Haar Wavelet method for numerical solution of two-dimensional partial fractional integro-differential equations

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Abstract
This article examines new methods for solving fractional integral differential equations of Fredholm using wavelets. In this research, first, fractional integral differential equations and their special properties are introduced. Then, the importance of using wavelets as a tool for analyzing and solving these equations is explained.

Wavelet methods have many advantages due to their ability to display signals and analyze nonlinear and indirect data, especially in complex and dynamic problems. The article describes various algorithms and techniques that, by utilizing the properties of wavelets, can be used to achieve numerical and analytical solutions of the above equations.

Convergence results and error evaluation are also presented in this article using examples to demonstrate the effectiveness and high efficiency of wavelet methods in solving fractional integral differential equations of Fredholm. It also reduces the variable-order fractional derivative theorem to a system of algebraic equations by approximating the Haar wavelet and integrating it.

Keywords

Haar wavelet method, derivative fractional, error analysis, numerical solution, two-dimensional integro-differential equation

1. Introduction

Solving partial differential equations has been of interest to scientists for a long time. Many researchers have proposed this method due to the applicability of fractional differential integral equations of singular type [19]. Recently, fractional integral differential equations have been used to model many physical phenomena in various fields of non-linear oscillation of earthquakes, fluid dynamics traffic, continuum, statistical mechanics of signal processing, control theory, dynamics, and the relationship between nanoparticles [9, 21]. Numerous numerical methods have been thoroughly researched to solve these equations, including Fourier transform, Laplace analysis, fractional differential transform, finite difference method, orthogonal functions, Adomian decomposition method, variable iteration method, and homotopy analysis method. They are used to obtain approximate solutions for fractional equations [10,12, 20]. Fractional integral differential equations are valuable in modeling many phenomena [13]. Over the past four decades, scientists have focused on the theory and applications of partial differential equations of fractional order, which generalize differential equations of the correct order. One such method that has gained attention is the modified homotopy analysis transformation method [15,17].

using Harr wavelets [4]. Islameh and Aziz also proposed a method for numerically solving one-dimensional equations using wavelets [6].

In 1909, Haar was the first person to mention wavelets. Later, Jean Morelet discovered that Fourier bases were not ideal tools for underground exploration, which led to the discovery of wavelets. Mir and Mallet then laid the foundations of orthogonal wavelets and created algorithms for wavelet decomposition and reconstruction. In 1990, Morenzi and Antonie expanded wavelets to two dimensions [3]. Wavelet analysis has been used to analyze transient signals that change rapidly. It has various applications, including analyzing sound and audio signals, electrical activity in the brain, and underwater sounds. It is also used to control power plants through the NMR display of computer spectroscopic data [14,3]. Today, Wavelets have various applications, including brain tissue separation, CT scanning in medical imaging, magnetic resonance imaging of nuclear energy, industries, agriculture, and computer software and hardware [16]. Over the last two decades, there have been advancements in wavelet theory. As a result, several studies have been conducted on solving integro-differential equations using wavelet methods. For instance, in 2004, Hibbert-Taylor solved Fredholm integral equations using wavelet methods [7]. In 2012, the Legendre wavelet method was employed to solve second-type Fredholm integral equations [8]. Wavelets were also used to solve partial fractional equations. The solution for binary systems of fractional integral differential equations has been achieved by utilizing Haar and Legendre wavelets. These wavelets have been employed in solving partial fractional equations as well as binary systems of fractional integral differential equations. However, applying Haar wavelets in solving 2D fractional differential integral equations is a new and unexplored phenomenon. Therefore, we aim to utilize the Haar wavelet method to solve the two-dimensional fractional Fredholm integrodifferential equations of the form,

$$D_t^{\beta} u(x,t) = f(x,t) + \int_0^1 \int_0^1 k(x,y,t,\omega) u(y,\omega) dy d\omega \qquad (1)$$

Where $D_t^{\beta} u(t)$is the fractional derivative and u(x, t) be a function defined over [0, 1] × [0, 1], and k(x, y, t, ω) be a continuous kernel; in addition, assume $0 < \beta < 1$. This article is written as follows: The concepts of Harr wavelet and related theorems are presented in section 2. The proposed method is presented in section 3. And finally, the accuracy and efficiency of the proposed design are shown using numerical solutions with some examples with tables and graphs in section 4.

