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Assessment of performance based on fuzzy Production Possibility Set

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Abstract

Data envelopment analysis is a non-parametric technique for measuring and evaluating the relative efficiency of a set of entities with deterministic and precise inputs and outputs. In fact, in a real evaluation problem, the input and output data have variability; this variable data can be represented in the form of linguistic variables defined by fuzzy numbers. The integration of deterministic data envelopment analysis models with concepts and approaches from the field of fuzzy mathematical programming in the form of fuzzy data envelopment analysis allows for the assessment of the efficiency of decision-making units in the presence of imprecise, ambiguous, and fuzzy data. Accordingly, in this paper, a fuzzy production possibility set in scale returns based on the established principles to address the efficiency evaluation problem with fuzzy input and output data is proposed, and subsequently, the fuzzy CCR model applied to this fuzzy production possibility set is introduced, followed by the calculation of efficiency in a fuzzy manner in the input-oriented state and the analysis of the FCCR model.

Keywords Data Envelopment Analysis, Fuzzy Data Envelopment Analysis, Efficiency, Production Possibility Set

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1. Introduction

Data Envelopment Analysis (DEA) was developed by Charnes, Cooper, and Rhodes based on a linear programming model to measure the efficiency of decision-making units [1]. DEA models are designed to evaluate the relative efficiency of a set of decision-making units using various inputs to produce different outputs, limited to precise data in the production possibility set. The CCR model estimates the efficiency score and efficiency frontier based on Farrell's ideas. Several models have been introduced for evaluating efficiency concerning the production possibility set (PPS) [2]. Belman and colleagues introduced the fuzzy concept to address the imprecision in decision-making [3]. Since the models are fundamentally a fuzzy linear programming problem, there is extensive research on solving fuzzy linear programming problems [4]. Some researchers have proposed several fuzzy models for evaluating decision-making units with fuzzy data. Without access to the fuzzy production possibility set (FPSS), Ramazani and colleagues considered fuzzy random variables for inputs and outputs in data envelopment analysis [5]. Vano and colleagues proposed two new fuzzy data envelopment models from a fuzzy computational perspective to address the fuzziness of input and output data in DEA. One fuzzy data envelopment model was proposed based on measuring efficiency and also for ranking all units to solve the fuzzy model. Liu and Chang developed a fuzzy DEA/AR method based on the principle of Develay, which formulated a pair of two-stage mathematical programs to compute the lower and upper bounds and the fuzzy efficiency score [6-8]. Guo and Tanako presented a fuzzy data envelopment model based on the fundamental CCR model to solve the efficiency evaluation problem with fuzzy input and output data, such that the accurate efficiency in the CCR model is extended to a fuzzy number to reflect the inherent uncertainty in real evaluation problems [9]. Peikani and colleagues examined some fuzzy data envelopment models based on their applications. Additionally, Peikani and colleagues proposed a new fuzzy data envelopment model based on general fuzzy measurement, where the perspective of decision-making units is defined by optimistic-pessimistic parameters [10,11]. Kao and Liu followed the main idea of transforming a fuzzy data envelopment model into a family of conventional clear data envelopment models and developed a solution methodology for measuring the efficiencies of units with fuzzy observations in the BCC model [12]. Kuo and Wang utilized fuzzy data envelopment analysis to evaluate the performance of multinational companies facing foreign exchange rate risk volatility [13]. Li and colleagues suggested a discriminative fuzzy data envelopment analysis method for classifying fuzzy observations into two groups based on Suoshi's work. They employed Kao and Liu's method and replaced the fuzzy linear programming models with a pair of parametric models to determine the lower and upper bounds of efficiency scores [14,15]. Using Kao and Liu's approach and fuzzy analytic hierarchy procedure, Chiang and Che proposed a new fuzzy data envelopment analysis method with weight constraints for ranking new product development projects in an electronics company in Taiwan [16,17]. Saati and colleagues proposed a fuzzy CCR model as a probabilistic programming problem and converted it into an interval programming problem using an α-level based approach [18]. Hatami-Marabini and colleagues extended a fuzzy CCR model for evaluating decision-making units from the perspective of the best and worst possible relative efficiency using Leon and colleagues' approach [19]. Hosseinzadeh

Lotfi and colleagues proposed a fuzzy data envelopment model for evaluating a set of decision-making units where all parameters and decision variables were fuzzy numbers [20]. Hatami-Marabini and colleagues developed a fully fuzzy CCR model to obtain the fuzzy efficiency of decision-making units using a completely fuzzy LP model in which all data and input-output variables (including their weights) were fuzzy numbers [21]. In this article, a collection of fuzzy production possibilities in terms of outputs is proposed based on the established principles for addressing the issue of efficiency evaluation with fuzzy input and output data.

Following that, the fuzzy CCR model is introduced on this collection of fuzzy production possibilities, and we then proceed to calculate efficiency in a fuzzy manner in the input-oriented case and analyze the FCCR model.

