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Ranking general two-stage feedback systems: Navigating undesirable exogenous inputs

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Abstract. Data Envelopment Analysis (DEA) is a widely used method for evaluating the efficiency of decision-making units in real-world applications. Network Data Envelopment Analysis (NDEA) extends this approach by assessing the efficiency of network systems, taking into account internal processes within departments. A key challenge in evaluating such systems is ranking efficient units, particularly when undesirable data and feedback mechanisms are present. While previous research has explored ranking methods for network systems, no study has addressed the ranking of systems that simultaneously involve undesirable data and feedback loops. This study proposes a novel model for ranking general two-stage feedback systems with undesirable exogenous inputs. A structural numerical example is provided to demonstrate the model's applicability and effectiveness. The results show that the proposed model successfully evaluates and ranks two-stage systems, offering valuable insights for decision-makers. By addressing this gap in the literature, the research provides a practical tool for analyzing complex network systems with undesirable data and feedback, advancing the field of DEA.

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Index to information contained in this paper

1. Introduction

Data Envelopment Analysis (DEA) is an approach to determine the performance of the units under evaluation. Basic models in data envelopment analysis determine the efficiency score of units and help inefficient units turn into efficient ones by reducing inputs or increasing outputs. One of the main concepts in DEA is the unit ranking. The rank of each unit provides information regarding the priority of the unit and defines its superiority in terms of efficiency over other units.

Numerous methods have been presented to rank efficient and inefficient units so far. Sexton et al. introduced a model for ranking efficient and inefficient units using the cross-efficiency method [1]. This model's challenge was to achieve the same efficiency results for some units. To solve this problem, Oral et al. introduced secondary goals [2]. After that, Torgersen et al. introduced efficient units which were the reference of a large number of inefficient units as the superior units in terms of ranking [3]. Also, Peterson and Anderson introduced a super-efficiency method for ranking units by removing the units under evaluation from the constraints [4]. The disadvantage of this model was that it might be infeasible from the input side. Researchers proposed some models to solve this challenge. For example, Mehrabian et al. introduced a non-radial model for ranking efficient units [5]. Jahanshahloo et al. introduced a model based on the rotation vectors of the hyperplanes of the production possibility set [6].

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Decision-making units may have different internal divisions in which processes are carried out. Traditional DEA methods ignore these processes and consider the system a black box that prevents access to valuable information and yields incorrect efficiency scores. Therefore, to study the efficiency of a DMU, it is necessary to identify its components so that the cause of any inefficiency can be identified. This idea was suggested by Charnes et al. In 1986, they found that military recruitment has two stages: creating awareness through advertising and signing contracts [7].

Downsizing operations to calculate efficiency and determine the true impact of various factors is very important. In 2000, Fare and Grosskopf introduced the Network Data Envelopment Analysis (NDEA) concept to better account for the operation of individual processes within a system [8]. This approach considers the internal structure of the evaluated system and examines a sequence of processes across different divisions.

Some methods have been provided for ranking network systems. To evaluate senior managers of US public banks, Khodabakhshi et al. presented an input-based evaluation model. [9]. Sadjadi et al. proposed a robust model for ranking different gas companies in Iran. [10]. Razavi et al. presented a radial model for evaluating and ranking two-stage network systems [11]. Gheisari et al. presented a model for ranking and calculating productivity changes in two-stage network systems [12].

It is common for real-world problems analyzed using DEA to have non-normal data. Such issues can involve special types of data that require unique approaches. For instance, there may be special data without physical value, like fuzzy, stochastic, interval, or ordinal data, present in the production process. Many researchers have examined such data using DEA. Fallah et al. (2020) conducted a study combining Discriminant Analysis and Data Envelopment Analysis for specific data [13]. Pourmahmoud and Norouzi (2022) proposed a new method for evaluating and ranking DMUs with ordinal data [14].

In some problems in the real world, we may have inputs that need to be increased to improve the system, as well as some outputs that need to be decreased. In this case, we have undesirable data in the system. This may even happen in feedback network systems, which means that undesirable data are present in the mentioned systems. In the following section, we will explain each of these undesirable factors and two-stage feedback systems in detail.