2. Haar Wavelets

Harr basis wavelet $(\psi_{j,i}\ (y))_{j \in N\ ,\ i \in Z}$ is a constant family function and an orthogonal subfamily of Hilbert space L^2 (R), a group of functions that arise from a constant function ψ called the mother wavelet. In the wavelet family, the following relations are established:

$$\Psi_{i,i}(y) = 2^{j/2} \psi(2^j y - i).$$

For the group of raging Haar wavelet in the interval [0, 1) we have,

$$h_1(x) = \begin{cases} 1 & , \quad for\ x \in [\alpha, \beta) \\ 0 & , \quad elsewhere \end{cases} \qquad (2)$$

and

$$h_i(\mathrm{x}) = \begin{cases} 1 \ , & for\ x \in [\alpha, \beta) \\ -1 \ , & for\ x \in [\beta, \gamma) \\ 0 \ , & otherwise\ , i = 2,3,4, \ldots \end{cases}$$

where

$$\alpha_k = \frac{k}{m} \quad , \quad \beta_k = \frac{k+0.5}{m} \quad , \quad \gamma_k = \frac{k+1}{m} \quad ,$$

$$m = 2^j \quad , \quad j = 0\ ,1,2, \ldots, j \quad , k = 0,1,2, \ldots, m-1$$

The connection between i, k, and m is given by i = k +m + 1. k is the transmission parameter.
In table 1, we calculate the correct values for i, j, and k up to level j=3

Table1. Calculation for Haar wavelet bases at j=3

k	0	0	1	0	1	2	3	0	1	2	3	4	....	7
j	0	1	1	2	2	2	2	3	3	3	3	3	....	3
i=k+m+1	2	3	4	5	6	7	8	9	10	11	12	13	....	16

The value of the number j denotes the maximum resolution level of the wavelet. Any specific integrable function f (x) in the space [0, 1) can be considered as a linear combination of the grades of the Harr wavelet, such as,

$$f(x) \approx \sum_{i=1}^{2M} c_i\, h_i(x)$$

Here, c_i is the real coefficient in the function. The upside series concludes at confined intervals if f (x) is a piece fixed [1].

According to the above explanation, We get a linear device from the following equations:

$$f(x_m) = \textstyle\sum_{i=1}^{2M} c_i\, h_i(x_m) \quad , \quad m = 1,2,3, \ldots, M \tag{3}$$

In the above text, the linear system of equations is a 2M×2M, which can be calculated using the following theorem to find the unknown coefficients c_i.

Theorem2.1. The answer to the system (3) is as follows*:*

$$c_1 = \frac{1}{2M}\sum_{m=1}^{2M} f(x_m),$$

$$c_i = \frac{1}{\mu}\left(\sum_{m=\alpha}^{\beta} f(x_m) - \sum_{m=\beta+1}^{\gamma} f(x_m\right) \quad , \quad i = 1,2,3, \dots ,2M$$

Where

$$\alpha = \mu(\lambda - 1) + 1,$$

$$\beta = \mu\left(\lambda - 1\right) + \frac{\mu}{2},$$

$$\gamma = \lambda\mu,$$

$$\mu = \frac{2M}{\theta},$$

$$\lambda = i - \theta,$$

$$\theta = 2^{[log_{2(i-1)}]}.$$

Proof . See [11]

Theorem2.2. With the variables x and y, a very good and real function *F(x, y)* can be estimated by two dimensional wavelets in an approximate form as,

$$F(x, y, s, t) \approx \sum_{i=1}^{2M}\sum_{J=1}^{2N} d_{i,j}(x, y) h_i(s) h_j(t)$$