2. Data Envelopment Analysis (DEA)

One of the most powerful, popular and widely used methods in the field of performance evaluation is the Data Envelopment Analysis approach, which has the ability to measure the efficiency, pattern finding, ranking and classification of decision-making units. It is necessary to explain that the Data Envelopment Analysis approach is one of the multi-criteria decision-making methods and non-parametric mathematical programming that calculates the efficiency of a set of homogeneous decision-making units using the two concepts of relative efficiency and production possibility set. Given the advantages and capabilities of the Data Envelopment Analysis approach, today the aforementioned approach is used significantly in various real-world areas and problems such as financial markets (stock markets, banks, insurance, investment funds), energy areas (electricity, gas, oil), service areas (hospitals, hotels), educational areas (schools, universities), transportation areas (airports, airlines), innovation areas (research and development projects), supply chain (blood, medicine, food), etc. One of the approaches to data envelopment analysis modeling is the production possibility set.

2.1 Production Possibility Set

All possible combinations of inputs and outputs are called Production Possibility Set. The boundary of this set is an approximation of the production function and the efficiency of the decision-making units is measured against this boundary. It is necessary to explain that in data envelopment analysis modeling under the envelopment form, the production possibility set approach is used. Suppose we have n observations in such a way that the input vector produces the output vector and it is assumed that and. In other words, at least one component of the input and output vectors is opposite to zero and positive. We consider the set T in such a way that the following principles are true [4][2]:

1. Principle of inclusion of observations: $\forall j\, (x_j, y_j) \in T$

2. Principle of infinite rays: If then3. Convexity Principle: If $(x, y) \in T$ then $\forall \lambda, \lambda \geq 0 \rightarrow (\lambda x, \lambda y) \in T$

3. Convexity Principle: If $(x, y) \in T, (x', y') \in T$ then $(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') \in T$ for every $\lambda \in [0,1]$

4. Feasibility Principle: If $(x, y) \in T, \bar{x} \geq x \;\&\; \bar{y} \leq y$ then $(\bar{x}, \bar{y}) \in T$.

5. Minimum Interpolation Principle: It is the smallest set that is true in the first to fourth principles.

We show a set with the above properties as follows:

$$T=\left\{\begin{array}{l}(x,y)\left|\sum_{j=1}^{n}\lambda_j x_j\le x,\right.\\ \sum_{j=1}^{n}\lambda_j y_j\ge y,\lambda_j\ge 0,\ j=1,\dots,\mathrm{n}\end{array}\right\} \quad (1)$$

2.2. CCR Model

The CCR model proposed by Charnes et al. [4] is as follows:

$$\begin{aligned} \min \quad & \theta \\ s.t \quad & \theta x_o \ge \sum_{j=1}^{n}\lambda_j x_j \\ & y_o \le \sum_{j=1}^{n}\lambda_j y_j \\ & \lambda_j \ge 0;\ j=1,\dots,n \end{aligned} \quad (2)$$

To construct the CCR model, we need $(\theta x_o, y_o)\in T$, when the goal is to find the minimum θ that decreases the input vector x_o radially towards θx_o while remaining at T. In the CCR model, we seek an activity at T that guarantees the output level Y_o of DMU_o in all components, while decreasing the input vector x_o periodically (radially) to a value as small as possible.

3. Data Envelopment Analysis and Uncertainty

One of the basic assumptions in using data envelopment analysis is the need for accurate and certain input and output data, because a small deviation in the data can lead to a significant change in the results related to the efficiency and ranking of decision-making units, especially when the efficiency of a particular unit is close to the efficiency of another unit. This is despite the fact that in many real-world situations and problems, such as financial markets, it is not possible to determine an exact numerical value for some inputs or outputs. Therefore, in such circumstances, there is a need to present and use data envelopment analysis models that can evaluate the efficiency of decision-making units in the presence of data uncertainty. To this end, researchers have presented uncertain data envelopment analysis models, considered the type of data uncertainty and used widely used approaches in the field of uncertain programming, such as stochastic optimization, fuzzy optimization, and robust optimization.

3.1 Introduction to Fuzzy Data Envelopment Analysis

Combining deterministic data envelopment analysis models with concepts and approaches from the field of fuzzy mathematical programming in the form of the fuzzy data envelopment analysis approach has made it possible to evaluate the efficiency of decision-making units in the presence of imprecise, ambiguous, and fuzzy data.

It should be noted that one of the very important capabilities of the fuzzy data envelopment analysis approach is its usability and implementation in the presence of variables and verbal expressions, which is why it has had many applications in different fields and areas. It is also important to note that a significant percentage and share of research conducted in the field of uncertain data envelopment analysis is related to the field of fuzzy data envelopment analysis, which itself has diverse and numerous approaches.

