phi(f(B)g(B))x, x \rangle + \langle \Phi(f^2(A))x, x \rangle \langle \Phi(g^2(B))x, x \rangle \left. \right)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
0 &\leq r \left( \langle \Phi(g^2(A))x, x \rangle \langle \Psi(f^2(A))x, x \rangle - 2 \langle \Phi(f(A)g(A))x, x \rangle \right. \\
&\quad \times \langle \Psi(f(A)g(A))x, x \rangle + \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(A))x, x \rangle \left. \right) \\
&\leq (1 - \nu) \langle \Phi(g^2(A))x, x \rangle \langle \Psi(f^2(A))x, x \rangle + \nu \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(A))x, x \rangle \\
&\quad - \langle \Phi(f^{2\nu}(A)g^{2(1-\nu)}(A))x, x \rangle \langle \Psi(f^{2(1-\nu)}(A)g^{2\nu}(A))x, x \rangle \\
&\leq R \left( \langle \Phi(g^2(A))x, x \rangle \langle \Psi(f^2(A))x, x \rangle - 2 \langle \Phi(f(A)g(A))x, x \rangle \right. \\
&\quad \times \langle \Psi(f(A)g(A))x, x \rangle + \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(A))x, x \rangle \left. \right)
\end{aligned} \tag{19}$$

for any  $x \in K$ . If we take in (18)  $B = A$ , then we get (8).

The inequality (8) has some particular inequalities of interest for power function. Indeed if  $p, q > 0$ ,  $A \in \mathcal{B}^+(H)$  and  $p > 0, q < 0$  or  $p < 0, q > 0$ , or  $p < 0, q < 0$  and  $A \in \mathcal{B}^{++}$ , then by (17) we have

$$\begin{aligned}
0 &\leq 2r \left( \langle \Phi(A^{2p})x, x \rangle \langle \Phi(A^{2q})x, x \rangle - \langle \Phi(A^{p+q})x, x \rangle^2 \right) \\
&\leq \langle \Phi(A^{2p})x, x \rangle \langle \Phi(A^{2q})x, x \rangle - \langle \Phi(A^{2\nu p + 2(1-\nu)q})x, x \rangle \langle \Phi(A^{2(1-\nu)p + 2\nu q})x, x \rangle \\
&\leq R \left( \langle \Phi(A^{2p})x, x \rangle \langle \Phi(A^{2q})x, x \rangle - \langle \Phi(A^{p+q})x, x \rangle^2 \right)
\end{aligned} \tag{20}$$

for any  $x \in K$ , where  $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ ,  $r = \min\{1 - \nu, \nu\}$  and  $R = \max\{1 - \nu, \nu\}$ . In particular, if we take  $p = s, q = -s$  with  $s > 0$ , in (20), then we have

$$\begin{aligned}
0 &\leq 2r \left( \langle \Phi(A^{2s})x, x \rangle \langle \Phi(A^{-2s})x, x \rangle - \langle \Phi(1_H)x, x \rangle^2 \right) \\
&\leq \langle \Phi(A^{2s})x, x \rangle \langle \Phi(A^{-2s})x, x \rangle - \langle \Phi(A^{2(2\nu-1)s})x, x \rangle \langle \Phi(A^{-2(2\nu-1)s})x, x \rangle \\
&\leq R \left( \langle \Phi(A^{2s})x, x \rangle \langle \Phi(A^{-2s})x, x \rangle - \langle \Phi(1_H)x, x \rangle^2 \right),
\end{aligned} \tag{21}$$

which by  $s = \frac{1}{2}$  gives the inequality

$$\begin{aligned} 0 &\leq 2r \left( \langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle - \langle \Phi(1_H)x, x \rangle^2 \right) \\ &\leq \langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle - \langle \Phi(A^{2\nu-1})x, x \rangle \langle \Phi(A^{-(2\nu-1)})x, x \rangle \\ &\leq R \left( \langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle - \langle \Phi(1_H)x, x \rangle^2 \right) \end{aligned} \tag{22}$$

for any  $x \in K$ , where  $A \in \mathcal{B}^{++}(H)$ ,  $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ ,  $r = \min\{1 - \nu, \nu\}$  and  $R = \max\{1 - \nu, \nu\}$ . If  $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$  and  $x \in K$  with  $\|x\| = 1$ , we have by (22)

$$\begin{aligned} 0 &\leq 2r \left( \langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle - 1 \right) \\ &\leq \langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle - \langle \Phi(A^{2\nu-1})x, x \rangle \langle \Phi(A^{-(2\nu-1)})x, x \rangle \\ &\leq R \left( \langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle - 1 \right). \end{aligned} \tag{23}$$