One of the key challenges in evaluating feedback network systems with undesirable data lies in ranking them effectively. Pourmahmoud and Norouzi (2023) assessed the efficiency of DMUs with undesirable feedback inputs [15], successfully ranking the units without accounting for their individual impacts on the frontier. This study introduces a novel model for two-stage feedback systems with undesirable feedback inputs. The proposed approach not only calculates the efficiency of network systems but also determines each system's influence on the PPS frontier by identifying the maximum deviations. This refinement addresses a critical request from managers, as we present, for the first time, a model to rank two-stage feedback systems in the presence of undesirable exogenous inputs.

In the following, the second section presents the basic conceptions. Section 3 provides the proposed model for evaluating and ranking mentioned systems. In the fourth section, we examine and analyze the results. We do this by presenting a numerical example. Finally, we present the conclusion in section 5.

2. Methodology

2.1. Undesirable data

Some systems produce products that are not desirable, such as environmental pollutants emitted from economic activities. These products are referred to as undesirable outputs. Traditional DEA methods improve the efficiency of units by reducing inputs or increasing outputs. However, reducing inputs and expanding outputs also include unwanted data.

Therefore, these methods ignore these data and may yield incorrect results during calculations. Several models have been introduced to deal with undesirable factors, such as data transformation, input-output swapping, loose-based measurements, and weak disposables [16].

In 1983, Pittman proposed the concept of undesirable outputs, which was studied by many researchers [17]. Sifford and Zhou (2002) developed a model based on the BCC, which included both desirable and undesirable data. In their model, the undesirable outputs were multiplied by a negative [18]. Fare and Grosskopf identified a challenge with the model as it produced different answers, which was eventually resolved by Sifford and Zhou in 2004 [19] through the definition of a directed distance function. Jahanshahloo et al. (2004) used multi-objective linear programming to solve issues with undesirable data [20]. Kordrostami and Amirteimoori (2005) presented a multi-stage model that used undesirable variables with negative signs to calculate weights [21]. In 2006, Amirteimoori et al. proposed a model to improve efficiency by increasing undesirable inputs and reducing undesirable outputs [22]. Akhtar et al. (2013) presented a model that minimized undesirable outputs while maximizing desirable outputs [23]. Homayounfar and Amirteimoori (2016) applied a fuzzy network model to study undesirable data [24].

The data envelopment analysis approach has also been used in network structures with undesirable data. Madadi et al. (2018) used an allocation method to evaluate branches of an I Tejarat bank with undesirable data [25]. Teimourzadeh et al. (2019) classified the selected road safety indicators into desirable and undesirable indicator groups [26]. Seihani Parashkouh et al. (2020) proposed two non-linear technologies based on weak for two-stage systems in the presence of undesirable outputs [27]. Mahboubi et al. (2021) estimated marginal rates of substitutions in two-stage processes with undesirable Factors [28]. Omrani et al. (2022) developed NDEA with negative input and undesirable outputs [29]. Pourmahmoud and Norouzi tried to extend of CCR model to evaluate two-stage network systems in the presence of undesirable and non-discretionary data. They evaluated general two-stage systems by defining a parameter for each division of the system [30]. Finally, Shirvani and Azizi presented a model to extend a Two-Stage NDEA model with desirable and undesirable data (2022) [31].

In this article we use input–output exchange approach to manage undesirable data. The major characteristic of an undesirable factor is that the conventional directions of increasing the outputs and decreasing the inputs have opposite effects. From this characteristic it seems reasonable to treat an undesirable input as an output, and an undesirable output as an input, so that they have the expected direction. The basic model used in this study for undesirable data is the model of Banker et al. (1984), which is referred to below.

$$
\begin{aligned}
& E_o = \max \sum_{r=1}^{s} u_r Y_{ro} - u_o \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{m} v_i X_{io} = 1 \qquad (1) \\
& \sum_{r=1}^{s} u_r Y_{rj} - u_o - \sum_{i=1}^{m} v_i X_{ij} \le 0, \qquad j = 1,2,\dots,n \\
& u_r, v_i \ge \varepsilon, \quad for\ all\ r, i \ \text{and } u_o \text{ is free.}
\end{aligned}
$$

In running the above model in the presence of undesirable data an undesirable input is considered as an output, and an undesirable output as an input.

2.2. General two-stage feedback system

In some systems, a stage may have several related sections in different structures. The simplest type of these systems is the basic structure with two stages, where all the outputs of the first section are consumed in the second section. These systems are known as basic two-stage systems. In two-stage network systems, some outputs of the second division are fed back to the first as a part of the inputs for production. In this case, we have a general two-stage feedback system. The structure of the general two-stage feedback system is shown in Figure 1.