3.2. Fuzzy number

A fuzzy number $\tilde{A}$ is a fuzzy set of real numbers with normality, convexity (fuzziness) and continuous membership function from the bounded boundary. The family of all fuzzy numbers is represented by F. Suppose $\tilde{A}$ that is a fuzzy number, then $\tilde{A}_\alpha$ it is a closed convex subset of the real numbers for all $\alpha \in [0,1]$ and

$$a_1(\alpha) = \min \tilde{A}_\alpha, \quad a_2(\alpha) = \max \tilde{A}_\alpha \quad (3)$$

In other words, it represents the left arc and the right view is -cut. Furthermore, the left and right view functions are respectively:

$$a_2 : [0,1] \to R, \quad a_1 : [0,1] \to R \qquad (4)$$

So, we have [6]: $\tilde{A}_\alpha = [a_1(\alpha), a_2(\alpha)]$ (5)

3.3. LR Fuzzy Numbers

Any fuzzy number $\tilde{A} \in F$ can be described as:

$$\mu_{\tilde{A}}(t) = \begin{cases} L(\frac{a-t}{\alpha}), & if \quad t \in [a-\alpha, a] \\ 1, & if \quad t \in [a,b] \\ R(\frac{t-b}{\beta}), & if \quad t \in [b, b+\beta] \\ 0 & o.w \end{cases} \qquad (6)$$

where [a,b] is the vertex or tip $\tilde{A}$. α and β are the right and left widths $\tilde{A}$, respectively.

$$\text{L} : [0,1] \to [0,1], \quad \text{R} : [0,1] \to [0,1] \, (7)$$

The continuous and non-increasing form of the functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ is. This fuzzy interval can be called LR-variant and is denoted by $\tilde{A} = (a, b, \alpha, \beta)_{LR}$. An LR fuzzy number is triangular if a=b and L=R-1. A triangular fuzzy number with center a=b can be viewed as the fuzzy value "x approximately equals a"[6].

4. Fuzzy Generation Feasibility Set and Fuzzy CCR Model

In this section, we are going to define the fuzzy production possibility set using the axioms in constant returns to scale. Also, the FCCR model is introduced in the input-driven case.

4.1. Fuzzy Production Possibility Set

By having a production function, it is easy to calculate the efficiency of a decision-making unit, but for various reasons, the production function is not easily calculated and even in some cases it is impossible to obtain it. Therefore, we construct a set called the production possibility set and take its boundary as an approximation of the production function.

We denote the fuzzy production possibility set by FT and it is defined as:

FT= { $(\tilde{x}, \tilde{y})$ |A non-negative vector $\tilde{x}$ can produce a non-negative vector $\tilde{y}$ }

1. Inclusion of fuzzy observations: $(j = 1, \dots, n)(\tilde{x}_j, \tilde{y}_j) \in \text{FT}$

Let X be an arbitrary reference set. The characteristic function of any normal subset A of X is {0,1}, which is defined as:

$$\chi_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases} \qquad (8)$$

According to the above relation for $x \in X$, $\chi_A(x)$ it will take only one of the values 0 or 1. Now if we extend the range of the characteristic function from the set of two members {0,1} to the interval [0,1], we will have a function that assigns to each x of X a number in the interval [0,1]. This function is called the membership function of A. Now it is no longer a normal set but something called a fuzzy set (more precisely, a fuzzy subset of X). A fuzzy set $\tilde{A}$ is a function that assigns to each element of X a number from the interval [0,1] as the degree of membership of that element in the fuzzy set $\tilde{A}$, and it is denoted by $\mu_{\tilde{A}}(x)$ The value $\mu_{\tilde{A}}(x)$ close to one indicates that x belongs more to the fuzzy set $\tilde{A}$, and vice versa, its closeness to zero indicates that x belongs less to $\tilde{A}$. If x is completely in the member $\tilde{A}$, we will have: $\mu_{\tilde{A}}(x)=1$

And if it is not in the member $\tilde{A}$ at all, we will have: $\mu_{\tilde{A}}(x)=0$

Definition: If X is a reference set who's each member is represented by x, the fuzzy set $\tilde{A}$ in X is expressed by ordered pairs as follows:

$$\tilde{A}=\{(x,\mu_{\tilde{A}}(x)\,|\,x\in X\} \qquad (9)$$

Where the first component represents the members and the second $\mu_{\tilde{A}}(x)$ component represents the degree of membership, which is the degree of belonging to the fuzzy set $\tilde{A}$.

Thus, for each fuzzy decision-making unit with input $(x,\mu_{\tilde{A}}(x))$ and output $(\mathrm{y},\mu_{\tilde{A}}(\mathrm{y}))$, it will be as follows:

$$\tilde{A}=\{(x,y),\min(\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(\mathrm{y}))\} \quad (10)$$

2. Infinity of fuzzy ray (constant return to scale):

If $\tilde{x}$ produces $\tilde{y}$, then $\lambda\tilde{x}$ produces $\lambda\tilde{y}$ $(\lambda>0)$.