Substituting the collocation point:

$$s_p = \frac{p - 0.5}{2M} , \qquad P = 1,2, \dots 2M$$

And

$$t_q = \frac{q - 0.5}{2N} , \qquad q = 1,2, \dots 2N$$

We get the following system of linear equations:

$$F(x,y,s_p,t_q) \approx \sum_{i=1}^{2M}\sum_{j=1}^{2N} d_{i,j}\,(x,y)h_i(s_p)h_j(t_q)\,, \quad p = 1,2,..,2M \;\; , \quad q = 1,2,\dots,2N$$

For each value of x, y ∈ [0, 1], the answer of this system is obtained as the following equation:

$$d_{1,1}(x,y) = \frac{1}{2M \times 2N}\sum_{p=1}^{2M}\sum_{q=1}^{2N} F(x,y,s_p,t_q),$$

$$d_{i,j}(x,y) = \frac{1}{\mu_1 \times 2N}(\sum_{P=\alpha_1}^{\beta_1}\sum_{q=1}^{2N} F(x,y,s_P,t_q) - \sum_{p=\beta_1}^{\gamma_1}\sum_{q=1}^{2N} F(x,y,s_p,t_q)) \quad ,i = 2,3,\dots,2M$$

$$d_{1,j}(x,y) = \frac{1}{\mu_2 \times 2M}(\sum_{P=1_1}^{2M}\sum_{q=\alpha_2}^{\beta_2} F(x,y,s_P,t_q) - \sum_{p=1}^{2M}\sum_{q=\beta_2+1}^{2N} F(x,y,s_p,t_q)) \quad ,j = 2,3,\dots,2N$$

$$d_{i,j}(x,y) = \frac{1}{\mu_1 \times \mu_2}(\sum_{P=\alpha_1}^{\beta_1}\sum_{q=1}^{\beta_2} F(x,y,s_P,t_q) - \sum_{p=\alpha_1}^{\beta_1}\sum_{q=P_2+1}^{\gamma_2} F(x,y,s_p,t_q)$$
$$- \sum_{P=\beta_1+1}^{\gamma_1}\sum_{q=\alpha_2}^{\beta_2} F(x,y,s_P,t_q) + \sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\beta_2+1}^{\gamma_2} F(x,y,s_p,t_q)) \;.$$

$$i = 2,3,\dots,2M \;\; ,j = 2,3,\dots,2N$$

Whither

$$\alpha_1 = \mu_1(\lambda_1 - 1) + 1,$$

$$\beta_1 = \mu_1\left(\lambda_1 - 1\right) + \frac{\mu_1}{2},$$

$$\gamma_\backslash = \mu_1\lambda_1, \qquad (4)$$

$$\mu_1 = \frac{2M}{\theta_1},$$

$$\lambda_1 = i - \theta_{1,}$$

$$\theta_1 = 2^{[log_{2(i-1)}]}\,.$$

And also

$$\alpha_2 = \mu_2(\lambda_2 - 1) + 1,$$

$$\beta_2 = \mu_2(\lambda_2 - 1) + \frac{\mu_2}{2},$$

$$\gamma_2 = \mu_2\lambda_2, \tag{5}$$

$$\mu_2 = \frac{2N}{\theta_2},$$

$$\lambda_2 = i - \theta_2$$

$$\theta_2 = 2^{[log_{2(i-1)}]}.$$

Proof . [2]

Consider the parameters t, y, x, and s from the function F(x, y, s, t). Let's assumption that the function F(x, y, s, t) is estimated by used to of a 2-dimensional Haar wavelet as follows:

$$F(x,y,s,t) \approx \sum_{i=1}^{2M}\sum_{J=1}^{2N} d_{i,j}\,(x,y)h_i(s)h_j(t)$$

We achieved the consequent system of linear equations.

