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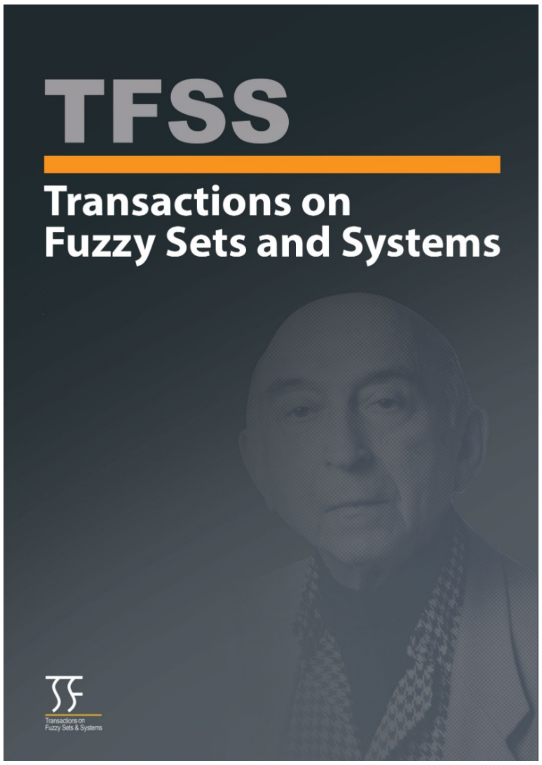

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ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.5, No.1, (2026), 26-53. DOI: https://doi.org/10.71602/tfss.2026.1196801

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.1, (2026), 26-53
DOI: https://doi.org/10.71602/tfss.2026.1196801

Article Type: Original Research Article
(Journal Pre-proof)

Energy and Laplacian Energy of Graphs under the Bipolar Valued Hesitant Fuzzy Framework

Sakshi Dev Pandey , **Abhay S. Ranadive** , **Sovan Samanta*** , **Leo Mrsic** , **Antonios Kalampakas**

(Special Issue on "Dynamical Systems in Fuzzy Environments".)

Abstract. Bipolar-valued hesitant fuzzy graphs (BVHFGs) provide a suitable framework for representing knowledge in situations characterized by uncertainty, imprecision, and hesitations. Current research shows a lack of studies on energy in contexts involving bipolarity, hesitations, and fuzzy data, motivating us to propose new definitions of energy in this area. In this study, we introduce innovative notions of graph energy and Laplacian energy within the framework of a bipolar valued hesitant fuzzy setting and scrutinize certain characteristics and various types of bounds of these concepts. Additionally, the investigation explores the interplay between the energy and Laplacian energy of BVHFGs. Consequently, a numerical illustration is provided, encompassing the identification of optimal alternatives to elucidate the pragmatic application of the proposed theoretical frameworks within the realm of decision-making. This empirical demonstration underscores the efficacy and relevance of the developed methodology in addressing real-world decision-making challenges.

AMS Subject Classification 2020: 05C72
Keywords and Phrases: Bipolar valued hesitant fuzzy graph, Spectrum, Energy, Laplacian energy.

1 Introduction

1.1 Research background

Zadeh [1] introduced the fuzzy set theory in 1965, which comprises a set of concepts that address the type of imprecisions that arise when the boundary of a class is not accurately defined. Following the emergence of fuzzy sets, study on them has garnered significant attention as a prominent area of research across multiple academic disciplines [2]. Numerous scholarly investigations have put up diverse expansions and applications of fuzzy sets in the realm of academic research [3, 4]. One such example is the introduction of bipolar fuzzy sets (BFSs) by Zhang [5]. This extension expands upon the notion of fuzzy sets, which are emphasized by membership values within the range of $[-1, 1]$. The BFS assigns membership degrees to elements based on their relevance according to specified criteria. We presume that BFS deals with the satisfaction exhibited by elements that meet the relevant property as well as some intrinsic contrast property associated with the

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Received: 17 January 2025; **Revised**: 1 March 2025; **Accepted**: 2 March 2025; **Available Online**: 6 July 2025; **Published Online**: 7 May 2026.

How to cite: Pandey SD, Ranadive AS, Samanta S, Mrsic L, Kalampakas A. Energy and laplacian energy of graphs under the bipolar valued hesitant fuzzy framework. *Transactions on Fuzzy Sets and Systems.* 2026; 5(1): 26-53. DOI: https://doi.org/10.71602/tfss.2026.1196801

provided criteria. However, the BFS does not consider hesitation in achieving this degree of satisfaction. This assertion remains impractical when applied to real-life scenarios. That is why it is important to continue researching efficient and reliable approaches.

1.2 Motivation and Contribution

As far as current scholarly research indicates, there is a lack of literature pertaining to the subject matter of energy in settings involving bipolarity, hesitations, and fuzzy data. On the other side, several studies have been conducted in the literature to present the utility of bipolar-valued hesitant fuzzy sets [6], dual hesitant fuzzy sets [7], and others. Notably, Pandey et al. [8] recently introduced the concept of the bipolar-valued hesitant fuzzy graph (BVHFG) in 2022 and expounded upon its fundamental operations. In this study, our main objective is to introduce the notion of bipolar-valued hesitant fuzzy graph energy and Laplacian energy and establish the relation between them. Additionally, we present various lower and upper bounds of BVHFG energy. Various methods and approaches are discussed in the literature related to decision-making issues [9]. To show the utility of the current study, we also discuss a problem related to group decision-making where the weight of experts is completely unknown [10, 11].

1.3 Framework of this study

The present manuscript is structured in a subsequent form: Section 2 represents an overview of the literature on fuzzy graphs, decision-making, and energy of different forms of graphs. Section 3 provides a brief summary of the historical context and fundamental characteristics of BVHFSs and BVHFGs. The fourth section primarily introduced the notion of the energy of BVHFGs and explored its characteristics. Section 5 presents an analysis of the Laplacian energy of BVHFGs and establishes a correlation between energy and Laplacian energy within the BVHF framework. Section 6 provides a quantitative illustration of the energy and Laplacian energy associated with BVHFG's in the context of decision-making considerations. The concluding part of this scholarly manuscript is expounded upon in the seventh section.

2 Literature Review

Rosenfeld [12] proposed the utilization of a fuzzy graph in 1975, drawing inspiration from Kauffman's [13] fundamental concepts. Presently, a significant amount of research is being conducted in the area related to fuzzy graphs. This includes the extensions and applications related to fuzzy graphs in various areas, such as link prediction under social media networks, the coloring of regions affected by Coronavirus disease 2019 (COVID-19) [14], centrality in bipolar fuzzy social networks [15], bipolar fuzzy bunch graphs [16] and many others. Akaram [17] presented the idea of bipolar fuzzy graphs (BFGs) as a means of addressing the bipolar nature of real-world problems, building upon Zhang's [5] bipolar fuzzy set theory. In 2015, Pathinathan et al. [18] developed hesitancy fuzzy graphs outlining fundamental ideas. Although Pathinathan coined the term hesitancy fuzzy graph, they didn't assign hesitant fuzzy elements (HFEs) to the graph's vertices and edges. Instead, they employed intuitionistic fuzzy (IF) [19] values, represented by triples indicating membership degree, hesitancy degree, and non-membership degree of vertices and edges. In 2019, Karaaslan [20] proposed hesitant fuzzy graphs (HFG), aligning with Torra's [21] original notion of heitant fuzzy set (HFS) by assigning HFEs to vertices and edges. In 2022, Pandey et al. [8] extended this concept by introducing score based bipolar-valued hesitant fuzzy graphs (BVHFGs), integrating bipolarity and defining fundamental properties. BVHFG is the generalization of HFG, which considers not only the satisfaction degree of units in a network but also the satisfaction degree to some implicit counter property of units with several bipolar fuzzy values.

The notion of energy has a close connection to the graph's spectrum. The idea of the graph's spectrum was initially introduced in a scholarly article authored by Collatz and Sinogowitz in 1957. The nomenclature of the subject matter draws inspiration from the concept of energy in the field of chemistry. The investigation of π-electron energy within the field of chemistry has a historical origin tracing back to the 1940's [22]. In 1978, Gutman [23] proposed a mathematical formulation for the concept of energy applicable to graphs of any nature. The investigation of specific limitations on energy is conducted in studies [24] and [25]. Gutman and Zhou [26] presented the notion of the Laplacian energy associated with the graph in 2006, which is computed by the summation for the absolute differences between the mean degree of the vertices in graph G and the eigenvalues of its Laplacian matrix. Distinct categories of graphs, specifically hypoenergetic, hyperenergetic, and equienergetic, have been classified based on their respective energy levels. Further information regarding these categories can be obtained from sources [27] and [28]. The concept of energy has been established for various types of graphs to solve the decision-making issues [29]. Specifically, it has been provided for weighted graphs in [30], for signed graphs in [31], for fuzzy graphs in [22], and for BFGs by Naz et al. [32]. Sharbaf and Fayazi [33] presented the definition of the Laplacian energy associated with the fuzzy graph in 2014. Energy of the bipolar-valued intuitionistic fuzzy digraph used to choose the COVID-19 vaccines is presented in [34]. Although the precise physical interpretation of energy application in graphs remains unclear, its inherent properties are of significant interest to mathematicians.

3 Preliminaries

This section provides an overview of fundamental concepts regarding BVHFS's and BVHFG's, which will aid in comprehending subsequent sections.