If we take  $p = \lambda \in [0, 1]$  and  $q = 1 - \lambda$  in (20), then we get

$$\begin{aligned} 0 &\leq 2r \left( \langle \Phi(A^{2\lambda})x, x \rangle \langle \Phi(A^{2(1-\lambda)})x, x \rangle - \langle \Phi(A)x, x \rangle^2 \right) \\ &\leq \langle \Phi(A^{2\lambda})x, x \rangle \langle \Phi(A^{2(1-\lambda)})x, x \rangle - \langle \Phi(A^{2(1-\nu-\lambda+2\nu\lambda)})x, x \rangle \langle \Phi(A^{2(\nu+\lambda-2\nu\lambda)})x, x \rangle \\ &\leq R \left( \langle \Phi(A^{2\lambda})x, x \rangle \langle \Phi(A^{2(1-\lambda)})x, x \rangle - \langle \Phi(A)x, x \rangle^2 \right) \end{aligned} \tag{24}$$

for any  $x \in K$ , where  $A \in \mathcal{B}^+(H)$ ,  $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ ,  $r = \min\{1 - \nu, \nu\}$  and  $R = \max\{1 - \nu, \nu\}$ .

**Remark 2** If in (8) we take the function  $g$  to be constant 1, then we have by (8) that

$$\begin{aligned} 0 &\leq 2r \left( \langle \Phi(1_H)x, x \rangle \langle \Phi(f^2(A))x, x \rangle - \langle \Phi(f(A))x, x \rangle^2 \right) \\ &\leq \langle \Phi(1_H)x, x \rangle \langle \Phi(f^2(A))x, x \rangle - \langle \Phi(f^{2\nu}(A))x, x \rangle \langle \Phi(f^{2(1-\nu)}(A))x, x \rangle \\ &\leq R \left( \langle \Phi(1_H)x, x \rangle \langle \Phi(f^2(A))x, x \rangle - \langle \Phi(f(A))x, x \rangle^2 \right) \end{aligned} \tag{25}$$

for any  $x \in K$ . Let  $T_j, j \in \{1, \dots, n\}$  be bounded linear operators in  $H$  and  $\Phi(A) := \sum_{i=1}^n T_j^* A T_j$  for  $A \in \mathcal{B}(H)$ . Then  $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(H)]$  and by (25) we have

$$\begin{aligned} 0 &\leq 2r \left( \left\langle \sum_{i=1}^n T_j^* A T_j x, x \right\rangle \left\langle \sum_{i=1}^n T_j^* f^2(A) T_j x, x \right\rangle - \left\langle \sum_{i=1}^n T_j^* f(A) T_j x, x \right\rangle^2 \right) \\ &\leq \left\langle \sum_{i=1}^n T_j^* A T_j x, x \right\rangle \left\langle \sum_{i=1}^n T_j^* f^2(A) T_j x, x \right\rangle - \left\langle \sum_{i=1}^n T_j^* f^{2\nu}(A) T_j x, x \right\rangle \left\langle \sum_{i=1}^n T_j^* f^{2(1-\nu)}(A) T_j x, x \right\rangle \\ &\leq R \left( \left\langle \sum_{i=1}^n T_j^* A T_j x, x \right\rangle \left\langle \sum_{i=1}^n T_j^* f^2(A) T_j x, x \right\rangle - \left\langle \sum_{i=1}^n T_j^* f(A) T_j x, x \right\rangle^2 \right), \end{aligned}$$

for any  $x \in K$ .