Corollary.2.3. Consider F(x, y) that includes two parameters y and x, which is estimated via the Harr wavelet access presented in Equation (1). Further suppose such F(x, y) at the points (x_m, y_n), n= 1, 2, …, 2N , m= 1, 2, . . . , 2M,. Therefore, at any point of the domain of the function F(x, y), its approximate value can be obtained as tracks:

$$F(x,y) = \frac{1}{2M \times 2N}\sum_{P=1}^{2M}\sum_{q=1}^{2N} F(x_m\ ,y_n)h_1(x)h_1(y)$$

$$+\sum_{I=1}^{2M}\frac{1}{\mu_1 \times 2N}\left(\sum_{P=\alpha_1}^{\beta_1}\sum_{q=1}^{2N} F(x_m\,,y_n) - \sum_{p=\beta_1}^{\gamma_1}\sum_{q=1}^{2N} F(x_m,y_n)\right)h_i(x)h_1(y)$$

$$+\sum_{I=1}^{2N}\frac{1}{\mu_2 \times 2M}\left(\sum_{P=1}^{2M}\sum_{q=\alpha_2}^{\beta_2} F(x_m\,,y_n) - \sum_{p=1}^{2M}\sum_{q=\beta_2+1}^{\gamma_2} F(x_m,y_n)\right)h_1(x)h_J(y)$$

$$+\sum_{i=1}^{2M}\sum_{j=1}^{2N}\frac{1}{\mu_1 \times \mu_2}\left(\sum_{P=\alpha_1}^{\beta_1}\sum_{q=\alpha_2}^{\beta_2} F(x_m\,,y_n) - \sum_{p=\alpha_1}^{\beta_1}\sum_{q=\beta_2+1}^{\gamma_2} F(x_m,y_n) - \sum_{P=\beta_1+1}^{\gamma_1}\sum_{q=\alpha_2}^{\beta_2} F(x_m,y_n\right.$$

$$\left. + \sum_{P=\beta_{1+1}}^{\gamma_1}\sum_{q=\beta_2+1}^{\gamma_2} F(x_m,y_n\right)h_i(x)h_j(y)$$

Where α_1, β_1, γ_1 and μ_1 are defined as in Eq. (4) and α_2 , β_2 , γ_2 and ρ_2 are defined as in Eq. (5)

3. Solution method

When dealing with both u(x) and its derivative $u^{'}(x)$ in differential integral equations of Haar wavelet in relation (2) is introduced as follows:

$$q_i = \int_0^x h_i dx = \begin{cases} x - \frac{k}{2^j} & , \quad \frac{k}{2^j} \le x \le \frac{k+0.5}{2^j} \\ \frac{k+1}{2^j} - x & , \quad \frac{k+0.5}{2^j} \le x \le \frac{k+1}{2^j} \\ 0 & , \quad elsewhere \end{cases} \tag{6}$$

Which if approximated $u^{'}(x) \approx \sum_{i=1}^{2^{J+1}} b_i h_i(x)$, as a result $u(x) - u(0) \approx \sum_{I=1}^{2^{J+1}} b_i h_i(x)$

First, we detect the level of clarity j to proximate U(x, t), then we assume,

$$\frac{\partial}{\partial t} U\,(x\,,t\,) \approx \sum_{i=1}^{2^{J+1}} \sum_{j=1}^{2^{J+1}} b_{ij} h_i(x) h_j(t) \tag{7}$$

Wherever $\{\, b_{ij}\}$ are to be found. From the initial condition u (x, 0) =0 and the composition t in $[o\,,t]$, can be written,

$$U\,(x\,,t\,) \approx \sum_{i=1}^{2^{J+1}} \sum_{j=1}^{2^{J+1}} b_{ij} h_i(x) q(t)$$

The following integral expression can be written as a result,

$$\int_0^1 \int_0^1 K(x,t\,,y,\omega) U(y,\omega) dy d\omega$$

$$\approx \sum_{i=1}^{2^{J+1}} \sum_{j=1}^{2^{J+1}} b_{ij} \int_0^1 \int_0^1 K(x,t,y,\omega) h_i q_j\,(\omega) dy d\omega$$

To evaluate the phrase $D_t^{\beta} u(x\,,t)$, we connection relation (7) into the $D_t^{\beta} u(x\,,t)$, obtained,

$$\frac{1}{\zeta(1-\beta)}\sum_{i=1}^{2^{J+1}}\sum_{j=1}^{2^{J+1}} b_{ij}h_i(x)\int_0^t h_j(\,\omega)(t-\omega)^{-\beta}\,d\omega$$

$$= f(x\,,t\,) + \sum_{i=1}^{2^{J+1}}\sum_{j=1}^{2^{J+1}} b_{ij}\int_0^1\int_0^1 k(x,t,y,\omega)h_i(y)q_j(\omega)dyd\omega$$