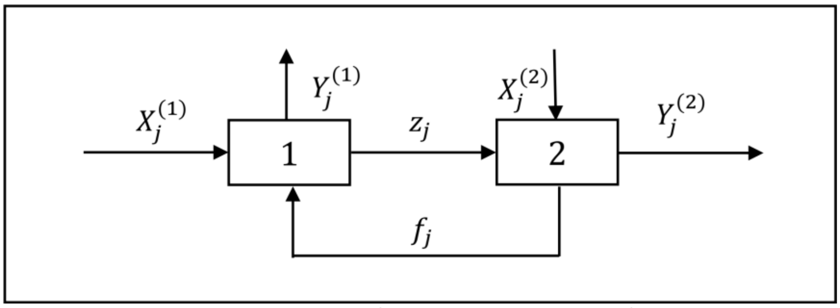

Figure 1. The structure of the general two-stage feedback system.

The first study of feedback systems in DEA is Liang et al. Their model examines the university performance [32]. Other examples of the feedback system are the recycling of waste materials and wastewater. Furthermore, several studies have been done on network systems with feedback data, some of which are mentioned here. Tavassoli et al. used an SBM model to evaluate Iran's domestic airlines [33]. They considered human resources as a shared input of both two stages. Wu et al. proposed a DEA model to evaluate two-stage network systems in the presence of feedback data and additional exogenous inputs [34]. Hu et al. presented a super-efficiency model to evaluate two-stage network systems in the presence of feedback data and shared input [35]. They considered invested capital as the shared input and reused water as feedback data. Wang et al. used cooperative and non-cooperative DEA models for two-stage systems in the presence of intermediate factors, shared inputs, and feedback factors, to evaluate China's high-tech industry [36].

In the following, we will refer to the basic model used in this paper. The model is presented for the evaluation of general network systems. The structure of the first part of this system is shown in Figure 2.

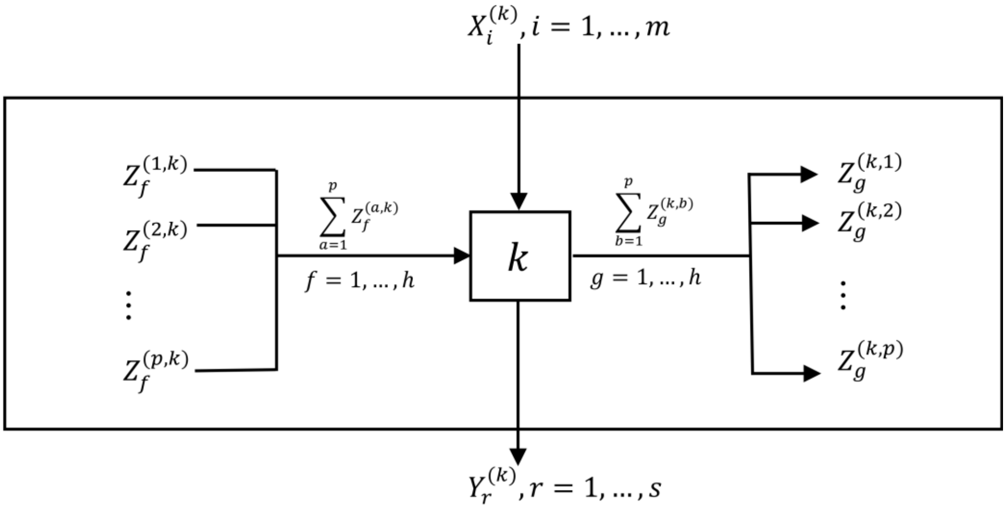

Figure 2. general structure for network systems.