The multiplication of fuzzy numbers may be defined as a α cut multiplication or using the expansion principle. Using the cut α , the multiplication of two numbers $\tilde{x}$ and $\tilde{y}$ is defined as follows:

$$\tilde{X}.\tilde{Y}=\left[a_1^{(\alpha)},a_2^{(\alpha)}\right].\left[b_1^{(\alpha)},b_2^{(\alpha)}\right]=\left[a_1^{(\alpha)}.b_1^{(\alpha)},a_2^{(\alpha)}.b_2^{(\alpha)}\right] \quad (11)$$

A special case of fuzzy multiplication is the multiplication of a definite number by a fuzzy number. λ Consider as a definite positive real number and $\tilde{x}$ as a fuzzy number defined on the reference set of positive real numbers. We define the multiplication λ by $\tilde{x}$ as interval multiplication or by the expansion principle. A definite number λ may be considered as a small interval whose left and right endpoints are the same and is $\lambda=[\lambda,\lambda]$ of the form:

$$\lambda.\tilde{x}=[\lambda,\lambda].[\tilde{x}_1^{(a)},\tilde{x}_2^{(a)}]=[\lambda\tilde{x}_1^{(a)},\lambda\tilde{x}_2^{(a)}] \quad (12)$$

3. Fuzzy Convexity:

If $(\tilde{x}_1,\tilde{y}_1)\in FT$ and $(\tilde{x}_2,\tilde{y}_2)\in FT$ then for every $0\le\lambda\le 1$ we have $\lambda(\tilde{x}_1,\tilde{y}_1)+(1-\lambda)(\tilde{x}_2,\tilde{y}_2)\in FT$

The algebraic sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A}+\tilde{B}$, is defined as a fuzzy set with the following membership function:

$$\tilde{A}+\tilde{B}: X\to[0,1]$$
$$(\tilde{A}+\tilde{B})(x)=\tilde{A}(x)+\tilde{B}(x)-\tilde{A}(x).\tilde{B}(x) \qquad (13)$$

Comparing triangular fuzzy numbers with respect to different criteria is also very difficult and time-consuming. As a result, the following method is proposed when dealing with triangular fuzzy numbers. Suppose that the values of the capacity of the arcs exist, the flows or the transportation costs are in the form of triangular fuzzy numbers. Then when adding (subtracting) the two original triangular fuzzy numbers, their centers are added (subtracted) and to calculate the deviations it is necessary to define the required value for the adjacent values. Suppose the fuzzy capacity (flow or transportation cost) "near" x is between two dependent values "near x_1" and "near x_2", ($x_1 \leq$ x $\leq x_2$) whose membership functions $\mu_{x_1}(x_1)$ and $\mu_{x_2}(x_2)$ are in the form of a triangular number, as shown in the figure below.

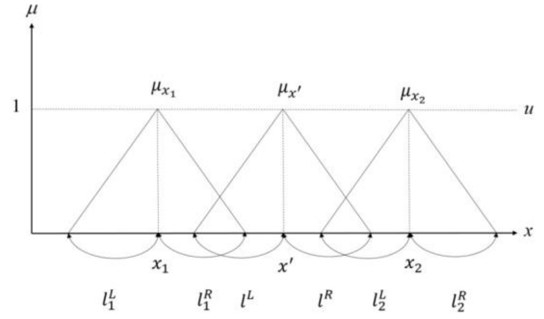

Figure 1: Convex combination of two triangular fuzzy numbers

Therefore, the set of membership function boundaries of the fuzzy arc capacity (flow or cost of transportation) "near" x can be represented based on the linear combination of the left and right boundaries of the adjacent values as follows:

$$l^L = \frac{(x_2 - x)}{(x_2 - x_1)} \times l_1^L + (1 - \frac{(x_2 - x)}{(x_2 - x_1)}) \times l_2^L$$
$$l^R = \frac{(x_2 - x)}{(x_2 - x_1)} \times l_1^R + (1 - \frac{(x_2 - x)}{(x_2 - x_1)}) \times l_2^R \quad (14)$$

In equation (14), l^L the left deviation boundary is the required fuzzy number, l^R the right deviation boundary is. In the case when the central value of the triangular number is repeated by adding (subtracting) the specified value on the number axis, its deviation boundaries coincide with the deviation boundaries of the specified number on the number axis. If the required central value is not between two numbers, but its value is specified for the first time on the number axis, its deviation boundaries are specified on the axis at the same time as the previously specified values. The same is true when the required central value is specified for the last value.

4. Fuzzy feasibility:

If $(\tilde{\bar{x}}, \tilde{\bar{y}}) \in T$ then for each $(\tilde{x}, \tilde{y})$ where $\tilde{x} \geq \tilde{\bar{x}}$ and $\tilde{y} \leq \tilde{\bar{y}}$ we have $(\tilde{x}, \tilde{y}) \in FT$.