A graph, denoted as $\zeta = (V, E)$, constitutes the mathematical framework made up of nodes V and links E. Each link is represented as an unordered combination of different nodes. An adjacency matrix, denoted as $M(\zeta)$, for a graph named ζ having n nodes, typically a square matrix of size n×n. Each element within the i-th row and j-th column of the matrix reflects the total amount of links connecting nodes i and j. The eigenvalues for graph ζ correspond to as eigenvalues of its adjacency matrix, denoted as λ_i, where $1 \leq i \leq n$. The spectrum for a graph ζ denoted as Spec(ζ), is defined as an accumulation of eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ associated with the adjacency matrix underlying ζ.

Definition 3.1. [24] *$E(\zeta)$ denotes the energy for the graph ζ, which is defined as the summation for the absolute magnitude of ζ's eigenvalues, i.e., $E(\zeta) = \sum_{i=1}^{n} |\lambda_i|$. A graph that consists only of isolated vertices has an energy of zero, whereas a complete graph has 2(n-1) energy.*

Definition 3.2. [26] *The Laplacian energy for the graph ζ, identified as $E_L(\zeta)$, can be mathematically expressed by the summation for the absolute differences between the Laplacian eigenvalues of ζ, denoted as $\{\mu_1, \mu_2, \cdots, \mu_n\}$, and the average degree of ζ, denoted as $\frac{2m}{n}$. Let γ_i be the auxiliary eigenvalues, we have,*

$$E_L(\zeta) = \sum_{i=1}^{n} |\gamma_i|$$

where, $\gamma_i = \mu_i - \frac{2m}{n}$, m denote the total amount of links and n denote the total amount of nodes.

Definition 3.3. [35] *Consider a real matrix P with dimensions $m \times n$. The matrix PP^T can be characterized as a positive semi definite matrix with a size of m. The matrix PP^T possesses eigenvalues that can be represented as $\sigma_1^2, \sigma_2^2, \cdots, \sigma_m^2$, where $\sigma_i \geq 0$ for $i = 1, 2, \cdots, m$. These values have been commonly referred to as the singular values for the matrix P. The sum $\sigma_1 + \sigma_2 + \cdots + \sigma_m$ is referred to as the singular energy for matrix P as will as is symbolically represented as $\sigma(P)$ for the sake of convenience.*

Definition 3.4. [36] *(Spanning Subgraph) A spanning subgraph is a graph that includes all the vertices of the parent graph but may not include all the links.*

Definition 3.5. [12] *The notion regarding the fuzzy graph denoted as $G^* = (V, \mu, \nu)$, constitutes a mathematical construct that builds upon the crisp graph $\zeta = (V, E)$. It is signified by the presence of membership functions, represented as $\mu : V \to [0, 1]$ and $\nu : V \times V \to [0, 1]$, which serve to define the characteristics of the graph, such that $\nu(v_1v_2) \leq \mu(v_1) \wedge \mu(v_2)$ for all $v_1, v_2 \in V$, where $\mu(v_1)$ and $\nu(v_1v_2)$ represents the value of membership of the node v_1 and link v_1v_2 within G^* correspondingly.*

Definition 3.6. [37] *Consider $X = \{x_1, x_2, \cdots, x_n\}$ as a non empty finite universe of discourse. Let $U = (\sigma_U(x_i), \delta_U(x_i))$ and $W = (\sigma_W(x_i), \delta_w(x_i))$ be two intuitionistic fuzzy sets on X. The CC (correlation coefficient) between U and W can be expressed mathematically as,*

$$K(U, W) = \frac{\sum_{i=1}^{n}[\sigma_U(x_i)\sigma_W(x_i) + \delta_U(x_i)\delta_W(x_i)]}{\sqrt{\sum_{i=1}^{n}[\sigma_U^2(x_i) + \delta_U^2(x_i)]}\sqrt{\sum_{i=1}^{n}[\sigma_V^2(x_i) + \delta_V^2(x_i)]}}.$$

In the present research article, the notation I^P will be employed to represent the interval $[0, 1]$, whereas I^N will denote the interval $[-1, 0]$.

Definition 3.7. [6] *Consider the universe of discourse denoted by X. A bipolar valued hesitant fuzzy set (BVHFS) $\mathbb{B}$ on the set X has been formally described as:*

$$\mathbb{B} = \{< x, H(x) > | x \in X\},$$

here $H(x)$ denote a collection containing values within $I^P \times I^N$. For simplicity, we convey $H(x)$ a bipolar valued hesitant fuzzy element (BVHFE) defined by:

$$H(x) = \{h_x | h_x \in I^P \times I^N\},$$

where the variable h_x can be expressed as (h_x^P, h_x^N) which is called a bipolar valued fuzzy number (BVFN). We have, $h_x^P \in I^P$ and $h_x^N \in I^N$.

Definition 3.8. [6] *Consider $h_x = (h_x^P, h_x^N) \in H(x)$ as a BVFN, where the Score of h_x, $S(h_x)$ can be computed as:*

$$S(h_x) = \frac{1}{2}(h_x^P - h_x^N).$$

Definition 3.9. [6] *consider $H(x)$ as a BVHFE, score function associated with $H(x)$, $S(H(x))$ can be computed by:*

$$S(H(x)) = \frac{1}{l(H(x))} \sum_{h_x \in H(x)} S(h_x),$$

here $l(H(x))$ represent the cardinality of the set of bipolar values in $H(x)$. Furthermore, h_x represent an element in $H(x)$, which is assumed to be in the form of a BVFN.

Definition 3.10. [8] *Consider A and B denote two BVHFS's defined over the universal set X. The function $A(x, y)$ is defined for any x and y belonging to X as follows: $A(x, y) : X \times X \to P(I^P \times I^N)$. Consider A as a bipolar valued hesitant fuzzy relation over set X. The relation A is referred to as the score based bipolar valued hesitant fuzzy relation on set B when, for all x and y in X, it holds that $S(A(x, y)) \leq S(B(x)) \wedge S(B(y))$.*

Pandey et al. [8] established the notion of the BVHFG's by bringing the concept of BVHFS's into graph theory. This paper will denote the link (x, y) as xy and the Cartesian product $V \times V$ as V^2, unless otherwise specified.

Definition 3.11. [8] *A bipolar valued hesitant fuzzy graph (BVHFG) can be characterized by the pair $\hat{G} = (A, B)$, with A and B represent BVHFS's defined on the reference set V and V^2 correspondingly. The given scenario involves two membership functions, namely $A : V \to P(I^P \times I^N)$ and $B : V^2 \to P(I^P \times I^N)$. It is stated that the inequalities*

$$\begin{aligned} S(B(xy)) &\leq S(A(x)) \wedge S(A(y)) \ for\ all\ xy \in V^2, \\ S(B(xy)) &= 0\ for\ all\ xy \in (V^2 - E), \end{aligned}$$

holds. Where $A(x)$ and $B(xy)$ represent the BVHFE's which is described as $B(xy) = \{(b^P_{xy}, b^N_{xy}) \mid (b^P_{xy}, b^N_{xy}) \in I^P \times I^N\}$ and $A(x) = \{(a^P_x, a^N_x) \mid (a^P_x, a^N_x) \in I^P \times I^N\}$.

Definition 3.12. [8] *Consider the BVHFG $\hat{G} = (A, B)$ over ζ. The score based degree associated with a vertex $v_i \in V$ belonging to the BVHFG has been represented as $deg(v_i)$. It is specified by the total of score of all links that are connected to the vertex v_i. Mathematically, this may be expressed as $deg(v_i) = \sum_{v \neq v_i \in V} S(B(v_i v))$.*

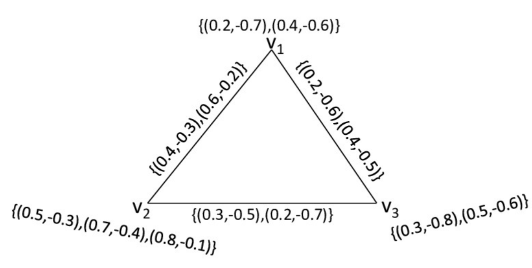
4 Energy of Bipolar valued hesitant fuzzy graph

In this following section, we introduce a comprehensive definition of energy of a BVHFG along with its associated bounds. We illustrate these bounds through the use of examples. The energy of BVHFG has broad applicability across diverse research domains.

Definition 4.1. *Adjacency matrix $M(\hat{G})$ of the BVHFG $\hat{G} = (V, A, B)$ is a square matrix with size $n \times n$, denoted as $M(\hat{G}) = [a_{ij}]_{n \times n}$. Each element a_{ij} in the matrix represents the bipolar valued hesitant membership grades of the links $v_i v_j$, specifically $a_{ij} = H(v_i v_j)$.*

$$H(v_i v_j) = \{(h^P_{ij}, h^N_{ij}) \mid (h^P_{ij}, h^N_{ij}) \in I^P \times I^N\}.$$

Example 4.2. Consider the BVHFG, suppose V represent the collection of three nodes $\{v_1, v_2, v_3\}$ and $E = \{v_1 v_2, v_2 v_3, v_3 v_1\}$ represent the collection of links, then the BVHFG and BVHFSs, A and B across V and V^2 are illustrated by Figure 1 and Table 1, accordingly. Additionally, the adjacency matrix $M(\hat{G})$ is specified according to:



A.jpg A.bb

Figure 1: Example of BVHFG

Table 1: BVHF table

	v_1	v_2	v_3
A	$\{(0.2,-0.7),(0.4,-0.6)\}$	$\{(0.5,-0.3),(0.7,-0.4),(0.8,-0.1)\}$	$\{(0.3,-0.8),(0.5,-0.6)\}$
Score	0.475	0.467	0.55

	v_1v_2	v_2v_3	v_3v_1
B	$\{(0.4,-0.3),(0.6,-0.2)\}$	$\{(0.3,-0.5),(0.2,-0.7)\}$	$\{(0.2,-0.6),(0.4,-0.5)\}$
Score	0.375	0.425	0.425

$$M(\hat{G}) = \begin{bmatrix} (0,0) & \{(0.4,-0.3),(0.6,-0.2)\} & \{(0.2,\text{-}0.6),(0.4,\text{-}0.5)\} \\ \{(0.4,-0.3),(0.6,-0.2)\} & (0,0) & \{(0.3,\text{-}0.5),(0.2,\text{-}0.7)\} \\ \{(0.2,\text{-}0.6),(0.4,\text{-}0.5)\} & \{(0.3,\text{-}0.5),(0.2,\text{-}0.7)\} & (0,0) \end{bmatrix}$$

Score of the adjacency matrix $M(\hat{G})$ is specified by

$$S(M(\hat{G})) = \begin{bmatrix} 0 & 0.375 & 0.425 \\ 0.375 & 0 & 0.425 \\ 0.425 & 0.425 & 0 \end{bmatrix}$$

The eigenvalues associated with the adjacency matrix $M(\hat{G})$ commonly denoted as eigenvalues of BVHFG $\hat{G}$. The collection of eigenvalues of $M(\hat{G})$ frequently referred by the term spectrum of $\hat{G}$ and is represented by $Spec(\hat{G})$.