With the help of nodes with equal distance $t_n = \frac{n}{2^{J+1}}$ and $x_m = \frac{m-0.5}{2^{J+1}}$ to create the system,

$$\frac{1}{\zeta(1-\beta)}\sum_{i=1}^{2^{J+1}}\sum_{j=1}^{2^{J+1}} b_{ij}h_i(x_m)\int_0^{t_n} h_j(\,\omega)(t_n-\omega)^{-\beta}\,d\omega$$

$$= f(x_m\,,t_n\,) + \sum_{i=1}^{2^{J+1}}\sum_{j=1}^{2^{J+1}} b_{ij}\int_0^1\int_0^1 k(x_m,t_n,y,\omega)h_i(y)q_j(\omega)dyd\omega$$

Where n, m=1, 2, 3, … , 2^{J+1}

By solving the system of equations $2^{J+1}\times 2^{J+1}$ in the above relation, the value of wavelet coefficients b_{ij} is obtained.

4. Numerical tests

In this section, we demonstrate the effectiveness, precision, application, and efficiency of the proposed method by providing several examples of a single weak PIDE. To do this, we utilize the definition of absolute error, denoted as e_M. It is defined as,

$$e_M(x,y) = |u(x,y) - u_M\,(x,y)|$$

Here, u (x, y) shows the approximate answer, and u(x,y) shows the exact result achieved using the suggested method.

Let us consider the mesh nodes on the square and the asymptotic spread powers of the step size h as,

$$x_{m=\frac{m-0.5}{2M}} \quad ,m = 1,2,3,\ldots,2M$$

$$y_{n=\frac{n-0.5}{2N}} \quad ,n = 1,2,3,\ldots,2N \quad , \quad 0 \le x\,,y\, \le 1$$

$$G(h) - G(0) = \beta h^k + o(h^s) \quad , \quad 0 < k < s \tag{8}$$

Here G (0) is the unknown correct value, G(h) means the quantity achieved with any numeric procedure with level range h, k is the theoretical order of exactness, and β is an unknown fixed independent of h.

Mean two numerical solutions established on the nested grid as follows,

$G_{i-1} = G(h_{i-1})$, $G_i = G(h_i)$. Applying (8) for these explanations, the following equality,

$$G_{i-1} - G(0) = \beta\, h^k{}_{i-1} + O(h^s{}_{i-1}) \quad (9)$$

$$G_i - G(0) = \beta\, h^k{}_i + O(h^s{}_i)$$

The error value can be obtained by combining these two relations. So, the value becomes an error as,

$$G(0) - G_i = \frac{G_i - G_{i-1}}{2^k - 1} - O(h_i{}^s) \quad (10)$$

Or another approximation of the value G (0) as,

$$E_i = G_i + \frac{G_i - G_{i-1}}{2^k - 1} = G(0) - O(h^s{}_i) \quad (11)$$

This simplified formula is known as the Richardson analogy formula. In essence, the approximate solutions E_i have more error than h of G_i. Therefore, if the numerical solutions for two grids and the theoretical order of accuracy k are known from the numerical method, as a simple analogical formula (11), it removes the preceding term from the error of the expansion equation (8) and leads us to an acceptable solution[11].

In this study, we employed Richardson's extrapolation method to assess the error in finite difference methods for various mathematical issues.