The following upper bound may be stated as well:

**Theorem 2.2** Let  $f, g : I \rightarrow \mathbb{R}$  be continuous functions on the interval  $I$  such that

$$0 < m \leq \frac{f}{g} \leq M < \infty \quad (26)$$

for some constants  $m, M$ , and  $A, B$  be two selfadjoint operators with  $Sp(A), Sp(B) \subset I$ . If  $\nu \in [0, 1]$  and  $\Phi, \Psi \in \mathfrak{F}[\mathcal{B}(H), \mathcal{B}(K)]$ , then

$$\begin{aligned} & (1 - \nu) \langle \Phi(g^2(A))x, x \rangle \langle \Psi(f^2(B))x, x \rangle + \nu \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(B))x, x \rangle \\ & - \langle \Phi(f^{2\nu}(A)g^{2(1-\nu)}(A))x, x \rangle \langle \Psi(f^{2(1-\nu)}(B)g^{2\nu}(B))x, x \rangle \\ & \leq R(M - m)^2 \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(B))x, x \rangle \end{aligned} \quad (27)$$

for any  $x \in K$ .

**Proof.** Let  $t, s \in I$  such that  $g(t), g(s) \neq 0$ . If we use the inequalities (2) for  $a = \frac{f^2(t)}{g^2(t)}$  and  $b = \frac{f^2(s)}{g^2(s)}$ , then we get

$$(1 - \nu) \frac{f^2(t)}{g^2(t)} + \nu \frac{f^2(s)}{g^2(s)} - \left( \frac{f^2(t)}{g^2(t)} \right)^{1-\nu} \left( \frac{f^2(s)}{g^2(s)} \right)^\nu \leq R(M - m)^2, \quad (28)$$

where  $\nu \in [0, 1]$ ,  $R = \max\{1 - \nu, \nu\}$ . If we multiply (28) by  $g^2(t)g^2(s)$ , then we get

$$\begin{aligned} & (1 - \nu) f^2(t) g^2(s) + \nu g^2(t) f^2(s) - f^{2(1-\nu)}(t) g^{2\nu}(t) f^{2\nu}(s) g^{2(1-\nu)}(s) \\ & \leq R(M - m)^2 g^2(t) g^2(s) \end{aligned} \quad (29)$$

for any  $t, s \in I$ . Now, by employing a similar argument to the one outlined in the proof of Theorem 2.1 we deduce the desired result (27).  $\blacksquare$

**Remark 3** In particular, we have the inequalities

$$\begin{aligned} & (1 - \nu) \langle \Phi(g^2(A))x, x \rangle \langle \Phi(f^2(B))x, x \rangle + \nu \langle \Phi(f^2(A))x, x \rangle \langle \Phi(g^2(B))x, x \rangle \\ & - \langle \Phi(f^{2\nu}(A)g^{2(1-\nu)}(A))x, x \rangle \langle \Phi(f^{2(1-\nu)}(B)g^{2\nu}(B))x, x \rangle \\ & \leq R(M - m)^2 \langle \Phi(f^2(A))x, x \rangle \langle \Phi(g^2(B))x, x \rangle \end{aligned} \quad (30)$$

and

$$\begin{aligned} & (1 - \nu) \langle \Phi(g^2(A))x, x \rangle \langle \Psi(f^2(A))x, x \rangle + \nu \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(A))x, x \rangle \\ & - \langle \Phi(f^{2\nu}(A)g^{2(1-\nu)}(A))x, x \rangle \langle \Psi(f^{2(1-\nu)}(A)g^{2\nu}(A))x, x \rangle \\ & \leq R(M - m)^2 \langle \Phi(f^2(A))x, x \rangle \langle \Psi(g^2(A))x, x \rangle \end{aligned} \quad (31)$$



and

$$\begin{aligned}
 0 &\leq \langle \Phi (g^2 (A)) x, x \rangle \langle \Phi (f^2 (A)) x, x \rangle \\
 &\quad - \langle \Phi (f^{2\nu} (A) g^{2(1-\nu)} (A)) x, x \rangle \langle \Phi (f^{2(1-\nu)} (A) g^{2\nu} (A)) x, x \rangle \\
 &\leq R (M - m)^2 \langle \Phi (f^2 (A)) x, x \rangle \langle \Phi (g^2 (A)) x, x \rangle
 \end{aligned} \tag{32}$$

for any  $x \in K$ . The inequality (32) can be also written as

$$\begin{aligned}
 0 &\leq 1 - \frac{\langle \Phi (f^{2\nu} (A) g^{2(1-\nu)} (A)) x, x \rangle \langle \Phi (f^{2(1-\nu)} (A) g^{2\nu} (A)) x, x \rangle}{\langle \Phi (g^2 (A)) x, x \rangle \langle \Phi (f^2 (A)) x, x \rangle} \\
 &\leq R (M - m)^2
 \end{aligned} \tag{33}$$

for any  $x \in K, x \neq 0$ .