Definition 4.3. *Let $\hat{G}$ be the BVHFG, $M(\hat{G})$ be an $n \times n$ adjacency matrix, $S(M(\hat{G}))$ be the score of the adjacency matrix and $\lambda_i's$, $i = 1, 2, \cdots, n$ are associated eigenvalues of $\hat{G}$. The energy of BVHFG, $E(G^*)$ is determined by $E(\hat{G}) = \sum_{i=1}^{n} |\lambda_i|$.*

Example 4.4. Regarding the graph depicted in Figure:1. $Spec(\hat{G})$={-0.4421, -0.3750, 0.8171}. The energy associated with the graph $\hat{G}$ is mathematically expressed by the total on the absolute values for eigenvalues of $\hat{G}$. This can be represented as $E(\hat{G}) = 0.4421 + 0.3750 + 0.8171 = 1.6342$.

Remark 4.5. *It can be observed that the energy value associated with the non trivial simple graph has been invariably higher than one [27]. However, it is noteworthy that this outcome does not hold true for the BVHFG, as seen in Example 4.6.*

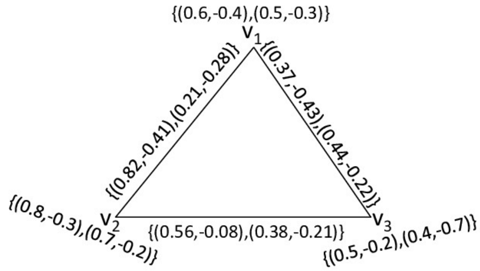
Example 4.6. From, Figure 2, the adjacency matrix can be observed as

$$M(\hat{G}) = \begin{bmatrix} (0,0) & \{(0.82,-0.41),(0.21,-0.28)\} & \{(0.37,\text{-}0.43),(0.44,\text{-}0.22)\} \\ \{(0.82,-0.41),(0.21,-0.28)\} & (0,0) & \{(0.56,\text{-}0.08),(0.38,\text{-}0.21)\} \\ \{(0.37,\text{-}0.43),(0.44,\text{-}0.22)\} & \{(0.56,\text{-}0.08),(0.38,\text{-}0.21)\} & (0,0) \end{bmatrix}$$

Score of the given adjacency matrix $M(G^*)$ has been specified by

$$S(M(\hat{G})) = \begin{bmatrix} 0 & 0.226 & 0.182 \\ 0.226 & 0 & 0.151 \\ 0.182 & 0.151 & 0 \end{bmatrix}$$

$Spec(\hat{G}) = \{0.374, -0.231, -0.143\}$, $E(G) = 0.748 < 1$.



B.jpg B.bb

Figure 2: BVHFG, $E(\hat{G}) < 1$

Theorem 4.7. *Let $\hat{G} = (V, A, B)$ be the BVHFG, where $|V| = n$ and λ_i's, $i = 1, 2, \cdots, n$ be the eigenvalues of $\hat{G}$. Consider $\{e_1, e_2 \cdots, e_m\}$ as the collection of links of $\hat{G}$ and $M = [a_{ij}]_{n\times n}$ is the adjacency matrix of $\hat{G}$, then $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 2\sum_{i=1}^{m}(S(B(e_i)))^2$.*

Proof. The total of the square matrix's eigenvalues corresponds to its trace, and the adjacency matrix of $\hat{G}$ has a trace of zero, it follows that $\sum_{i=1}^{n} \lambda_i = 0$.
Let P and P^t be the square matrix and its transpose respectively, by the property of square matrix, $trace(PP^t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ij}$.
Now, since M is the symmetric matrix so $MM^t = M^2$.

$$\begin{aligned} trace(M^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} S(B(a_{ij}))S(B(a_{ij})), \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} S(B(a_{ij}))^2, \\ &= 2\sum_{i=1}^{m} S(B(e_i))^2. \end{aligned}$$

Thus, $\sum_{i=1}^{n} \lambda_i^2 = trace(M^2) = 2\sum_{i=1}^{m} S(B(e_i))^2$. □

Lemma 4.8. [35] (The Ky Fan inequality) *Let us consider the matrices P, Q, and R that allows the equation $R = P + Q$ holds. In this context, $\sigma(M)$ denotes the singular energy of matrix M. It can be stated that the singular energy of matrix R is less than or equal to the total of singular energies of matrices P and Q, i.e., $\sigma(R) \leq \sigma(P) + \sigma(Q)$.*

Theorem 4.9. [38] *Let ζ be a weighted network having n nodes in which each link possesses a non-zero weight. Let e_i, $i = 1, 2, \cdots, m$ denote all of the links for ζ. Then, we have, the energy of ζ less than or equal to the twice into total of the weights of all links in ζ, i.e., $E(\zeta) \leq 2\sum_{i=1}^{m} |w(e_i)|$.*

Proof. Let ζ_e be the spanning (weighted) subgraph of ζ containing a single link e, then $M(\zeta) = \sum_{i=1}^{m} M(\zeta_{e_i})$

and let $\sigma(M)$ denote the singular energy of M. Now by the Ky Fan inequality,

$$\begin{aligned} E(\zeta) &= \sigma(M(\zeta)), \\ &\leq \sum_{i=1}^{m} \sigma(M(\zeta_{e_i})), \\ &= 2\sum_{i=1}^{m} |w(e_i)|. \end{aligned}$$

Therefore, $E(\zeta) \leq 2\sum_{i=1}^{m} |w(e_i)|$. □

The previous theorem applies to weighted graphs. In this case, $w(e_i)$ stands for the weight of link e_i. If we consider a BVHFG to be the weighted graph having a weighted score within the range I^P, one possible rewording of Theorem 4.9 would be as:

Theorem 4.10. *Consider $\hat{G} = (V, A, B)$ as the BVHFG having $|V| = n$ and $B^* = \{e_1, e_2, \cdots, e_m\}$. Then, we have, $E(\hat{G}) \leq 2\sum_{i=1}^{m} S(B(e_i))$.*

Example 4.11. According to the illustration of Theorem 4.10, it can be observed from Figure:1 that the value of $E(\hat{G})$ is 1.6342 and $2\sum_{i=1}^{3} S(B(e_i))$ is equal to 2.45. It is evident that $2\sum_{i=1}^{3} S(B(e_i))$ is greater than $E(\hat{G})$.

Theorem 4.12. *Consider $\hat{G} = (V, A, B)$ as the BVHFG having $|V| = n$ and eigenvalues $\lambda_i's, i = 1, 2\cdots, n$. Suppose $B^* = \{e_1, e_2, \cdots, e_m\}$ and $M = [a_{ij}]_{n\times n}$ is adjacency matrix associated with $\hat{G}$. Then, we have,*

$$2\sqrt{\sum_{i=1}^{m} S(B(e_i))^2} \leq E(\hat{G}) \leq 2\sum_{i=1}^{m} S(B(e_i)).$$

Proof. For lower bound,

$$\begin{aligned} |E(\hat{G})|^2 &= (\sum_{i=1}^{n} |\lambda_i|)^2, \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{1\leq i<j\leq n} |\lambda_i\lambda_j|, \\ &= 2\sum_{i=1}^{m} S(B(e_i))^2 + 2\sum_{1\leq i<j\leq n} |\lambda_i\lambda_j|. \end{aligned} \tag{1}$$

Now, while comparing the coefficient of λ^{n-2} in the equation

$$\prod_{i=1}^{n} (\lambda - \lambda_i) = |M - \lambda I|.$$

We have,

$$\sum_{1=i<j=n} \lambda_i\lambda_j = -\sum_{i=1}^{m} S(B(e_i))^2.$$

Since $\sum_{i<j}^{n} |\lambda_i\lambda_j| \geq |\sum_{i<j}^{n} \lambda_i\lambda_j|$, we get from Equation (1).

$$\begin{aligned} |E(\hat{G})|^2 &\geq 2\sum_{i=1}^{m} S(B(e_i))^2 + 2\sum_{i=1}^{m} S(B(e_i))^2. \\ E(\hat{G}) &\geq 2\sqrt{\sum_{i=1}^{m} S(B(e_i))^2}. \end{aligned}$$

Also from Theorem 4.10, we have, $E(\hat{G}) \leq 2\sum_{i=1}^{m} S(B(e_i))$.