From solving the real value of G (0) in (9), provide a simple method for assessing the convergence rate of the numerical approach as,

$$\frac{G_{i-1} - G(0)}{G_i - G(0)} = 2^k + \mathrm{O}(h_i{}^{s-k})$$

$$= \frac{\log\left(\frac{G_{i-1} - G(0)}{G_i - G(0)}\right)}{log2} \qquad (12)$$

It is possible to guess and estimate the accuracy of visionary content using three paths on a series of nested networks,

$$G_{i-2}\,, G_{i-1}\,, G_i\,, \frac{h_{i-2}}{h_{i-1}} = \frac{h_{i-1}}{h_i} = 2.$$

The beneath relation can be achieved from three relationships a like to parity(12),

$$\omega_i = \frac{G_{i-2} - G_{i-1}}{G_{i-1} - G_i} = 2^k + 0\left(h_i{}^{s-k}\right) \qquad (13)$$

With the help of relationship (18), the order of accuracy k can be evaluated and specified [16],

$$k \cong k_i = \frac{\log(\omega_i)}{log2} \qquad (14)$$

Here, k_i is an amount of discovered degree of precision, and relation (14) grants the pilot procedure for determinative or evident relation (14) can be used only for ω_i>0.

Further, the following formula can be applied to evaluate the order of convergence for the advanced value E_i .

$$s \cong s_i = \frac{\log\left(\frac{E_{i-2} - E_{i-1}}{E_{i-1} - E_i}\right)}{log2} \qquad (15)$$

Example4. 1

Notice the following two-dimensional fractional integro-differential equation:

$$\begin{cases} D_t^{\frac{5}{2}} u(x,t) = -2\,\pi^{\frac{5}{2}}\,(-6+\pi^2) + 12\,\sqrt{t}\,\sin(\pi x) \int_0^1 \int_0^1 y\, e^{\omega}\, u(y\,,\omega) t\; d\omega dy \\ \quad u(x,0) = 0 \end{cases} \qquad (16)$$

The exact solution of Equation (16) is $u(x,t) = \sqrt{\pi}\, t^3 \sin(\pi x)$.

We use the Haar wavelet method the level of resolution J=4. The following discrete system is given by the proposed method.

$$\sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij} h_i(x_m) h_j(t_n) = \frac{16}{15}\pi^2(6-\pi^2)(t_n)^{\frac{5}{2}} + \sqrt{\pi}(t_n)^3)\sin(\pi x_m)$$

$$+\frac{8\pi^5\sqrt{\pi}}{15}\sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij}(t_n)^{\frac{5}{2}} q_i \int_0^1 \sin(\pi t) h_j\,(t) dt$$

By approximate solution,

$$u(x,t) \approx \sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij} h_i(x_m) h_j(t_n).$$

Table 2 . Approximate, absolute error and exact for distinct of t_n and x_m in example 4.1 with J=4.

x_m	t_n	approximate value	exact value	absolute error
0.984375		0.00010996	0.000113795	3.83569×10^{-6}
0.734375		0.00171454	0.00171838	3.83569×10^{-6}
0.484375		0.00231252	0.00231636	3.83569×10^{-6}
0.359375		0.00209265	0.00209649	3.83569×10^{-6}
0.234375	0.109375	0.00155361	0.00155745	3.83569×10^{-6}
0.109375		0.000777463	0.000781299	3.83569×10^{-6}
0.046875		0.00033645	0.00034029	3.83569×10^{-6}
0.015625		0.00109392	0.00111971	2.57826×10^{-5}
0.046875		0.00332255	0.00334834	2.57826×10^{-5}
0.234375	0.234375	0.015299	0.0153247	2.57826×10^{-5}
0.484375		0.0227664	0.0227922	2.57826×10^{-5}
0.734375		0.0168825	0.0169082	2.57826×10^{-5}
0.984375		0.00109392	0.00111971	2.57826×10^{-5}

The chart approximate and exact solution for example 4.2. with purpose J=4.