### 3. Hölder’s type inequalities

**Theorem 3.1** Let  $f, g : I \rightarrow (0, \infty)$  be continuous functions on the interval  $I$ , and  $A, B$  be two selfadjoint operators with  $Sp(A), Sp(B) \subset I$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $\Phi, \Psi \in \mathfrak{P}[B(H), B(K)]$ , then

$$\begin{aligned}
 &s \left( \frac{\langle \Psi (f^p (B)) x, x \rangle}{\langle \Phi (f^p (A)) x, x \rangle} + \frac{\langle \Psi (g^q (B)) x, x \rangle}{\langle \Phi (g^q (A)) x, x \rangle} - \frac{2 \langle \Psi (f^{\frac{p}{2}} (B) g^{\frac{q}{2}} (B)) x, x \rangle}{\sqrt{\langle \Phi (f^p (A)) x, x \rangle \langle \Phi (g^q (A)) x, x \rangle}} \right) \\
 &\leq \frac{1}{p} \frac{\langle \Psi (f^p (B)) x, x \rangle}{\langle \Phi (f^p (A)) x, x \rangle} + \frac{1}{q} \frac{\langle \Psi (g^q (B)) x, x \rangle}{\langle \Phi (g^q (A)) x, x \rangle} - \frac{\langle \Phi (f (B) g (B)) x, x \rangle}{\langle \Phi (f^p (A)) x, x \rangle^{1/p} \langle \Phi (g^q (A)) x, x \rangle^{1/q}} \\
 &\leq S \left( \frac{\langle \Psi (f^p (B)) x, x \rangle}{\langle \Phi (f^p (A)) x, x \rangle} + \frac{\langle \Psi (g^q (B)) x, x \rangle}{\langle \Phi (g^q (A)) x, x \rangle} - \frac{2 \langle \Psi (f^{\frac{p}{2}} (B) g^{\frac{q}{2}} (B)) x, x \rangle}{\sqrt{\langle \Phi (f^p (A)) x, x \rangle \langle \Phi (g^q (A)) x, x \rangle}} \right)
 \end{aligned} \tag{34}$$

for any  $x \in K, x \neq 0$ , where  $s = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$  and  $S = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ . In particular, we have

$$\begin{aligned}
 2s \left( 1 - \frac{\langle \Phi (f^{\frac{p}{2}} (A) g^{\frac{q}{2}} (A)) x, x \rangle}{\sqrt{\langle \Phi (f^p (A)) x, x \rangle \langle \Phi (g^q (A)) x, x \rangle}} \right) &\leq 1 - \frac{\langle \Phi (f (A) g (A)) x, x \rangle}{\langle \Phi (f^p (A)) x, x \rangle^{1/p} \langle \Phi (g^q (A)) x, x \rangle^{1/q}} \\
 &\leq 2S \left( 1 - \frac{\langle \Phi (f^{\frac{p}{2}} (A) g^{\frac{q}{2}} (A)) x, x \rangle}{\sqrt{\langle \Phi (f^p (A)) x, x \rangle \langle \Phi (g^q (A)) x, x \rangle}} \right)
 \end{aligned} \tag{35}$$

for any  $x \in K, x \neq 0$ .