The basic model used in this study for network systems is the relational model in multiplier form, which is referred to below.

$$E_o = \max \sum_{k=1}^{p} \sum_{r=1}^{s} u_r \, {Y^{(k)}}_{ro}$$

$$\text{s.t.} \quad \sum_{k=1}^{p} \sum_{r=1}^{s} u_i \, {X^{(k)}}_{io} = 1$$

$$\sum_{k=1}^{p} \sum_{r=1}^{s} u_r \, {Y^{(k)}}_{rj} - \sum_{k=1}^{p} \sum_{r=1}^{s} v_i \, {X^{(k)}}_{ij} \le 0, \qquad j = 1,2, \dots, n$$

$$\left[\sum_{r=1}^{s} u_r\, Y^{(k)}{}_{rj} + \sum_{g=1}^{h} w_g\, (\sum_{b=1}^{p} Z_{gj}^{(k,b)})\right] -$$
$$\left[\sum_{r=1}^{s} v_i\, X^{(k)}{}_{ij} + \sum_{f=1}^{h} w_f \left(\sum_{a=1}^{p} Z_{fj}^{(a,k)}\right)\right] \leq 0,\ j = 1,2,\dots,n \quad k = 1,2,\dots,p \qquad (2)$$
$$u_r, v_i, w_g, w_f \ \geq \varepsilon \qquad for\ all\ \ r,i,g,f.$$

Here, the number of divisions of each system is assumed to be p, and Figure 2 shows the structure of part k of the system under evaluation. The objective function is the system efficiency. Each constraint in the second and third constraint sets corresponds to one system and one division, respectively.

As mentioned in the introduction no study has been done on ranking general two-stage feedback systems with undesirable data. For this reason, we address this issue for the first time in this article. In the next section, we will present a model that can rank effective two-stage feedback network systems in the presence of unfavorable data.

3. Proposed model

In this section, we present a model that solves the challenge raised in the introduction section, ranking two-stage feedback systems in the presence of undesirable data. The systems evaluated by the proposed model have the structure of Figure 1.

Assume n two-stage feedback systems shown in Figure 1 with undesirable exogenous inputs. In this case, we present the proposed model assuming the following assumptions.
Suppose for the division 1 of system $j,\ j = 1,\dots,n$.
Number of inputs $= m_1$.
Number of undesirable inputs $= d_1$.
Number of desirable inputs $= m_1 - d_1$.
Number of outputs $= s_1$.
Number of intermediate products $= p$.
Number of intermediate products $= q$.
and division 2 of system $j,\ j = 1,\dots,n$.
Number of inputs $= m_2$.
Number of undesirable inputs $= d_2$.
Number of desirable inputs $= m_2 - d_2$.
Number of outputs $= s_2$.
Number of intermediate products $= p$.
Number of intermediate products $= q$.
For the case where the systems are under variable returns to scale technology, Charnes et al. have presented the BCC model to determine the efficiency of decision-making units (DMUs). We have developed this model to evaluate two-stage feedback systems with undesirable exogenous inputs.

$$E_o = \max\Big(\sum_{r=1}^{s_1} u_r\, Y^{(1)}{}_{ro} + \sum_{r=1}^{s_2} q_r\, Y^{(2)}{}_{ro} + \sum_{i=1}^{d_1} v_i\, X^{(1)}{}_{io} + \sum_{i=1}^{d_2} p_i\, X^{(2)}{}_{io} - (u_0 + q_0 + v_0 + p_0)\Big)$$

s. t.

$$\Big(\sum_{r=1}^{s_1} u_r\, Y^{(1)}{}_{rj} + \sum_{g=1}^{p} w_g\, Z_{gj} + \sum_{i=1}^{d_1} v_i\, X^{(1)}{}_{ij}\Big) - (u_o + v_o) - \left(\sum_{i=d_1+1}^{m_1} v_i\, X^{(1)}{}_{ij} + \sum_{l=1}^{b} c_l\, F_{lj}\right) \leq 0, \quad j = 1,\dots,n \qquad (3)$$
$$\Big(\sum_{r=1}^{s_2} q_r\, Y^{(2)}{}_{rj} + \sum_{l=1}^{b} c_l\, F_{lj} + \sum_{i=1}^{d_2} p_i\, X^{(2)}{}_{ij}\Big) - (q_o + p_o) - \left(\sum_{i=d_2+1}^{m_2} p_i\, X^{(2)}{}_{ij} + \sum_{g=1}^{p} w_g\, Z_{gj}\right) \leq 0.\ \ j = 1,\dots,n$$
$$\sum_{i=d+1}^{m_1} v_i\, X^{(1)}{}_{io} + \sum_{i=d+1}^{m_2} p_i\, X^{(2)}{}_{io} = 1$$

$u_r, q_r, v_i, p_i, c_l, w_g \geq \varepsilon, \quad for\ all\ r, q, i, l, w$ and u_0, q_0, v_0, p_0 are free.