An effective method for comparing fuzzy numbers is to use the ranking function $\mathfrak{R}: \tilde{F}(R) \rightarrow R$, which $\tilde{F}(R)$ is a set of fuzzy numbers defined on the set of real numbers that maps each fuzzy number to the real line and must exist in a natural order, in other words:

$$\begin{aligned} &(i)\ \tilde{A} > \tilde{B} \Leftrightarrow \mathfrak{R}(\tilde{A}) > \mathfrak{R}(\tilde{B}) \\ &(ii)\ \tilde{A} < \tilde{B} \Leftrightarrow \mathfrak{R}(\tilde{A}) < \mathfrak{R}(\tilde{B}) \\ &(iii)\ \tilde{A} = \tilde{B} \Leftrightarrow \mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B}) \end{aligned} \quad (15)$$

Suppose $\tilde{A} = (a, b, c)$ a fuzzy number is triangular, then we have: $\mathfrak{R}(\tilde{A}) = \frac{a + 2b + c}{4}$

And if $\tilde{A} = (a, b, c, \mathrm{d})$ a fuzzy number is trapezoidal, then we have: $\Re(\tilde{A}) = \dfrac{a+b+c+d}{4}$

Suppose we have n observations $(j = 1,\ldots,n)(\tilde{x}_j, \tilde{y}_j)$ such that the input vector $\tilde{x}_j$ produces the output vector $\tilde{y}_j$ and it is assumed that $(j = 1,\ldots,n)\tilde{x}_j \geq 0, \tilde{x}_j \neq 0$ and $(j = 1,\ldots,n)\,\tilde{y}_j \geq 0, \tilde{y}_j \neq 0$ we consider the set of possibilities of production (T) in such a way that the following principles hold:

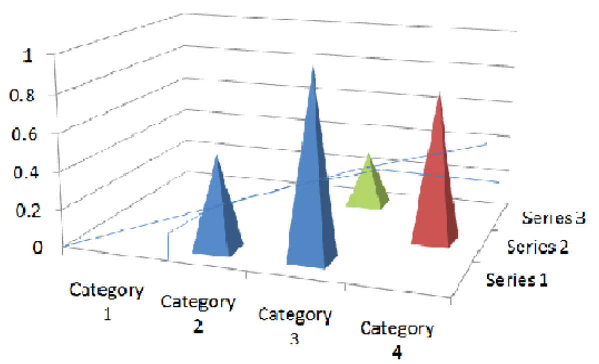

Figure 2: The region of the smallest possible set of fuzzy production with one input and one output

Suppose we have n observations $(j = 1,\ldots,n)(\tilde{x}_j, \tilde{y}_j)$ such that the input vector $\tilde{x}_j$ produces the output vector $\tilde{y}_j$ and it is assumed that $(j = 1,\ldots,n)\tilde{x}_j \geq 0, \tilde{x}_j \neq 0$ and $(j = 1,\ldots,n)\,\tilde{y}_j \geq 0, \tilde{y}_j \neq 0$ we consider the set of possibilities of production (T) in such a way that the following principles hold:

1. Principle of inclusion of observations: $\forall j\,(\tilde{x}_j, \tilde{y}_j) \in FT$ (according to the definition of inclusion of fuzzy observations)

2. Principle of infinity of rays: if $(\tilde{x}, \tilde{y}) \in FT$ then $\forall \lambda, \lambda \geq \cdot \rightarrow (\lambda\tilde{x}, \lambda\tilde{y}) \in FT$ (according to the definition of infinity of fuzzy rays).

3. Principle of convexity: if $(\tilde{x}, \tilde{y}) \in T, (\tilde{x}', \tilde{y}') \in FT$ then $(\lambda\tilde{x} + (1-\lambda)\tilde{x}', \lambda\tilde{y} + (1-\lambda)\tilde{y}') \in T$ for each $\lambda \in [0,1]$ (according to the definition of convexity in fuzzy).

4. Feasibility principle: If $(\tilde{x}, \tilde{y}) \in FT, \tilde{\tilde{x}} \geq \tilde{x} \;\&\; \tilde{\tilde{y}} \leq \tilde{y}$ then $(\tilde{\tilde{x}}, \tilde{\tilde{y}}) \in FT$ (according to the definition of feasibility in fuzzy).

5. Minimum interpolation principle: By accepting this principle, we accept that FT is the smallest set that is valid in the first to fourth principles.

4.2. FCCR model

Suppose we have n decision-making units, each of which produces s fuzzy outputs $\tilde{y}_{rj} = (y^l_{rj}, y^m_{rj}, y^u_{rj})$ using m fuzzy inputs $\tilde{x}_{rj} = (x^l_{rj}, x^m_{rj}, x^u_{rj})$, all of which are positive. Fuzzy efficiency models are used to calculate the lower bound, the middle value, and the upper bound of the efficiency with respect to different boundaries of a decision-making unit, whose covering form is as follows:

$$\theta^{*l} = \min \theta$$
$$s.t \quad \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j x_{ij}^{l} + \lambda_p x_{ip}^{u} \le \theta x_{ip}^{u}$$
$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j y_{rj}^{u} + \lambda_p y_{rp}^{l} \ge y_{rp}^{l} \qquad (16)$$
$$\lambda_j \ge 0$$

and

$$\theta^{*m} = \min \theta$$
$$s.t \quad \sum_{j=1}^{n} \lambda_j x_{ij}^{m} \le \theta x_{ip}^{m}$$
$$\sum_{j=1}^{n} \lambda_j y_{rj}^{m} \ge y_{rp}^{m} \qquad (17)$$
$$\lambda_j \ge 0$$

and

$$\theta^{*u} = \min \theta$$
$$s.t \quad \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j x_{ij}^{u} + \lambda_p x_{ip}^{l} \le \theta x_{ip}^{l}$$
$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j y_{rj}^{l} + \lambda_p y_{rp}^{u} \ge y_{rp}^{u} \qquad (18)$$
$$\lambda_j \ge 0$$

Definition 4.1. Suppose $x_1, x_2 \in S$ that in this case the point x_1 is better than x_2 if and only if $F(x_1) \underset{\neq}{\ge} F(x_2)$.

$$\max \quad F(x) = \{ f_1(x), \ldots, f_n(x)\}$$
$$s.t. \quad x \in S \qquad (19)$$

Theorem 4.1. $(\tilde{x}_p, \tilde{y}_p)$ In the fuzzy CCR model, is efficient if and only if $\theta^{*l} \le \theta^{*m} \le \theta^{*u} = 1$.