**Proof.** From (2), we have

$$s \left( a + b - 2\sqrt{ab} \right) \leq \frac{1}{p} a + \frac{1}{q} b - a^{\frac{1}{p}} b^{\frac{1}{q}} \leq S \left( a + b - 2\sqrt{ab} \right), \tag{36}$$

where  $a, b \geq 0$ . If we choose in (36)

$$a = \frac{f^p(t)}{\langle \Phi(f^p(A))x, x \rangle}, b = \frac{g^q(t)}{\langle \Phi(g^q(A))x, x \rangle}, t \in I,$$

then we get

$$\begin{aligned} & s \left( \frac{f^p(t)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{g^q(t)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{2f^{\frac{p}{2}}(t)g^{\frac{q}{2}}(t)}{\sqrt{\langle \Phi(f^p(A))x, x \rangle \langle \Phi(g^q(A))x, x \rangle}} \right) \\ & \leq \frac{1}{p} \frac{f^p(t)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{1}{q} \frac{g^q(t)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{f(t)g(t)}{\langle \Phi(f^p(A))x, x \rangle^{1/p} \langle \Phi(g^q(A))x, x \rangle^{1/q}} \\ & \leq S \left( \frac{f^p(t)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{g^q(t)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{2f^{\frac{p}{2}}(t)g^{\frac{q}{2}}(t)}{\sqrt{\langle \Phi(f^p(A))x, x \rangle \langle \Phi(g^q(A))x, x \rangle}} \right) \end{aligned}$$

for any  $t \in I$  and  $x \in K, x \neq 0$ . If we use the functional calculus for  $B$ , we get

$$\begin{aligned} & s \left( \frac{f^p(B)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{g^q(B)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{2f^{\frac{p}{2}}(B)g^{\frac{q}{2}}(B)}{\sqrt{\langle \Phi(f^p(A))x, x \rangle \langle \Phi(g^q(A))x, x \rangle}} \right) \\ & \leq \frac{1}{p} \frac{f^p(B)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{1}{q} \frac{g^q(B)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{f(B)g(B)}{\langle \Phi(f^p(A))x, x \rangle^{1/p} \langle \Phi(g^q(A))x, x \rangle^{1/q}} \\ & \leq S \left( \frac{f^p(B)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{g^q(B)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{2f^{\frac{p}{2}}(B)g^{\frac{q}{2}}(B)}{\sqrt{\langle \Phi(f^p(A))x, x \rangle \langle \Phi(g^q(A))x, x \rangle}} \right), \end{aligned}$$

and by taking the map  $\Psi$  and the inner product over  $y \in K, y \neq 0$  we get

$$\begin{aligned} & s \left( \frac{\langle \Psi(f^p(B))y, y \rangle}{\langle \Phi(f^p(A))x, x \rangle} + \frac{\langle \Psi(g^q(B))y, y \rangle}{\langle \Phi(g^q(A))x, x \rangle} - \frac{2\langle \Psi(f^{\frac{p}{2}}(B)g^{\frac{q}{2}}(B))y, y \rangle}{\sqrt{\langle \Phi(f^p(A))x, x \rangle \langle \Phi(g^q(A))x, x \rangle}} \right) \\ & \leq \frac{1}{p} \frac{\langle \Psi(f^p(B))y, y \rangle}{\langle \Phi(f^p(A))x, x \rangle} + \frac{1}{q} \frac{\langle \Psi(g^q(B))y, y \rangle}{\langle \Phi(g^q(A))x, x \rangle} - \frac{\langle \Phi(f(B)g(B))y, y \rangle}{\langle \Phi(f^p(A))x, x \rangle^{1/p} \langle \Phi(g^q(A))x, x \rangle^{1/q}} \\ & \leq S \left( \frac{\langle \Psi(f^p(B))y, y \rangle}{\langle \Phi(f^p(A))x, x \rangle} + \frac{\langle \Psi(g^q(B))y, y \rangle}{\langle \Phi(g^q(A))x, x \rangle} - \frac{2\langle \Psi(f^{\frac{p}{2}}(B)g^{\frac{q}{2}}(B))y, y \rangle}{\sqrt{\langle \Phi(f^p(A))x, x \rangle \langle \Phi(g^q(A))x, x \rangle}} \right). \end{aligned}$$