In this model, $u_r, q_r, v_i, p_i, c_l, w_g$ are assumed to be greater than ε so that they are not considered to be equal to zero in calculations and are not ignored. u_0, q_0, v_0, p_0 are intercepts of constant returns to scale technology so those are free in the sign. The above model usually identifies more than one system as efficient with a perfect efficiency score of one. This makes it challenging to rank them. Here, the proposed model is introduced to rank the mentioned efficient systems.

$$R_o = \max \left(\sum_{r=1}^{s_1} u_r Y^{(1)}{}_{ro} + \sum_{r=1}^{s_2} q_r Y^{(2)}{}_{ro} + \sum_{i=1}^{d_1} v_i X^{(1)}{}_{io} + \sum_{i=1}^{d_2} p_i X^{(2)}{}_{io} - (u_0 + q_0 + v_0 + p_0)\right)$$

$s.t.$

$$\left(\sum_{r=1}^{s_1} u_r Y^{(1)}{}_{rj} + \sum_{g=1}^{p} w_g Z_{gj} + \sum_{i=1}^{d_1} v_i X^{(1)}{}_{ij}\right) - (u_0 + v_0) - \left(\sum_{i=d_1+1}^{m_1} v_i X^{(1)}{}_{ij} + \sum_{l=1}^{b} c_l F_{lj}\right) \leq 0,\ j = 1, \dots, n,\ j \neq o$$

$$\left(\sum_{r=1}^{s_2} q_r Y^{(2)}{}_{rj} + \sum_{l=1}^{b} c_l F_{lj} + \sum_{i=1}^{d_2} p_i X^{(2)}{}_{ij}\right) - (q_0 + p_0) - \left(\sum_{i=d_2+1}^{m_2} p_i X^{(2)}{}_{ij} + \sum_{g=1}^{p} w_g Z_{gj}\right) \leq 0,$$

$j = 1, \dots, n,\ j \neq o$

$$\sum_{i=d+1}^{m_1} v_i X^{(1)}{}_{io} + \sum_{i=d+1}^{m_2} p_i X^{(2)}{}_{io} = 1 \tag{4}$$

$$\left(\sum_{r=1}^{s_1} u_r Y^{(1)}{}_{rj} + \sum_{r=1}^{s_2} q_r Y^{(2)}{}_{rj} + \sum_{i=1}^{d_1} v_i X^{(1)}{}_{ij} + \sum_{i=1}^{d_2} p_i X^{(2)}{}_{ij} - (u_0 + q_0 + v_0 + p_0)\right) - (E_j{}^* - \alpha)\left(\sum_{i=d+1}^{m_1} v_i X^{(1)}{}_{ij} + \sum_{i=d+1}^{m_2} p_i X^{(2)}{}_{ij}\right) \geq 0, \quad j = 1, \dots, n,,\ j \neq o$$

$u_r, q_r, v_i, p_i, c_l, w_g \geq \varepsilon, \quad for\ all\ r, q, i, l, w$

u_0, q_0, v_0, p_0 are free $0 < \alpha < 1$

Where $E_j{}^*$ is efficiency score of DMU_j Which is calculated from model (1) and $0 < \alpha < 1$. This model allows us to rank units by creating an artificial border by the most appropriate α.

Theorem 3.1. There is an α in $(0, 1)$ for which the model (4) is feasible.

Proof. Assuming that all of the inputs and outputs are positive, Let:

$v_1 = v_2 = \dots = v_{m_{1-1}} = 0\ ,\ v_{m_1} = \frac{1}{2X^{(1)}{}_{m_1 o}}$

$p_1 = p_2 = \dots = p_{m_{1-1}} = 0\ ,\ p_{m_2} = \frac{1}{2X^{(2)}{}_{m_2 o}}$

$u_1 = u_2 = \dots = u_{s_1} = 0$

$q_1 = q_2 = \dots = q_{s_2} = 0$

$w_1 = w_2 = \dots = w_p = 0$

$c_1 = c_2 = \dots = c_b = 0$

In this case, the third constraint of model (4) is valid. With the above assumptions, we have for the first and second constraints, respectively.

$$-(u_0 + v_0) \leq \frac{X^{(1)}{}_{m_1 j}}{2X^{(1)}{}_{m_1 o}},\ j = 1, \dots, n,\ j \neq o. \tag{5}$$

$$-(q_0 + p_0) \leq \frac{X^{(2)}{}_{m_2 j}}{2X^{(2)}{}_{m_2 o}}, \quad j = 1, \dots, n,\ j \neq o. \tag{6}$$