Therefore, $2\sqrt{\sum_{i=1}^{m} S(B(e_i))^2} \leq E(\hat{G}) \leq 2\sum_{i=1}^{m} S(B(e_i))$. □

Theorem 4.13. *Consider* $\hat{G} = (V, A, B)$ *as the BVHFG, where* $V = A^* = \{v_1, v_2, \cdots, v_n\}$ *and* $B^* = \{e_1, e_2, \cdots, e_m\}$*. Then, we have,* $E(\hat{G}) \leq (n-1)\sum_{i=1}^{n} S(A(v_i))$.

Proof.

$$\begin{aligned} \text{Since, } E(\hat{G}) &\leq 2\sum_{i=1}^{m} S(B(e_i)), \\ &\leq 2\sum_{i=1}^{\frac{n(n-1)}{2}} S(B(e_i)), \end{aligned}$$

where $m \leq \frac{n(n-1)}{2}$ (maximum possible number of links). Now,

$$\begin{aligned} E(\hat{G}) &\leq \sum_{i=1}^{\frac{n(n-1)}{2}} S(B(e_i)) + S(B(e_i)), \\ &= \sum_{1\leq i<j\leq n} (S(B(v_iv_j)) + S(B(v_iv_j))). \end{aligned}$$

Since, we have $S(B(v_iv_j)) \leq \min\{S(A(v_i)), S(A(v_j))\}$ for all $v_i, v_j \in V$. Hence,

$$\begin{aligned} E(\hat{G}) &\leq \sum_{1\leq i<j\leq n} S(A(v_i)) + S(A(v_j)), \\ &= (n-1)\sum_{i=1}^{n} S(A(v_i)). \end{aligned}$$

Thus, $E(\hat{G}) \leq (n-1)\sum_{i=1}^{n} S(A(v_i))$. □

Subsequently, a finding is presented that provides an improved lower bound and upper bound on the energy associated with the BVHFG. These bounds are expressed in relation to the total amount of nodes in BVHFG and the determinant of its adjacency matrix.

Theorem 4.14. *Consider $\hat{G} = (V, A, B)$ as the BVHFG, where $|V| = n$ and $B^* = \{e_1, e_2, \cdots, e_m\}$. Let $k_i = B(e_i)$ represent the membership grade for the i^{th} link and $|M|$ represent the determinant for adjacency matrix underlying $\hat{G}$, then, we have,*

$$\sqrt{2\sum_{i=1}^{m} S(k_i)^2 + n(n-1)|M|^{\frac{2}{n}}} \leq E(\hat{G}) \leq \sqrt{2n\sum_{i=1}^{m} S(k_i)^2}.$$

Proof. Suppose $(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)$ is the modulus of eigenvalues of the adjacency matrix of $\hat{G}$ and (1,1,$\cdots$,1) is the vector number.

For upper bound, using the Cauchy Schwarz inequality on the set of numbers $(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)$ and (1,1,$\cdots$,1), we get,

$$\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n}\sqrt{\sum_{i=1} |\lambda_i|^2}. \tag{2}$$

Applying Theorem 4.7, we also get

$$\sum_{i=1}^{n} |\lambda_i|^2 = 2\sum_{i=1}^{m} S(k_i)^2. \tag{3}$$

From equation (3) and equation (2), we get

$$\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n}\sqrt{2\sum_{i=1}^{m} S(k_i)^2} = \sqrt{2\sum_{i=1}^{m} S(k_i)^2 n}.$$
$$E(\hat{G}) \leq \sqrt{2\sum_{i=1}^{m} S(k_i)^2 n}.$$

Now, for lower bound,

$$\begin{aligned} |E(\hat{G})|^2 &= (\sum_{i=1}^{n} |\lambda_i|)^2, \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{1\leq i<j\leq n} |\lambda_i \lambda_j|, \\ &= 2\sum_{i=1}^{m} S(k_i)^2 + 2\frac{n(n-1)}{2} AM\{|\lambda_i \lambda_j|\}. \end{aligned}$$

Since, $AM\{|f_i f_j|\} \geq GM\{|f_i f_j|\}$, $1 \leq i < j \leq n$. Now,

$$E(\hat{G}) \geq \sqrt{2\sum_{i=1}^{m} S(k_i)^2 + n(n-1)GM\{|\lambda_i \lambda_j|\}},$$

Where,

$$\begin{aligned} GM\{\lambda_i\lambda_j\} &= (\prod_{1\leq i<j\leq n} |\lambda_i\lambda_j|)^{\frac{2}{n(n-1)}}, \\ &= (\prod_{i=1}^{n} |\lambda_i|^{n-1})^{\frac{2}{n(n-1)}}, \\ &= (\prod_{i=1}^{n} |\lambda_i|)^{\frac{2}{n}} \\ &= |M|^{\frac{2}{n}}. \end{aligned}$$

$$E(\hat{G}) \geq \sqrt{2\sum_{i=1}^{m} S(k_i)^2 + n(n-1)|M|^{\frac{2}{n}}}.$$

Thus, $\sqrt{2\sum_{i=1}^{m} S(k_i)^2 + n(n-1)|M|^{\frac{2}{n}}} \leq E(\hat{G}) \leq \sqrt{2\sum_{i=1}^{m} S(k_i)^2 n}.$ □

Example 4.15. Based on the depiction of Theorem 4.14, it is evident from Figure:1 that the numerical value of $E(\hat{G})$ is 1.6342, with the lower bound being 1.608 and the upper bound being 1.735. The inequality $1.608 < E(\hat{G}) < 1.735$ clearly holds.

Theorem 4.16. *consider $\hat{G} = (V, A, B)$ denote the BVHFG, where $|V| = n$ and $B^* = \{e_1, e_2, \cdots, e_m\}$. Let $k_i = B(e_i)$ denote the membership grade for the i^{th} link and $n \leq 2\sum_{i=1}^{m} S(k_i)^2$. Let $M(\hat{G})$ represent the adjacency matrix underlying $\hat{G}$, also $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$ are the eigenvalues of M, then, we have,*

$$E(\hat{G}) \leq \frac{2\sum_{i=1}^{m} S(k_i)^2}{n} + \sqrt{(n-1)\{2\sum_{i=1}^{m} S(k_i)^2 - (\frac{2\sum_{i=1}^{m} S(k_i)^2}{n})^2\}}.$$

Proof. By using the Gershgorin circle theorem and the property of the symmetric matrix, we have

$$\lambda_{max} = \lambda_1 \geq \frac{2\sum_{i=1}^{m} S(k_i)}{n},$$

where, λ_{max} represents the maximum eigenvalues of $M(\hat{G})$.

Now, since

$$\sum_{i=1}^{n} \lambda_i^2 = 2\sum_{i=1}^{m} S(k_i)^2. \tag{4}$$

$$\sum_{i=2}^{n} \lambda_i^2 = 2\sum_{i=1}^{m} S(k_i)^2 - \lambda_1^2. \tag{5}$$

By utilising the Cauchy Schwarz inequality, we can apply it to the set of numbers $(\lambda_2, \lambda_3, \cdots, \lambda_n)$ and the set (1,1,$\cdots$,1), resulting in the following expression:

$$E(\hat{G}) - \lambda_1 = \sum_{i=2}^{n} |\lambda_i| \leq \sqrt{(n-1)\sum_{i=2}^{n} |\lambda_i|^2}. \tag{6}$$

Based on equation (5) and equation (6), it can be inferred that

$$\begin{aligned} E(\hat{G}) - \lambda_1 &\leq \sqrt{(n-1)(2\sum_{i=1}^{m} S(k_i)^2 - \lambda_1^2)}, \\ E(\hat{G}) &\leq \sqrt{(n-1)(2\sum_{i=1}^{m} S(k_i)^2 - \lambda_1^2)} + \lambda_1. \end{aligned}$$

Let $f(x) = \sqrt{(n-1)(2\sum_{i=1}^{m} S(k_i)^2 - x^2)} + x$. It is clear that $f(x)$ is decreasing in the interval $[\sqrt{\frac{2\sum_{i=1}^{m} s(k_i)^2}{n}}, \sqrt{2\sum_{i=1}^{m} S(k_i)^2}]$.

Now, since $n \leq 2\sum_{i=1}^{m} S(k_i)^2$, so $1 \leq \frac{2\sum_{i=1}^{m} S(k_i)^2}{n}$, then we have,

$$\sqrt{\frac{2\sum_{i=1}^{m} S(k_i)^2}{n}} \leq \frac{2\sum_{i=1}^{m} S(k_i)^2}{n} \leq \frac{2\sum_{i=1}^{m} S(k_i)}{n} \leq \lambda_1 \leq \sum_{i=1}^{m} |\lambda_i| = \sqrt{2\sum_{i=1}^{m} S(k_i)^2}. \tag{7}$$

Therefore from Equation (7), $f(x)$ satisfies the inequality

$$f(\lambda_1) \leq \sqrt{(n-1)\{2\sum_{i=1}^{m} S(k_i)^2 - (\frac{2\sum_{i=1}^{m} S(k_i)^2}{n})^2\}} + \frac{2\sum_{i=1}^{m} S(k_i)^2}{n}.$$

Finally, we get $E(\hat{G}) \leq \sqrt{(n-1)\{2\sum_{i=1}^{m} S(k_i)^2 - (\frac{2\sum_{i=1}^{m} S(k_i)^2}{n})^2\}} + \frac{2\sum_{i=1}^{m} S(k_i)^2}{n}$. □

5 Laplacian energy concept of BVHFG

Our objective in this section is to formulate a BVHFG energy like quantity that is defined based on Laplacian eigenvalues, while maintaining the key characteristics of the original BVHFG energy.