Now, if we put  $y = x$  in this inequality, then we get (34). ■

**Remark 4** If we take  $f(t) = t^\alpha, g(t) = t^\beta$ , then from (34) we have for  $p, q > 1$  with

$\frac{1}{p} + \frac{1}{q} = 1$  that

$$2s \left( 1 - \frac{\langle \Phi \left( A^{\frac{\alpha p + \beta q}{2}} \right) x, x \rangle}{\sqrt{\langle \Phi(A^{\alpha p}) x, x \rangle \langle \Phi(A^{\beta q}) x, x \rangle}} \right) \leq 1 - \frac{\langle \Phi(A^{\alpha + \beta}) x, x \rangle}{\langle \Phi(A^{\alpha p}) x, x \rangle^{1/p} \langle \Phi(A^{\beta q}) x, x \rangle^{1/q}} \tag{37}$$

$$\leq 2S \left( 1 - \frac{\langle \Phi \left( A^{\frac{\alpha p + \beta q}{2}} \right) x, x \rangle}{\sqrt{\langle \Phi(A^{\alpha p}) x, x \rangle \langle \Phi(A^{\beta q}) x, x \rangle}} \right)$$

for any  $x \in K, x \neq 0$ , where  $\alpha, \beta \in \mathbb{R} \setminus \{0\}$  and  $A \in \mathcal{B}^{++}$ . If we take in (37)  $\alpha = \beta = 1$ , then we have

$$2s \left( 1 - \frac{\langle \Phi \left( A^{\frac{p+q}{2}} \right) x, x \rangle}{\sqrt{\langle \Phi(A^p) x, x \rangle \langle \Phi(A^q) x, x \rangle}} \right) \leq 1 - \frac{\langle \Phi(A^2) x, x \rangle}{\langle \Phi(A^p) x, x \rangle^{1/p} \langle \Phi(A^q) x, x \rangle^{1/q}} \tag{38}$$

$$\leq 2S \left( 1 - \frac{\langle \Phi \left( A^{\frac{p+q}{2}} \right) x, x \rangle}{\sqrt{\langle \Phi(A^p) x, x \rangle \langle \Phi(A^q) x, x \rangle}} \right)$$

for any  $x \in K, x \neq 0$ .

We have the following reverse of Hölder’s inequality:

**Theorem 3.2** Let  $f, g : I \rightarrow \mathbb{R}$  be continuous functions on the interval  $I$  such that

$$0 < m_1 \leq f \leq M_1 < \infty, \quad 0 < m_2 \leq g \leq M_2 < \infty, \tag{39}$$

for some constants  $m_1, m_2, M_1$  and  $M_2, A, B$  be two selfadjoint operators with  $Sp(A), Sp(B) \subset I$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$ , then

$$0 \leq 1 - \frac{\langle \Phi(f(A)g(A)) x, x \rangle}{\langle \Phi(f^p(A)) x, x \rangle^{1/p} \langle \Phi(g^q(A)) x, x \rangle^{1/q}} \tag{40}$$

$$\leq S \left( \max \left\{ \left( \frac{M_1}{m_1} \right)^{\frac{p}{2}}, \left( \frac{M_2}{m_2} \right)^{\frac{q}{2}} \right\} - \min \left\{ \left( \frac{m_1}{M_1} \right)^{\frac{p}{2}}, \left( \frac{m_2}{M_2} \right)^{\frac{q}{2}} \right\} \right)^2$$

for any  $x \in K, x \neq 0$ , where  $s = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$  and  $S = \max \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ .

**Proof.** It suffices to prove the inequality for  $x \in K, \|x\| = 1$ . By (39) we have

$$\langle \Phi(f^p(A)) x, x \rangle \leq \langle M_1^p \Phi(1_H) x, x \rangle = M_1^p \langle x, x \rangle = M_1^p,$$

$$m_1^p = m_1^p \langle x, x \rangle = \langle m_1^p \Phi(1_H) x, x \rangle \leq \langle \Phi(f^p(A)) x, x \rangle$$

showing that  $m_1^p \leq \langle \Phi(f^p(A)) x, x \rangle \leq M_1^p$  for  $x \in K$  with  $\|x\| = 1$ . Similarly, we have  $m_2^q \leq \langle \Phi(g^q(A)) x, x \rangle \leq M_2^q$  for  $x \in K$  with  $\|x\| = 1$ . These imply

$$\left( \frac{m_1}{M_1} \right)^p \leq \frac{f^p}{\langle \Phi(f^p(A)) x, x \rangle} \leq \left( \frac{M_1}{m_1} \right)^p \quad \text{and} \quad \left( \frac{m_2}{M_2} \right)^q \leq \frac{g^q}{\langle \Phi(g^q(A)) x, x \rangle} \leq \left( \frac{M_2}{m_2} \right)^q.$$