As a result of 3 and 4, the following relationship is established.

$$-(u_0+v_0+q_0+p_0) \le \frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}} \Rightarrow -(u_0+v_0+q_0+p_0) \le \min_j \{\frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}}\},\ \ j=1,\dots,n,\ \ j\ne o \qquad (7)$$

By applying (5), (6), and (7) in the fifth constraints we will have the following relationship.

$$-(u_0+v_0+q_0+p_0) \ge \left(E_j{}^* - \alpha\right)(\frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}}),\quad j=1,\dots,n,\ \ j\ne o \qquad (8)$$

Thus

$$\alpha \le E_j{}^* - \min_j \{\frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}}\} \Big/ \left(\frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}}\right),\ j=1,\dots,n,\ \ j\ne 0 \qquad (9)$$

It is enough to consider α as follows.

$$\alpha = \max_j \{E_j{}^* - \min_j \{\frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}}\} \Big/ \left(\frac{X^{(1)}{}_{m_1j}}{2X^{(1)}{}_{m_1o}} + \frac{X^{(2)}{}_{m_2j}}{2X^{(2)}{}_{m_2o}}\right)\},$$
$$j=1,\dots,n,\ \ j\ne o \qquad (10)$$

With the above selection, it can be seen that the model (4) is feasible. □

Choose the most suitable α

Consider the following model.

$$\max\ \ \alpha$$
$$s.t.$$
$$\left(\sum_{r=1}^{s_1} u_r\, Y^{(1)}{}_{rj} + \sum_{g=1}^{p} w_g\, Z_{gj} + \sum_{i=1}^{d_1} v_i\, X^{(1)}{}_{ij}\right) - (u_0+v_0) - \left(\sum_{i=d_1+1}^{m_1} v_i\, X^{(1)}{}_{ij} + \sum_{l=1}^{b} c_l\, F_{lj}\right) \le 0,\ j=1,\dots,n,\ \ j\ne o$$
$$\left(\sum_{r=1}^{s_2} q_r\, Y^{(2)}{}_{rj} + \sum_{l=1}^{b} c_l\, F_{lj} + \sum_{i=1}^{d_2} p_i\, X^{(2)}{}_{ij}\right) - (q_0+p_0) - \left(\sum_{i=d_2+1}^{m_2} p_i\, X^{(2)}{}_{ij} + \sum_{g=1}^{p} w_g\, Z_{gj}\right) \le 0,\ j=1,\dots,n,\ \ j\ne o$$
$$\sum_{i=d+1}^{m_1} v_i\, X^{(1)}{}_{io} + \sum_{i=d+1}^{m_2} p_i\, X^{(2)}{}_{io} = 1 \qquad (11)$$
$$\left(\sum_{r=1}^{s_1} u_r\, Y^{(1)}{}_{rj} + \sum_{r=1}^{s_2} q_r\, Y^{(2)}{}_{rj} + \sum_{i=1}^{d_1} v_i\, X^{(1)}{}_{ij} + \sum_{i=1}^{d_2} p_i\, X^{(2)}{}_{ij} - (u_0+q_0+v_0+p_0)\right) - \left(E_j{}^* - \alpha\right)\left(\sum_{i=d+1}^{m_1} v_i\, X^{(1)}{}_{ij} + \sum_{i=d+1}^{m_2} p_i\, X^{(2)}{}_{ij}\right) \ge 0,\ j=1,\dots,n,,\ \ j\ne o$$
$$u_r, q_r, v_i, p_i, c_l, w_g \ge \varepsilon,\quad for\ all\ r,q,i,l,w$$
$$u_0, q_0, v_0, p_0\quad \text{are free } 0<\alpha<1,$$

by solving the above model for all efficient units and assuming that we have efficient units, the most suitable will be as follows.

$$\bar{\alpha} = \max\{\alpha_1^*, \alpha_2^*, \dots, \alpha_k^*\}$$

Where $\alpha_j^*, j=1,\ 2,\dots,\ k$ is the optimal value of model (3) for efficient systems. In fact, the $\bar{\alpha}$ is the value for which we can produce the largest changes on the frontier of the PPS by eliminating the system we are looking to rank.