Proof: Suppose $\tilde{y}_p = (y_p^l, y_p^m, y_p^u), \tilde{x}_p = (x_p^l, x_p^m, x_p^u)$ that in this case we have; In model (17) if $\theta^* = 1$ since it is a minimization problem then definitely the optimal solution will be less than equal to 1, so $\theta^* \le 1$. On the other hand, $\theta^* > 0$, since then $x_p \underset{\neq}{\ge} 0$ at least one such as $x_{1p} > 0$ so the left side of the first constraint must be negative, but considering that $x_{ij} > 0, \lambda_j > 0$ it is not possible. Therefore, $\theta^* > 0$ on the other hand, $\theta^* \ne 0$ because if becomes zero, the right side of the first condition becomes zero and it is not possible due to the fact that it is positive x_{ij}, λ_j . Therefore $\theta^* > 0$, in the model (16) and (18) we have:

If $\theta^* < 1$ then $\lambda_p = 0$ and if $\theta^* = 1$ in this case $\lambda_p^* = 0$ the answer will be different or $\lambda_p^* = 1$.

In other words, if $0 < \theta^* < 1$ and $\lambda_p^* = 0$ then $(\theta^* - \lambda_p^*) \ge 0$, if $\theta^* = 1$ and $\lambda_p^* = 0$ then $(\theta^* - \lambda_p^*) \ge 0$

And if $\theta^* = 1$ and $\lambda_p^* = 1$ then $(\theta^* - \lambda_p^*) \ge 0$ Now we must prove $\theta^{*l} < \theta^{*m} < \theta^{*u}$

First, we show $\theta^{*l} < \theta^{*m}$ (i.e., the optimal answer θ^{*m} is a feasible answer θ^{*l}). Suppose (λ^*, θ^{*m}) the optimal answer to problem (17) is therefore true in conditions (17).

$$\sum_{j\neq p}\lambda_j^* x_{ij}^l \le \sum_{j\neq p}\lambda_j^* x_{ij}^m \le (\theta^{*^m}-\lambda_p^*)x_{ip}^m \le (\theta^{*^m}-\lambda_p^*)x_{ip}^u \quad (20)$$
$$\sum_{j\neq p}\lambda_j^* y_{rj}^u \ge \sum_{j\neq p}\lambda_j^* y_{rj}^m \ge (1-\lambda_p^*)\,y_{rp}^m \ge (1-\lambda_p^*)\,y_{rp}^l$$

As a result, we have:

$$\begin{aligned}&\sum_{j\neq p}\lambda_j^* x_{ij}^l + \lambda_p^* x_{ip}^u \le \theta^{*m} x_{ip}^u \qquad ;\forall i\\ &\sum_{j\neq p}\lambda_j^* y_{rj}^u + \lambda_p^* y_{rp}^l \ge y_{rp}^l \qquad ;\forall r\end{aligned} \quad (21)$$

In this order, (λ^*,θ^{*m}) a feasible answer to problem (16) is therefore $\theta^{*^l} \le \theta^{*^m}$. Now we show $\theta^{*^m} < \theta^{*^u}$ (i.e., the optimal answer θ^{*^u} is a feasible answer θ^{*^m}). For this purpose, (λ^*,θ^{*^u}) suppose the optimal answer to problem (18) is therefore true in conditions (18).

$$\sum_{j\neq p}\lambda_j^* x_{ij}^m \le \sum_{j\neq p}\lambda_j^* x_{ij}^u \le (\theta^{*^u}-\lambda_p^*)x_{ip}^l \le (\theta^{*^u}-\lambda_p^*)x_{ip}^m \quad (22)$$
$$\sum_{j\neq p}\lambda_j^* y_{rj}^m \ge \sum_{j\neq p}\lambda_j^* y_{rj}^l \ge (1-\lambda_p^*)\,y_{rp}^u \ge (1-\lambda_p^*)\,y_{rp}^m$$

As a result, we have:

$$\begin{aligned}&\sum_{j\neq p}\lambda_j^* x_{ij}^m + \lambda_p^* x_{ip}^m \le \theta^{*^m} x_{ip}^m \qquad ;\forall i\\ &\sum_{j\neq p}\lambda_j^* y_{rj}^m + \lambda_p^* y_{rp}^m \ge y_{rp}^m \qquad ;\forall r\end{aligned} \quad (23)$$

Therefore, (λ^*,θ^{*^u}) a feasible solution to problem (17) is, so $\theta^{*^m} \le \theta^{*^u}$ for each decision-making unit, an efficiency is obtained in triplicate $(\theta^l,\theta^m,\theta^u)$.