Therefore,

$$\min \left\{ \left( \frac{m_1}{M_1} \right)^p, \left( \frac{m_2}{M_2} \right)^q \right\} \leq \frac{f^p}{\langle \Phi(f^p(A))x, x \rangle}, \frac{g^q}{\langle \Phi(g^q(A))x, x \rangle} \leq \max \left\{ \left( \frac{M_1}{m_1} \right)^p, \left( \frac{M_2}{m_2} \right)^q \right\}$$

for  $x \in K$ ,  $\|x\| = 1$ . Let  $t \in I$ . By (3) we have for  $\nu = \frac{1}{q}$ ,  $a = \frac{f^p(t)}{\langle \Phi(f^p(A))x, x \rangle}$ ,  $b = \frac{g^q(t)}{\langle \Phi(g^q(A))x, x \rangle}$ ,  $m = \min \left\{ \left( \frac{m_1}{M_1} \right)^p, \left( \frac{m_2}{M_2} \right)^q \right\}$  and  $M = \max \left\{ \left( \frac{M_1}{m_1} \right)^p, \left( \frac{M_2}{m_2} \right)^q \right\}$  that

$$\frac{1}{p} \frac{f^p(t)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{1}{q} \frac{g^q(t)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{f(t)g(t)}{[\langle \Phi(f^p(A))x, x \rangle]^{1/p} [\langle \Phi(g^q(A))x, x \rangle]^{1/q}} \quad (41)$$

$$\leq S \left( \max \left\{ \left( \frac{M_1}{m_1} \right)^{\frac{p}{2}}, \left( \frac{M_2}{m_2} \right)^{\frac{q}{2}} \right\} - \min \left\{ \left( \frac{m_1}{M_1} \right)^{\frac{p}{2}}, \left( \frac{m_2}{M_2} \right)^{\frac{q}{2}} \right\} \right)^2$$

for  $x \in K$ ,  $\|x\| = 1$ . Using the functional calculus for  $A$ , we have

$$\frac{1}{p} \frac{f^p(A)}{\langle \Phi(f^p(A))x, x \rangle} + \frac{1}{q} \frac{g^q(A)}{\langle \Phi(g^q(A))x, x \rangle} - \frac{f(A)g(A)}{[\langle \Phi(f^p(A))x, x \rangle]^{1/p} [\langle \Phi(g^q(A))x, x \rangle]^{1/q}} \quad (42)$$

$$\leq S \left( \max \left\{ \left( \frac{M_1}{m_1} \right)^{\frac{p}{2}}, \left( \frac{M_2}{m_2} \right)^{\frac{q}{2}} \right\} - \min \left\{ \left( \frac{m_1}{M_1} \right)^{\frac{p}{2}}, \left( \frac{m_2}{M_2} \right)^{\frac{q}{2}} \right\} \right)^2$$

for  $x \in K$ ,  $\|x\| = 1$ . Taking the normalized map  $\Phi$  in (42) and then the inner product for  $y \in K$  with  $\|y\| = 1$ , we get

$$\frac{1}{p} \frac{\langle \Phi(f^p(A))y, y \rangle}{\langle \Phi(f^p(A))x, x \rangle} + \frac{1}{q} \frac{\langle \Phi(g^q(A))y, y \rangle}{\langle \Phi(g^q(A))x, x \rangle} - \frac{\langle \Phi(f(A)g(A))y, y \rangle}{[\langle \Phi(f^p(A))x, x \rangle]^{1/p} [\langle \Phi(g^q(A))x, x \rangle]^{1/q}}$$

$$\leq S \left( \max \left\{ \left( \frac{M_1}{m_1} \right)^{\frac{p}{2}}, \left( \frac{M_2}{m_2} \right)^{\frac{q}{2}} \right\} - \min \left\{ \left( \frac{m_1}{M_1} \right)^{\frac{p}{2}}, \left( \frac{m_2}{M_2} \right)^{\frac{q}{2}} \right\} \right)^2,$$