Definition 5.1. *Degree matrix $D(\hat{G})$ under the BVHFG $\hat{G} = (V, A, B)$ has been a diagonal matrix with size $n \times n$, denoted as $D(\hat{G}) = [d_{ij}]_{n\times n}$, and defined by*

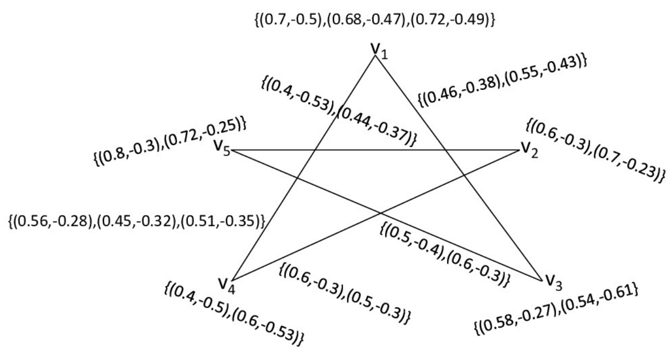
$$d_{ij} = \begin{cases} deg(v_i), & if \ \ i = j, \\ 0, & otherwise. \end{cases}$$

The degree matrix for Figure 3 is provided by

$$D(\hat{G}) = \begin{bmatrix} 0.867 & 0 & 0 & 0 & 0 \\ 0 & 0.860 & 0 & 0 & 0 \\ 0 & 0 & 0.905 & 0 & 0 \\ 0 & 0 & 0 & 0.837 & 0 \\ 0 & 0 & 0 & 0 & 0.885 \end{bmatrix}$$

Definition 5.2. *consider $M(\hat{G})$ and $D(\hat{G})$ are the adjacency matrix and degree matrix of BVHFG $\hat{G} = (V, A, B)$. The score based Laplacian matrix $S(L(\hat{G})) = D(\hat{G}) - S(M(\hat{G}))$ represent the square matrix with size $n \times n$, denoted as $L(\hat{G}) = [l_{ij}]_{n \times n}$.*

Example 5.3. Consider the BVHFG, let V be a set of five vertices $\{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_3, v_1v_4, v_2v_4, v_3v_5, v_2v_5\}$ be the collection of links, then BVHFG and BVHFS's, A and B over V and V^2 are given by Figure 3 and Table 2, respectively, also Laplacian matrix $L(\hat{G})$ is defined as follows:



C.jpg C.bb

Figure 3: The BVHFG

Table 2: Bipolar valued hesitant fuzzy table

	v_1	v_2	v_3
A	$\{(0.7, -0.5), (0.68, -0.47), (0.72, -0.49)\}$	$\{(0.6, -0.3), (0.7, -0.23)\}$	$\{(0.58, -0.27), (0.54, -0.61)\}$
Score	0.795	0.457	0.500

v_4	v_5
$\{(0.4, -0.5), (0.6, -0.53)\}$	$\{(0.8, -0.3), (0.72, -0.25)\}$
0.508	0.517

	v_1v_3	v_1v_4	v_2v_4
B	$\{(0.46, -0.38), (0.55, -0.43)\}$	$\{(0.4, -0.53), (0.44, -0.37)\}$	$\{(0.6, -0.3), (0.5, -0.3)\}$
Score	0.455	0.412	0.425

v_3v_5	v_2v_5
$\{(0.5, -0.4), (0.6, -0.3)\}$	$\{(0.4, -0.53), (0.44, -0.37)\}$
0.45	0.435

The score based adjacency matrix and Laplacian matrix under the BVHFG depicted in Figure 3 have been presented below.

$$S(M(\hat{G})) = \begin{bmatrix} 0 & 0 & 0.455 & 0.412 & 0 \\ 0 & 0 & 0 & 0.425 & 0.435 \\ 0.455 & 0 & 0 & 0 & 0.45 \\ 0.412 & 0.425 & 0 & 0 & 0 \\ 0 & 0.435 & 0.45 & 0 & 0 \end{bmatrix}$$

$$S(L(\hat{G})) = \begin{bmatrix} 0.867 & 0 & -0.455 & -0.412 & 0 \\ 0 & 0.860 & 0 & -0.425 & -0.435 \\ -0.455 & 0 & 0.905 & 0 & -0.45 \\ -0.412 & -0.425 & 0 & 0.837 & 0 \\ 0 & -0.435 & -0.45 & 0 & 0.885 \end{bmatrix}$$

Theorem 5.4. *consider $\hat{G} = (V, A, B)$ is the BVHFG, where $|V| = n$. suppose $\{\mu_1, \mu_2, \cdots, \mu_n\}$ and $\{e_1, e_2 \cdots, e_m\}$ is the eigenvalues associated with Laplacian matrix and the collection of links underlying $\hat{G}$, respectively. Suppose $L(\hat{G}) = [l_{ij}]_{n\times n}$ is the Laplacian matrix. Then, we have, $\sum_{i=1}^{n} \mu_i = 2\sum_{i=1}^{m} S(B(e_i))$ and $\sum_{i=1}^{n} \mu_i^2 = 2\sum_{i=1}^{m} S(B(e_i))^2 + \sum_{i=1}^{n} deg^2(v_i)$.*

Proof. (1) The trace associated with Laplacian matrix $L(\hat{G})$ is given by

$$\begin{aligned} trace(L) &= \sum_{i=1}^{n} S(B(l_{ii})), \\ &= \sum_{i=1}^{n} deg(v_i), \\ &= 2\sum_{i=1}^{m} S(B(e_i)). \end{aligned}$$

Also the trace associated with the square matrix has been equivalent of the total of its eigenvalues. Therefore, $trace(L) = \sum_{i}^{n} \mu_i = 2\sum_{i=1}^{m} S(B(e_i))$.

(2) Now, since Laplacian matrix $L(\hat{G})$ is the symmetric matrix, so

$$\begin{aligned} trace(LL^t) = trace(L^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} S(B(l_{ij}))S(B(l_{ij})), \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} S(B(l_{ij}))^2, \\ &= 2\sum_{1=i<j=n} S(B(l_{ij}))^2 + \sum_{i=j=1}^{n} S(B(l_{ii}))^2, \\ &= 2\sum_{i=1}^{m} S(B(e_i))^2 + \sum_{i=1}^{n} deg^2(v_i). \end{aligned}$$

Therefore, $trace(L^2) = \sum_{i=1}^{n} \mu_i^2 = 2\sum_{i=1}^{m} S(B(e_i))^2 + \sum_{i=1}^{n} deg^2(v_i)$. Hence the proof is completed. □

Remark 5.5. *In scenarios where the BVHFG $\hat{G}$ consists of k components ($k \geq 1$), and assuming that the Laplacian eigenvalues are ordered such that $\mu_1 \geq \mu_2, \cdots, \geq \mu_n$, it may be deduced that $\mu_{n-i} = 0$ for $i = 0, \cdots, k-1$ and $\mu_{n-k} > 0$.*

Definition 5.6. Consider $\hat{G}$ denote a BVHFG. Let $L(\hat{G})$ be an $n \times n$ Laplacian matrix, $S(L(\hat{G}))$ denote the score of the Laplacian matrix and $\mu_i's$, $i = 1, 2, \cdots, n$ are the Laplacian eigenvalues of $\hat{G}$, then, we have, the Laplacian energy associated with BVHFG is defined as

$$E_L(\hat{G}) = \sum_{i=1}^{n} |\gamma_i|,$$

where, $\gamma_i = \mu_i - \frac{2 \sum_{1=i<j=n} S(B(l_{ij}))}{n}$.

Remark 5.7. *According to the analogy presented in theorem 4.7, it can be inferred that*

$$\sum_{i=1}^{n} \gamma_i = 0; \quad \sum_{i=1}^{n} \gamma_i^2 = 2M,$$

where, $M = \sum_{1=i<j=n} S(B(l_{ij}))^2 + \frac{1}{2} \sum_{i=1}^{n} (deg(v_i) - \frac{2 \sum_{1=i<j=n} S(B(l_{ij}))}{n})^2$.

Example 5.8. The Laplacian eigenvalues and Laplacian energy associated with the BVHFG, as presented in Figure 3, are provided below.
Laplacian $Spec(\hat{G})$=$\{1.61154, 1.54033, 0.608674, 0.593459, 0\}$
$\gamma_1 = 0.74074$, $\gamma_2 = 0.66953$, $\gamma_3 = -0.262126$, $\gamma_4 = -0.277341$, $\gamma_5 = -0.8708$. Therefore, $E_L(\hat{G}) = 2.820537$. Moreover, according to Remark 5.7, we have $\sum_{i=1}^{n} \gamma_i = 0$ and $\sum_{i=1}^{n} \gamma_i^2 = 2(0.949119 + 0.0013224) = 1.90088$

Remark 5.9. *The Laplacian matrices have non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ with*

$$E_L(\hat{G}) = \sum_{i=1}^{n} |\mu_i - \frac{2 \sum_{i<j}^{n} S(B(l_{ij}))}{n}| = 2 \sum_{i=1}^{\alpha} \mu_i - \frac{4 \sum_{i<j}^{n} S(B(l_{ij}))\alpha}{n},$$

where α is the greatest positive integer satisfying the condition $\mu_\alpha \geq \frac{2 \sum_{i<j}^{n} S(B(l_{ij}))}{n}$.