4.3. Fuzzy Efficiency Analysis

The efficiency of each fuzzy decision-making unit is in the form of a triple $(\theta^l,\theta^m,\theta^u)$. Considering this, four cases can be defined:

$$\begin{aligned}&E^{+++}=\{DMU_j \mid \theta_j^l=\theta_j^m=\theta_j^u=1\}\\ &E^{++}=\{DMU_j \mid \theta_j^l<1, \theta_j^m=\theta_j^u=1\}\\ &E^{+}=\{DMU_j \mid \theta_j^l<1, \theta_j^m<1, \theta_j^u=1\}\\ &E^{-}=\{DMU_j \mid \theta_j^l<1, \theta_j^m<1, \theta_j^u<1\}\end{aligned} \quad (24)$$

In this way, it can be said that if E^- only $\theta_j^u<1$ is written in is sufficient because it is proved that $\theta^{*^l} \le \theta^{*^m} \le \theta^{*^u}$ .in E^+ is sufficient $\theta_j^m<1, \theta_j^u=1$. in E^{++} is sufficient $\theta_j^l<1, \theta_j^m=1$ and in E^{+++} is sufficient $\theta_j^l=1$.

5. Example

27 branches of Tehran Social Security Organization have been evaluated. Each branch produces three outputs with three inputs. The labels of inputs and outputs are presented in the table below. The general fuzzy triangular data can be seen in Tables 1 and 2. "L" is considered as the lower bound, "M" as the middle value, "U" as the upper bound. We run the FCCR model on 27 DMUs and show its results in Table 3. Table 3 shows the relative efficiency of all DMUs. In this table, $(\theta^{*^l},\theta^{*^m},\theta^{*^u}) \in (0.1]$ the efficiency value is relative and the higher it is, the better. If it is,then the $\theta_j^l=\theta_j^m=\theta_j^u=1$ decision-making unit is efficient in all relevant parts and if it is $\theta_j^l<1, \theta_j^m<1, \theta_j^u=1$ or $\theta_j^l<1, \theta_j^m=\theta_j^u=1$, then there is a part or parts in the decision-making unit that is inefficient and if it is $\theta_j^l<1, \theta_j^m<1, \theta_j^u<1$ then the decision-making unit is inefficient.
Thus, units 8 and 10 are considered inefficient and units 9, 18, 20 are considered fully efficient.

6. Conclusion

In the real world, there are many problems with fuzzy parameters, such as DEA models for evaluating the relative efficiency of a set of decision data with fuzzy data. Since DEA models are basically proposed by the production possibility set, in this paper, the production possibility set with fuzzy data in the fixed-scale efficiency is proposed according to the axioms, and then the fuzzy CCR model is introduced on this fuzzy production possibility set. One of the most important advantages of this work is that the FCCR is constructed using the axioms in the fuzzy production possibility set. We hope that other DEA topics of fuzzy data will be used through the use of the fuzzy production possibility set.

Table 1: Inputs of triangular fuzzy numbers for 27 branches of the Social Security Organization

	(L_{I1}, M_{I1}, U_{I1})	(L_{I2}, M_{I2}, U_{I2})	(L_{I3}, M_{I3}, U_{I3})
DMU1	(59153.05,73504.25,86523)	(84,86,87)	(3998.34,4000,4000)
DMU2	(36216,36582.75,37112)	(88,89.78,92)	(2565,2565,2565)
DMU3	(24566,25002.25,25369)	(85,87,89)	(1343,1344,1345)
DMU4	(36078,36684.75,37722)	(93,94,96)	(1499.7,1500,1500)
DMU5	(34734,36834.5,39445)	(83,83.44,87)	(1680,1681,1682)
DMU6	(58344,62611.63,73005)	(97,97,97)	(3748,3750,3750)
DMU7	(38849.02,41572.77,42573.02)	(90,90.67,92)	(3312.78,3313,3313)
DMU8	(51410,55949.63,63341)	(92,92,92)	(1500,1500,1500)
DMU9	(91930,95522.75,100220)	(84,89.56,92)	(1600,1601,1603.33)
DMU10	(52695,59080.63,61767)	(95,95.22,97)	(1725,1725,1725)
DMU11	(35985,40736.12,54521)	(78,78.33,79)	(1919.78,1920,1920)
DMU12	(26687,27300.63,27712)	(89,89.67,91)	(4430,4430,4430)
DMU13	(61716,63295,65026)	(103,106.33,111)	(2500,2500,2500)
DMU14	(93116,94969,96821)	(92,93,95)	(2800,2801,2802)
DMU15	(44305,50062,52856)	(92,93.44,95)	(1628.67,1630,1632)
DMU16	(41418,45926.25,48429)	(85,85,85)	(1127,1127,1127)
DMU17	(81588,82202.5,82923)	(104,104,104)	(3399,3400,3400)
DMU18	(72553,88782.5,98678)	(91,92.33,95)	(1304,1304,1304)
DMU19	(84905,87247.25,898844)	(95,96.11,98)	(4206,4206,4206)
DMU20	(26927,33196.5,37990)	(100,100.44,101)	(1340,1340,1343)
DMU21	(27744,28402.75,29058)	(88,89.33,90)	(1392.78,1393,1393)
DMU22	(107513,122897.88,129100)	(120,121.89,123)	(2191,2191,2191)
DMU23	(27704,32587.75,35194)	(100,100,100)	(2140,2140,2142.25)
DMU24	(60140,60866.38,61760)	(91,92,93)	(1231,1231,1231)
DMU25	(83940,86429.88,88038)	(90,91,92)	(1960,1960,1960)
DMU26	(58765,66803.13,69733)	(87,93.56,98)	(1603,1603,1603)
DMU27	(39142,40156.13,40860)	(81,82.33,86)	(2300,2300,2300)