which for  $y = x$  produces the desired result (40). ■

**Remark 5** Assume that the operator  $A$  satisfy the inequality  $k1_H \leq A \leq K1_H$  for some numbers  $k, K$  with  $0 < k < K$ . If  $\alpha, \beta > 0$  and we take  $f(t) = t^\alpha$  and  $g(t) = t^\beta$  in Theorem 3.2, then  $m_1 = k^\alpha$ ,  $M_1 = K^\alpha$ ,  $m_2 = k^\beta$  and  $M_2 = K^\beta$  and by the inequality (40) we have

$$0 \leq 1 - \frac{\langle \Phi(A^{\alpha+\beta})x, x \rangle}{\langle \Phi(A^{\alpha p})x, x \rangle^{1/p} \langle \Phi(A^{\beta q})x, x \rangle^{1/q}} \quad (43)$$

$$\leq S \left( \max \left\{ \left( \frac{K}{k} \right)^{\frac{\alpha p}{2}}, \left( \frac{K}{k} \right)^{\frac{\beta q}{2}} \right\} - \min \left\{ \left( \frac{k}{K} \right)^{\frac{\alpha p}{2}}, \left( \frac{k}{K} \right)^{\frac{\beta q}{2}} \right\} \right)^2$$

for any  $x \in K$ ,  $x \neq 0$ .

## References

- [1] D. K. Callebaut, Generalization of Cauchy-Schwarz inequality, *J. Math. Anal. Appl.* 12 (1965), 491-494.
- [2] M. D. Choi, Positive linear maps on  $C^*$ -algebras, *Canad. J. Math.* 24 (1972), 520-529.
- [3] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* 74 (3) (2006), 417-478.
- [4] S. S. Dragomir, Čebyšev type inequalities for positive linear maps of selfadjoint operators in Hilbert spaces, *Math. Mora.* 21 (1) (2017), 1-15.
- [5] S. S. Dragomir, Grüss' type inequalities for positive linear maps of selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.* (2016), 19:62.
- [6] S. S. Dragomir, Operator quasilinearity of some functionals associated to Davis-Choi-Jensen's inequality for positive maps, *Bull. Austral. Math. Soc.* 95 (2) (2017), 322-332.
- [7] S. S. Dragomir, Some Hermite-Hadamard type inequalities for operator convex functions and positive maps, *Spec. Matrices.* 7 (2019), 38-51.
- [8] X. Fu, C. He, Some operator inequalities for positive linear maps, *Linear Multilinear Algebra.* 63 (3) (2015), 571-577.
- [9] F. Kittaneh, Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* 361 (2010), 262-269
- [10] F. Kittaneh, Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.* 59 (2011), 1031-1037.
- [11] J. S. Matharu, M. S. Moslehian, Grüss inequality for some types of positive linear maps, *J. Operator Theory.* 73 (1) (2015), 265-278.
- [12] H. R. Moradi, M. E. Omidvar, S. S. Dragomir, An operator extension of Čebyšev inequality, *An. St. Univ. Ovid. Cons.* 25 (2) (2017), 135-147.
- [13] H. R. Moradi, M. E. Omidvar, S. S. Dragomir, More operator inequalities for positive linear maps, *Bull. Math. Anal. Appl.* 9 (1) (2017), 65-73.
- [14] M. Niezgodá, Shannon like inequalities for  $f$ -connections of positive linear maps and positive operators, *Linear Algebra Appl.* 481 (2015), 186-201.
- [15] J. Pečarić, T. Furuta, J. Mičić, Y. Seo, *Mond-Pečarić Method in Operator Inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
- [16] R. Sharma, A. Thakur, More inequalities for positive linear maps, *J. Math. Inequal.* 7 (1) (2013), 1-9.
- [17] J. Xue and X. Hu, Some generalizations of operator inequalities for positive linear maps, *J. Inequal. Appl.* (2016), 2016:27.
- [18] P. Zhang, More operator inequalities for positive linear maps, *Banach J. Math. Anal.* 9 (1) (2015), 166-172.