5.1 BVHFG Energy and Laplacian energy relationship

Lemma 5.10. [39] *Let P and Q denote the real symmetric matrices with order $n \times n$. For any $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \lambda_i(P + Q) \leq \sum_{i=1}^{k} \lambda_i(P) + \sum_{i=1}^{k} \lambda_i(Q),$$

holds. Where $\lambda_i(R)$ represents the i^{th} largest eigenvalue underlying matrix R.

Theorem 5.11. *Let $\hat{G} = (V, A, B)$ denote the BVHFG and $|V| = n$. Let $\{\mu_1, \mu_2, \cdots, \mu_n\}$ and $\{e_1, e_2 \cdots, e_m\}$ denote the eigenvalues associated with Laplacian matrix and the collection of links of $\hat{G}$, respectively. Then,*

$$E_L(\hat{G}) \leq E(\hat{G}) + 2 \sum_{i=1}^{\alpha} (deg(v_i) - \frac{2 \sum_{i<j}^{n} S(B(l_{ij}))}{n}),$$

where α is the greatest positive integer satisfying the condition in Remark 5.9.

Proof. Consider L, D, M denote the Laplacian matrix, Degree matrix and Adjacency matrix of $\hat{G}$, respectively. From Lemma 5.10 for every $1\leq k \leq n$, we have,

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i(D-M) &\leq \sum_{i=1}^{k}\lambda_i(D)+\sum_{i=1}^{k}\lambda_i(-M).\\ \sum_{i=1}^{k}\mu_i &\leq \sum_{i=1}^{k}deg(v_i)-\sum_{i=1}^{k}\lambda_{n-i+1}, \end{aligned} \tag{8}$$

because, $\sum_{i=1}^{k}\lambda_i(-M)=-\sum_{i=1}^{k}\lambda_{n-i+1}(M)$.
Now,

$$\begin{aligned} E(\hat{G}) &= \sum_{i=1}^{n}|\lambda_i|,\\ &= -2\sum_{\lambda_i<0}\lambda_i,\\ &= 2\max\{-\sum_{i=1}^{k}\lambda_{n-i+1}:\quad 1\leq k\leq n\},\\ &\geq -2\sum_{i=1}^{k}\lambda_{n-i+1}\quad for\ any\ 1\leq k\leq n. \end{aligned}$$

putting in Equation (8). We have,

$$\begin{aligned}\sum_{i=1}^{k}\mu_i &\leq \sum_{i=1}^{k}deg(v_i)+\frac{E(\hat{G})}{2}.\\ 2\sum_{i=1}^{k}\mu_i &\leq 2\sum_{i=1}^{k}deg(v_i)+E(\hat{G}).\end{aligned}$$

Since, α is the greatest positive integer satisfying the condition $\mu_\alpha \geq \frac{2\sum_{i<j}^{n}S(B(l_{ij}))}{n}$, so we can write here,

$$2\sum_{i=1}^{\alpha}\mu_i-\frac{4\alpha\sum_{i<j}^{n}S(B(l_{ij}))}{n} \leq 2\sum_{i=1}^{\alpha}deg(v_i)+E(\hat{G})-\frac{4\alpha\sum_{i<j}^{n}S(B(l_{ij}))}{n}.$$

Therefore, by using Remark 5.9, we get, $E_L(\hat{G})\leq E(\hat{G})+2\sum_{i=1}^{\alpha}(deg(v_i)-\frac{2\sum_{i<j}^{n}S(B(l_{ij}))}{n})$. □

Theorem 5.12. *Assuming the BVHFG $\hat{G}$ constitutes a regular graph, then $E_L(\hat{G})=E(\hat{G})$.*

Proof. Given that $\hat{G}$ is a regular BVHFG with degree k. Let $\lambda_1,\lambda_2,\cdots,\lambda_n$ and $\mu_1,\mu_2,\cdots,\mu_n$ denote the eigenvalues of adjacency and Laplacian matrix of $\hat{G}$, respectively.
BVHFG is k-regular so the degree matrix associated with $\hat{G}$ denote the scalar (k) multiple of the identity

matrix. We have, $\mu_i = k - \lambda_i$.
Now,

$$\begin{aligned} E_L(\hat{G}) &= \sum_{i=1}^{n} |\mu_i - \frac{2\sum\limits_{i\leq j}^{n} S(B(l_{ij}))}{n}|, \\ &= \sum_{i=1}^{n} |k - \lambda_i - \frac{2\sum\limits_{i\leq j}^{n} S(B(l_{ij}))}{n}|, \\ &= \sum_{i=1}^{n} |k - \lambda_i - k|, \end{aligned}$$

since $\hat{G}$ is k-regular so $\frac{2\sum\limits_{i\leq j}^{n} S(B(l_{ij}))}{n} = k$.

Therefore, $E_L(\hat{G}) = \sum\limits_{i=1}^{n} |\lambda_i| = E(\hat{G})$. □

6 Numerical illustration and discussion of BVHFG energy and Laplacian energy to a decision-making issue

Table 3: An algorithm for choosing the best facade dressing option for covering building surfaces.

Input:	A set of surface covering alternatives $x_1, x_2, \cdots, x_n$; a group of experts $E_1, E_2, \cdots, E_m$; and BVHF preference relations from each expert $(M^k = (r^k_{ij})_{n\times n})$.
Output:	Best alternative for facade dressing.
1.	Begin.
2.	Compute the energy and correlation coefficient of each BVHFG G_k for $k = 1, 2, \cdots, m$.
3.	Determine the weight vector for each expert using energy and correlation: $w^a_k = \frac{E(G_k)}{\sum_{k=1}^{m} E(G_k)}$, $w^b_k = \frac{K(G_k)}{\sum_{i=1}^{m} K(G_i)}$.
4.	Compute the objective weight for each expert: $w_k = \gamma w^a_k + (1-\gamma) w^b_k, \quad \gamma \in I^P$.
5.	Determine net preference degree for each alternative: $\theta(x_i) = \sum_{k=1}^{m} w_k \left(\sum_{i\neq j\leq n} \left(S(r^k_{ij})^2 - S(r^k_{ji})^2\right)\right)$, for $i = 1, 2, \cdots, n$.
6.	Rank all alternatives based on $\theta(x_i)$.
7.	Repeat the decision-making process using Laplacian matrices.
8.	Select the best alternative.
9.	End.

6.1 Illustration of the proposed approach

A committee of decision-makers evaluates several facade dressing alternatives for a building's surface covering based on their practical qualities. For $1 \leq k \leq 5$, consider the five experts E_k are in the group: E_1 is a civil engineer, E_2 is a builder, E_3 is an architect, E_4 is a contractor, and E_5 is a decorator. The specialists contrast three different alternatives: x_1 = plastic painting, x_2 = compact laminate clothing, and x_3 = wood clothing. The following bipolar valued hesitant fuzzy preference relations $M^k = (r^k_{ij})_{3\times 3}$ are developed after each expert compares each set of criteria (alternatives) x_i and x_j individually and provides his or her bipolar valued hesitant fuzzy preference value $r_{ij} = \{(r^P_{ij}, r^N_{ij}) |\, r^P_{ij} \in I^P \; and \; r^N_{ij} \in I^N\}$, which is made up of a certainty degree r^p_{ij} in which x_i has preference over x_j as well as a certainty degree r^N_{ij} in which x_i has not preference over x_j:

$$M^1 = \begin{bmatrix} (0.5,-0.5) & \{(0.51,-0.41),(0.48,-0.40)\} & \{(0.37,-0.43),(0.44,-0.42)\} \\ \{(0.49,-0.59),(0.52,-0.60)\} & (0.5,-0.5) & \{(0.56,-0.28),(0.61,-0.26)\} \\ \{(0.63,-0.57),(0.56,-0.58)\} & \{(0.44,-0.72),(0.39,-0.74)\} & (0.5,-0.5) \end{bmatrix}$$

$$M^2 = \begin{bmatrix} (0.5,-0.5) & \{(0.65,-0.56),(0.52,-0.41)\} & \{(0.48,-0.36),(0.51,-0.46)\} \\ \{(0.42,-0.44),(0.48,-0.52)\} & (0.5,-0.5) & \{(0.72,-0.57),(0.55,-0.49)\} \\ \{(0.52,-0.64),(0.49,-0.54)\} & \{(0.28,-0.63),(0.45,-0.79)\} & (0.5,-0.5) \end{bmatrix}$$

$$M^3 = \begin{bmatrix} (0.5,-0.5) & \{(0.63,-0.54),(0.59,-0.51)\} & \{(0.65,-0.48),(0.57,-0.55)\} \\ \{(0.37,-0.66),(0.41,-0.69)\} & (0.5,-0.5) & \{(0.66,-0.31),(0.61,-0.49)\} \\ \{(0.55,-0.77),(0.59,-0.78)\} & \{(0.44,-0.69),(0.39,-0.71)\} & (0.5,-0.5) \end{bmatrix}$$

$$M^4 = \begin{bmatrix} (0.5,-0.5) & \{(0.71,-0.47),(0.63,-0.48)\} & \{(0.49,-0.51),(0.47,-0.39)\} \\ \{(0.53,-0.41),(0.55,-0.38)\} & (0.5,-0.5) & \{(0.52,-0.39),(0.69,-0.54)\} \\ \{(0.39,-0.28),(0.46,-0.31)\} & \{(0.37,-0.39),(0.41,-0.59)\} & (0.5,-0.5) \end{bmatrix}$$

$$M^5 = \begin{bmatrix} (0.5,-0.5) & \{(0.56,-0.51),(0.59,-0.33)\} & \{(0.61,-0.59),(0.67,-0.58)\} \\ \{(0.51,-0.38),(0.44,-0.39)\} & (0.5,-0.5) & \{(0.54,-0.49),(0.59,-0.44)\} \\ \{(0.38,-0.42),(0.34,-0.41)\} & \{(0.65,-0.48),(0.71,-0.47)\} & (0.5,-0.5) \end{bmatrix}$$