We introduced a model in section 3. It can evaluate two-stage feedback systems. It does this in the presence of undesirable exogenous inputs. To check the performance of the model, an example is provided in the next section. It is a structural example designed by the authors. It defines 6 hypothetical two-stage feedback systems in the presence of undesirable exogenous inputs.

4. Numerical example

Consider two-stage feedback systems A, B, C, D, E, and F, with the data shown in Tables 1 and 2.

Table 1. Data of the first division of the systems.

system	$X_1^{(1)}$	$X_2^{(1)}$	$X_3^{(1)}$	$Z_1^{(1)}(output)$	$F_1^{(1)}(input)$	$Y_1^{(1)}$	$Y_2^{(1)}$
A	0.500	1.000	0.500	2.000	0.800	0.700	1.000
B	0.700	1.000	5.000	1.000	0.500	2.000	3.000
C	1.000	2.000	2.000	3.000	0.900	1.500	2.000
D	9.000	2.000	3.000	5.000	2.000	1.000	0.010
E	4.000	1.000	1.000	4.000	4.000	0.400	0.500
F	0.700	0.998	5.001	1.000	0.500	2.001	2.998

Table 2. Data of the second division of the systems.

system	$X_1^{(1)}$	$X_2^{(1)}$	$X_3^{(1)}$	$Z_1^{(1)}(output)$	$F_1^{(1)}(input)$	$Y_1^{(1)}$	$Y_2^{(1)}$
A	1.000	1.500	0.400	1.000	0.900	0.400	0.500
B	0.400	0.800	3.000	1.000	0.500	3.000	1.000
C	0.800	3.000	2.000	2.000	1.000	2.000	3.000
D	3.000	2.000	1.000	4.000	3.000	1.000	4.000
E	5.000	3.000	0.500	3.000	5.000	0.700	2.000
F	0.400	0.801	3.001	1.000	0.500	2.998	1.000

Tables 1 and 2 show data of the first and second stages respectively. The first two inputs are assumed desirable and the third input is assumed undesirable. Every system has an intermediate product and feedback data. Also, the outputs are considered desirable. The models (3) and (4), and relationship (10). are applied to the data, and the results are given in Table 3.

Table 3. The result of models (3), (5), and relationship (10).

system	E_o	α	R_o	Ranking
A	0.4896699	---	---	6
B	1.0000000	0.186629	1.001762	2
C	0.9998720	---	---	3
D	0.9977020	---	---	4
E	0.5752257	---	---	5
F	1.0000000	0.4500000	1.002498	1

As previously stated, systems B and F have an efficiency score of 1 and are regarded as efficient. The remaining systems are deemed inefficient. By solving model (5) for efficient units B and F, the most suitable will be as follows.

$$\bar{\alpha} = \max\{\alpha_B^*, \alpha_F^*\} = \max\{0.186629, 0.450000\} = 0.450000$$

Model (4) introduces ranking the efficient systems, resulting in the determination of their respective ranks. The results obtained from models (3) and (4) are shown in Table 3. According to the results, system B has rank 2 and system F has rank 1, which shows that the proposed model can rank systems.

5. Conclusion

This study introduces a groundbreaking model designed to evaluate and rank general two-stage feedback systems in the presence of undesirable exogenous inputs, addressing a critical challenge in efficiency analysis. The fundamental principle underpinning this model is its ability to account for controlled reductions in system efficiency, ensuring a balanced and thorough evaluation process. This innovative approach enables the model to effectively rank all efficient systems while maintaining reliability and fairness in its assessments.

Moreover, the model refines the efficiency evaluation of inefficient decision-making units (DMUs) by generating scores and rankings that are not only more reasonable but also better aligned with practical and real-world scenarios. By capturing the impact of undesirable feedback inputs with greater accuracy, this methodology significantly enhances the precision of ranking processes. Such improvements make the model a valuable tool for stakeholders aiming to optimize system performance and address inefficiencies across diverse applications.

Future research directions could explore integrating a broader range of data types into general multi-stage feedback systems, extending the model's applicability to more complex and dynamic environments. These expansions might include examining systems with intertwined stages or addressing new categories of undesirable inputs. Additionally, efforts to generalize this approach across different industries and decision-making frameworks would further contribute to the ongoing development of ranking methodologies, ensuring their adaptability to evolving needs.

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