Table 2: Outputs of triangular fuzzy numbers for 27 branches of the Social Security Organization

	(L_{o1}, M_{o1}, U_{o1})	(L_{o2}, M_{o2}, U_{o2})	(L_{o3}, M_{o3}, U_{o3})
DMU1	(55830,57029.56,58487)	(30,40.89,58)	(1117,1269,1350)

DMU2	(36740,36872,37110)	(5,18.56,35)	(8385,8543.33,8776)
DMU3	(38004,38680,39449)	(11,20.22,32)	(6588,6594.78,6603)
DMU4	(35469,35933,36651)	(10,32.44,59)	(8083,10516.56,10821)
DMU5	(52927,54457.78,56082)	(9,30.11,44)	(9493,9684.67,9955)
DMU6	(70253.89,72277,78573.89)	(1,11.78,19)	(7536,8022,8752)
DMU7	(32585,36625,39539)	(47,101.22,129)	(13121,14513.33,15264)
DMU8	(42900,46360.33,50028)	(11,17.11,27)	(1563,1622.56,1661)
DMU9	(84531,86063.33,87858)	(43,71.44,111)	(10206,10645,11080)
DMU10	(46924,47242.11,47800)	(9,26.22,36)	(6608,6824.33,7472)
DMU11	(31554,38977.78,44298)	(81,184.11,242)	(11996,12226,12582)
DMU12	(27169,38214.89,39620)	(11,21.78,31)	(7422,7561.78,7731)
DMU13	(56144,58340.44,60545)	(30,45.22,77)	(7380,7584,7936)
DMU14	(80425,88472,91461)	(28,40,66)	(630,661,707)
DMU15	(50210,50499,50728)	(6,14.33,24)	(10247,10264.11,10293)
DMU16	(40166,47907.22,49855)	(15,25,37)	(7302,7491.56,7786)
DMU17	(52923,59579,82222)	(14,19.89,29)	(4740,4952.8,5205)
DMU18	(77340,83075.11,89111)	(13,23.44,33)	(4745,4917,5151)
DMU19	(46154,51026.65,86330)	(13,17.56,25)	(825,1528.33,1636)
DMU20	(27978,29658.11,33038)	(23,77.33,325)	(14473,14766.33,15125)
DMU21	(27128,27735,28297)	(0.1,19.22,36)	(921,940.67,973)
DMU22	(102175,102855,103641)	(31,47.56,77)	(252,2510.44,3577)
DMU23	(31819,34063.67,36205)	(12,23.11,32)	(1963,2110.89,2257)
DMU24	(51345,53731.33,56514)	(35,63.33,73)	(10157,10219.56,10344)
DMU25	(72915,75776,85431)	(40,53.89,96)	(4193,4480.33,4705)
DMU26	(71743,72552.67,74218)	(50,75.89,96)	(8762,12091.22,13015)
DMU27	(38054,38630.78,39065)	(13,24.89,39)	(1405,1460.56,1516)

Table 3: Efficiency results for 27 branches of the Social Security Organization

	L	M	U
DMU01	0.627	0.7731	1
DMU02	0.6749	0.7572	1
DMU03	0.9278	1	1
DMU04	0.6782	0.8431	1
DMU05	0.9741	1	1
DMU06	0.7821	0.9842	1
DMU07	0.8841	1	1
DMU08	0.5227	0.7216	0.8638
DMU09	1	1	1
DMU10	0.579	0.6741	0.8685
DMU11	0.9528	1	1
DMU12	0.6727	0.9632	1
DMU13	0.6288	0.7575	1
DMU14	0.8227	1	1
DMU15	0.7424	0.8566	1
DMU16	0.7788	0.9498	1
DMU17	0.5322	0.69	1
DMU18	1	1	1
DMU19	0.4503	0.5946	1

DMU20	1	1	1
DMU21	0.5802	0.6611	1
DMU22	0.8077	0.9055	1
DMU23	0.5599	0.6986	1
DMU24	0.9611	1	1
DMU25	0.7808	0.919	1
DMU26	0.8796	1	1
DMU27	0.6137	0.6965	1

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