Now, we find the score of the preference relations:

$$S(M^1) = \begin{bmatrix} 0.5 & 0.451 & 0.414 \\ 0.553 & 0.5 & 0.43 \\ 0.59 & 0.572 & 0.5 \end{bmatrix}$$

$$S(M^2) = \begin{bmatrix} 0.5 & 0.533 & 0.453 \\ 0.465 & 0.5 & 0.583 \\ 0.548 & 0.538 & 0.5 \end{bmatrix}$$

$$S(M^3) = \begin{bmatrix} 0.5 & 0.568 & 0.563 \\ 0.533 & 0.5 & 0.414 \\ 0.673 & 0.558 & 0.5 \end{bmatrix}$$

$$S(M^4) = \begin{bmatrix} 0.5 & 0.573 & 0.465 \\ 0.468 & 0.5 & 0.535 \\ 0.36 & 0.428 & 0.5 \end{bmatrix}$$

$$S(M^5) = \begin{bmatrix} 0.5 & 0.498 & 0.613 \\ 0.43 & 0.5 & 0.515 \\ 0.388 & 0.578 & 0.5 \end{bmatrix}$$

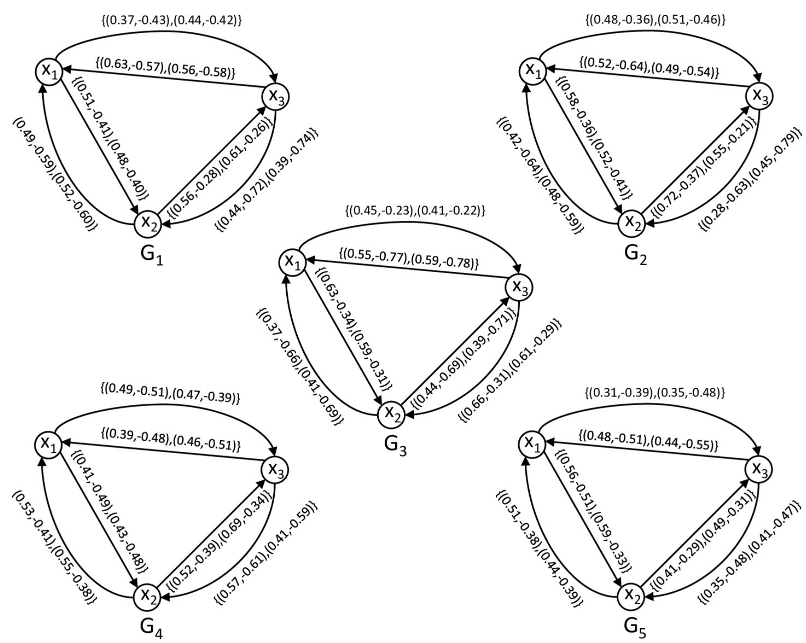


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Figure 4: BVHFG of preference relations M^1, M^2, M^3, M^4 and M^5

The BVHFGs denoted as G_k, which correspond to the bipolar valued hesitant fuzzy preference relations presented in matrices M^ks, are illustrated in Figure 4. In order to determine the objective weight of each expert, we compute the energy and correlation coefficient of BVHFG's.

The energy values of the BVHFG's are as follows: $E(G_1) = 1.5321$, $E(G_2) = 1.6389$, $E(G_3) = 1.7543$, $E(G_4) = 1.5581$, and $E(G_5) = 1.5744$. Subsequently, the weight of each expert through energy may be computed in the following manner:

$$w_k^a = \frac{E(G_k)}{\sum\limits_{k=1}^{m} E(G_k)} \tag{9}$$

The values of w_1^a, w_2^a, w_3^a, w_4^a and w_5^a are $0.1901, 0.2034, 0.2177, 0.1934$, and 0.1954, respectively.

Now, we compute the score base correlation coefficients $K(G_s, G_t)$ between G_s and G_t for $s \neq t = 1, 2, \cdots, 5$, using the following equation.

$$K(G_s, G_t) = K(S(M^s), S(M^t)) = \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} r_{ij}^s r_{ij}^t}{\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}(r_{ij}^s)^2}\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}(r_{ij}^t)^2}} \tag{10}$$

The values are given by $K(G_1, G_2) = 0.9911$, $K(G_1, G_3) = 0.9977$, $K(G_1, G_4) = 0.9760$, $K(G_1, G_5) = 0.9772$, $K(G_2, G_3) = 0.9885$, $K(G_2, G_4) = 0.9903$, $K(G_2, G_5) = 0.9874$, $K(G_3, G_4) = 0.9695$, $K(G_3, G_5) = 0.9795$, $K(G_4, G_5) = 0.9891$.

Moreover, the average correlation coefficient $K(G_t)$ is calculated as follows:

$$K(G_t) = \frac{1}{m-1} \sum_{s \neq t = 1}^{m} K(G_t, G_s) \tag{11}$$

The values are given by $K(G_1) = 0.9855$, $K(G_2) = 0.9893$, $K(G_3) = 0.9838$, $K(G_4) = 0.9812$, $K(G_5) = 0.9833$. Consequently, the determination of the weight of each expert based on the correlation coefficient can be calculated in the following way:

$$w_k^b = \frac{K(G_k)}{\sum_{i=1}^{m} K(G_i)} \tag{12}$$

The values of w_1^b, w_2^b, w_3^b, w_4^b and w_5^b are $0.2002, 0.2020, 0.1998, 0.1993$, and 0.1997, respectively.

Ultimately, the objective weight of each expert is determined through the utilization of the subsequent equation:

$$w_k = \gamma w_k^a + (1-\gamma) w_k^b, \;\; \gamma \in I^P. \tag{13}$$

Consider the values of γ as $\gamma = 0$, 0.5, and 1. When $\gamma = 0$, the objective weight is solely determined by the correlation coefficient. When $\gamma = 0.5$, the objective weight is equally influenced by both the weight determined by energy and the correlation coefficient. When $\gamma = 1$, the objective weight is solely influenced by the weight determined by energy.

In order to ascertain the appropriate alternative, i.e., the overall degree of precedence of x_i compared to the remaining alternatives, the following definition can be utilized:

$$\theta(x_i) = \sum_{k=1}^{m} w_k \Big(\sum_{i \neq j \leq n} (S(r_{ij}^k)^2 - S(r_{ji}^k)^2) \Big), \quad i = 1, 2, \cdots, n. \tag{14}$$

The respective net flows of the three alternatives for various degrees of γ are presented in Table 4.

Table 4: The net degree of alternatives for different values of γ

γ	weight	θ
0	$w_1 = 0.2002$, $w_2 = 0.2020$, $w_3 = 0.1998$, $w_4 = 0.1993$, $w_5 = 0.1997$	$x_1 = 0.01589$ $x_2 = -0.0747$ $x_3 = 0.0590$
0.5	$w_1 = 0.1952$, $w_2 = 0.2027$, $w_3 = 0.2088$, $w_4 = 0.1964$, $w_5 = 0.1976$	$x_1 = 0.015109$ $x_2 = -0.075892$ $x_3 = 0.060782$
1	$w_1 = 0.1901$, $w_2 = 0.2034$, $w_3 = 0.2177$, $w_4 = 0.1934$, $w_5 = 0.1954$	$x_1 = 0.01424$ $x_2 = -0.08109$ $x_3 = 0.06272$

Based on these values presented in Table 4, the alternatives can be ranked accordingly, $x_3 > x_1 > x_2$. The ranking order remains consistent across all values of γ. Therefore, x_3 is considered the optimal choice. Next, we follow the above procedure of decision-making through Laplacian matrices.

The following bipolar valued hesitant fuzzy Laplacian matrices correspond to the BVHFG's (Figure 4) is given by $L^k = (l^k_{ij})_{3\times 3}$, where,

$$l^k_{ij} = \begin{cases} deg(x_i)^k, & \text{if } i = j, \\ -r^k_{ij}, & \text{if } i \neq j, \end{cases}$$

The score based Laplacian matrices of the BVHFG's depicted in Figure 4 are presented below.

$$S(L^1) = \begin{bmatrix} 0.865 & -0.451 & -0.414 \\ -0.553 & 0.983 & -0.43 \\ -0.59 & -0.572 & 1.162 \end{bmatrix}$$

$$S(L^2) = \begin{bmatrix} 0.986 & -0.533 & -0.453 \\ -0.465 & 1.048 & -0.583 \\ -0.548 & -0.538 & 1.086 \end{bmatrix}$$

$$S(L^3) = \begin{bmatrix} 1.131 & -0.568 & -0.563 \\ -0.533 & 0.947 & -0.414 \\ -0.673 & -0.558 & 1.231 \end{bmatrix}$$

$$S(L^4) = \begin{bmatrix} 1.038 & -0.573 & -0.465 \\ -0.468 & 1.003 & -0.535 \\ -0.36 & -0.428 & 0.788 \end{bmatrix}$$

$$S(L^5) = \begin{bmatrix} 1.111 & -0.498 & -0.613 \\ -0.43 & 0.945 & -0.515 \\ -0.388 & -0.578 & 0.966 \end{bmatrix}$$

The Laplacian energy values of the BVHFG's G_k(Figure 4) are as follows: $E_L(G_1) = 2.5083$, $E_L(G_2) = 2.6$, $E_L(G_3) = 2.757$, $E_L(G_4) = 2.3575$, and $E_L(G_5) = 2.5247$. Subsequently, the Laplacian weight of each experts may be computed in the following manner:

$$L(w_k) = \frac{E_L(G_k)}{\sum\limits_{k=1}^{m} E_L(G_k)} \tag{15}$$

The values of $L(w^a_1)$, $L(w^a_2)$, $L(w^a_3)$, $L(w^a_4)$ and $L(w^a_5)$ are $0.1968, 0.2040, 0.2162, 0.1849$, and 0.1981, respectively. Now, we compute the score base Laplacian correlation coefficients $K_L(G_s, G_t)$ between G_s and G_t for $s \neq t$, using the following equation.

$$K_L(G_s, G_t) = K(S(L^s), S(L^t)) = \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} l^s_{ij} l^t_{ij}}{\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} (l^s_{ij})^2}\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} (l^t_{ij})^2}} \tag{16}$$

The values are given by $K_L(G_1, G_2) = 0.9934$, $K_L(G_1, G_3) = 1.1828$, $K_L(G_1, G_4) = 0.9696$, $K_L(G_1, G_5) = 0.9777$, $K_L(G_2, G_3) = 0.9910$, $K_L(G_2, G_4) = 0.9950$, $K_L(G_2, G_5) = 0.9898$, $K_L(G_3, G_4) = 0.9751$, $K_L(G_3, G_5) = 0.9864$, $K_L(G_4, G_5) = 1.2204$.

Moreover, the average correlation coefficient $K_L(G_t)$ is calculated as follows:

$$K_L(G_t) = \frac{1}{m-1}\sum_{t\neq s=1}^{m} K_L(G_t, G_s) \tag{17}$$

The values are given by $K_L(G_1) = 1.0309$, $K_L(G_2) = 0.9923$, $K_L(G_3) = 1.0338$, $K_L(G_4) = 1.0400$, $K_L(G_5) = 1.0436$. Consequently, the determination of the Laplacian weight of each expert based on the correlation coefficient can be calculated in the following way:

$$L(w_k^b) = \frac{K_L(G_k)}{\sum\limits_{i=1}^{m} K_L(G_i)} \tag{18}$$

The values of $L(w_1^b)$, $L(w_2^b)$, $L(w_3^b)$, $L(w_4^b)$ and $L(w_5^b)$ are $0.2005, 0.1930, 0.2011, 0.2023$, and 0.2030, respectively.

Ultimately, the objective weight of each expert is determined through the utilization of the subsequent equation:

$$L(w_k) = \gamma L(w_k^a) + (1-\gamma)L(w_k^b),\ \ \gamma \in I^P. \tag{19}$$

In order to ascertain the appropriate alternative through Laplacian matrices, the following definition can be utilized:

$$\theta_L(x_i) = \sum_{k=1}^{m} L(w_k)\Big(\sum_{i\neq j\leq n} (S(l_{ij}^k)^2 - S(l_{ji}^k)^2)\Big), \quad i = 1, 2, \cdots, n. \tag{20}$$

The respective net flows of the three alternatives through Laplacian matrices for various degrees of γ have been presented in Table 5.

Table 5: The Laplacian degree of alternatives

γ	Laplacian weight	θ_L
0	$w_1 = 0.2005$, $w_2 = 0.1930$, $w_3 = 0.2011$, $w_4 = 0.2023$, $w_5 = 0.2030$	$x_1 = 0.0173577$ $x_2 = -0.0796645$ $x_3 = 0.0579519$
0.5	$w_1 = 0.1987$, $w_2 = 0.1985$, $w_3 = 0.2087$, $w_4 = 0.1936$, $w_5 = 0.2006$	$x_1 = 0.0146$ $x_2 = -0.0806$ $x_3 = 0.0617$
1	$w_1 = 0.1968$, $w_2 = 0.2040$, $w_3 = 0.2162$, $w_4 = 0.1849$, $w_5 = 0.1981$	$x_1 = 0.01175$ $x_2 = -0.08136$ $x_3 = 0.06550$

These values, which are shown in Table 5, allow the alternatives to be rated as follows: $x_3 > x_1 > x_2$. The ranking order remains consistent across all values of γ. Therefore, x_3 is considered the optimal choice. The entire process of selecting the best alternative is illustrated in Figure 5.

Note: The time complexity of the algorithm represented in Table 3 is $O(kn^2)$, where k is the number of experts and n is the number of alternatives. This is because the algorithm has two nested loops that iterate over the alternatives (lines 5 and 6), and each iteration involves a summation over the experts (line 5). The rest of the steps are either constant or linear in terms of time complexity.

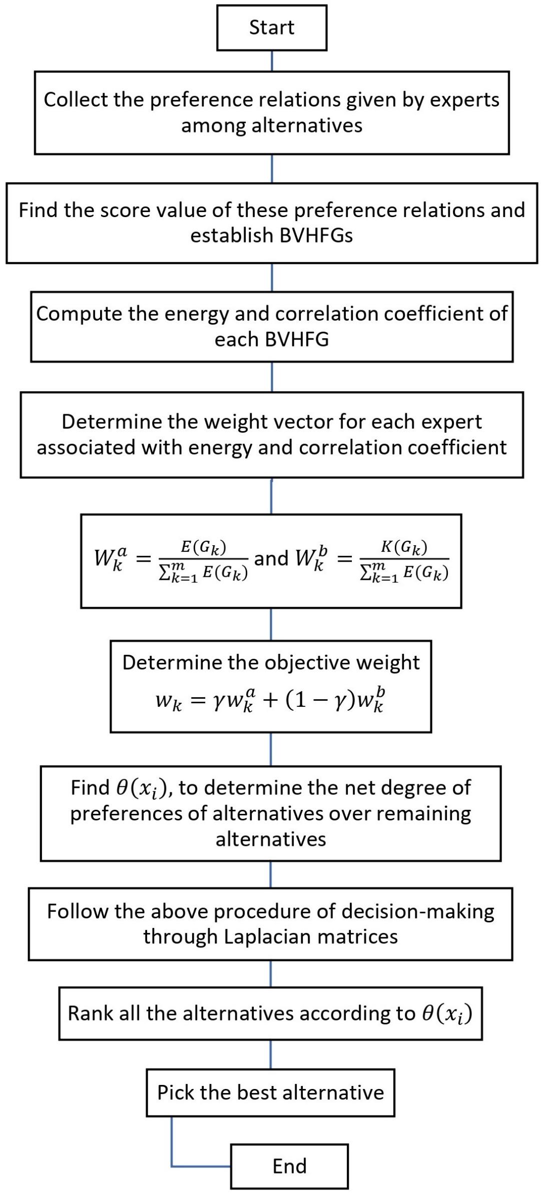

Figure 5: Flowchart of the whole procedure

6.2 Discussion

The bipolar-valued hesitant fuzzy graphs (BVHFGs) have an edge over existing fuzzy graphs like hesitant fuzzy graphs and bipolar fuzzy graphs, which are discussed in [8]. In this study, based on the above numerical illustration it can be seen that, both the energy and correlation coefficient of BVHFG's effectively rank the alternatives in practical applications. Additionally, the evaluation values obtained from the formula in both procedures can be utilized for further analysis. However, other methods do not provide this specific information as they offer hesitant fuzzy values or interval values instead of crisp values. The selection of parameter γ can be made in practical scenarios, taking into consideration the decision-maker's inclination towards either subjective or objective weighted information provided by experts. In addition, this approach demonstrates improved efficiency compared to the method described in [40], because the above research work [40] has been done in the framework of hesitancy fuzzy graphs [18]. Despite bringing the concept of hesitant fuzzy elements (HFEs) to the vertices and edges of the graph, they did not implement it. Instead of using HFEs, they employ Intuitionistic fuzzy (IF) values [19], which are represented by triplets including the membership, hesitancy and non-membership degree of vertices and edges. The present study has been done in the framework of BVHFGs which considers fuzziness, bipolarity as well as hesitation of the elements.

7 Conclusion

The utilization of a bipolar valued hesitant fuzzy model offers enhanced adaptability, coherence, and accuracy to the system in comparison to alternative fuzzy models. This paper provides the definitions of the adjacency matrix, energy, correlation coefficient, and Laplacian energy for bipolar valued hesitant fuzzy graphs. The article also presents derived outcomes concerning energy and Laplacian energy bounds for BVHFG's. In addition, we establish the energy and Laplacian energy relationship within the context of a BVHF background. Furthermore, we make reference to numerical examples pertaining to the energy and Laplacian energy of BVHFG's.

In decision making tasks, this study introduces a methodology for evaluating the relative significance of an expert's weights that incorporates both subjective and objective factors. This research demonstrates the utilization of correlation coefficient and energy as effective tools for addressing group decision making issues in scenarios where the experts' weights are entirely unknown. This paper also presents an algorithm that aims to comprehensively analyse the process of identifying the optimal alternative.

Nonetheless, our work has some limitations, we assume a modest-size network here, but manually calculating the whole procedure for complex or big networks is quite tough. For this computation, we must devise an appropriate pseudo code. Subsequent investigation into the energy and Laplacian energy of BVHFG's may yield additional comparable outcomes of this nature and will be expounded upon in forthcoming papers.

Acknowledgements: "The first author is thankful to the Council of Scientific and Industrial Research India for award of senior research fellowship (Award No. 09/1041(0016)/2019-EMR-I) to meet up the financial expenditure to carry out the research work. The authors are also thankful to the Department of Science and Technology, New Delhi, India for approving the proposal under the scheme FIST program (Ref. No. SR/FST/MS/2022 dated 19.12.2022)."

Conflict of Interest: "The authors declare that they have no conflict of interest."

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