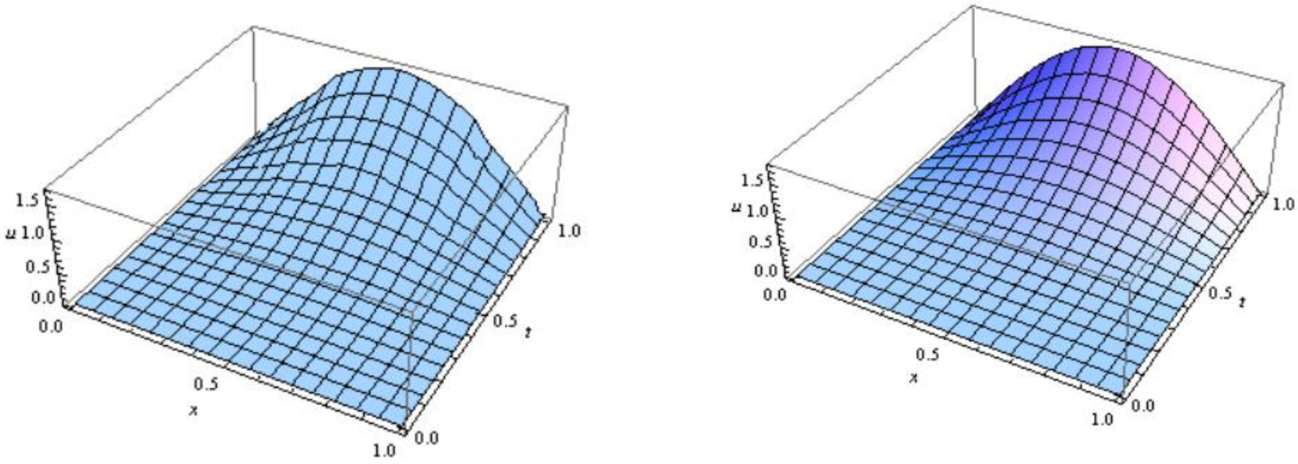

Figure 1. The approximate (to the left) and the exact (to the right) solutions for example 4.1 with J=4 using Haar wavelet method.

Example4. 2

Notice the following 2-D linear fractional integro-differntial equation:

$$\begin{cases} D_t^{0.5}u(x,t) = \frac{\sqrt{\pi}}{2}\, e^{x} - \frac{2(e-1)}{3} + \int_0^1 \int_0^1 u(y\,,\omega)t\; d\omega dy \\ u(x,0) = 0 \end{cases} \tag{17}$$

The precise solution of Equation (17) is u (x, t)=$\sqrt{t}\, e^{x}$. We solve this example differently using the Haar wavelet.

To estimate the value of $D_t^{0.5}\, u(x,t)$, we use,

$$\frac{\partial}{\partial t}u(x,t) \approx \sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij}h_j(x)h_j\;(t) \tag{18}$$

Considering the condition u (x ,0)=0 ,to find that,

$$u(x,t) \approx \sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij}h_i(x)q_j(t) \tag{19}$$

By using the estimates in (18, (19) and nods $t_n = \frac{n-0.5}{32}$ *and* $x_m = \frac{m-0.5}{32}$, following system is obtained,

$$\frac{1}{\sqrt{\pi}}\sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij}h_i(x)\int_0^{t_n} h_j(\omega)(t_n-\omega)^{-0.5}d\omega = \sum_{j=1}^{32} b_{ij}t_n q_i(1)s_j(1) + \frac{\sqrt{\pi}}{2}e^{x_m} - \frac{2(e-1)t_n}{3}$$

Where,

$$s_j(x) = \int_0^x q_j(y)dy \quad and \quad n,m = 1,2,3,\dots,32$$

Table 3. Approximate, absolute error and exact for distinct of t_n and x_m in example 4.2 with J=4.

x_m	t_n	approximate value	exact value	absolute error
0.984375		1.86251	1.86228	2.37762×10^{-4}
0.734375		1.45053	1.45029	2.38002×10^{-4}
0.48437		1.12943	1.12967	2.38189×10^{-4}
0.234375	0.484375	0.879551	0.879789	2.38335×10^{-4}
0.109375		0.776173	0.776411	2.38395×10^{-4}
0.015625		0.706692	0.70693	2.38435×10^{-4}
0. 984375		1.29551	1.29558	7.13023×10^{-5}
0.734375		1.00893	1.009	7.32756×10^{-5}
0.234375	0.234375	0.611913	0.611989	7.60092×10^{-5}
0.484375		0.785734	0785809	7.48123×10^{-5}
0.109375		0.540002	0.540078	7.65043×10^{-5}
0.015625		0.49157	0.491747	7.68371×10^{-5}

The chart approximate and exact solution for example 4.2 with purpose J=4.

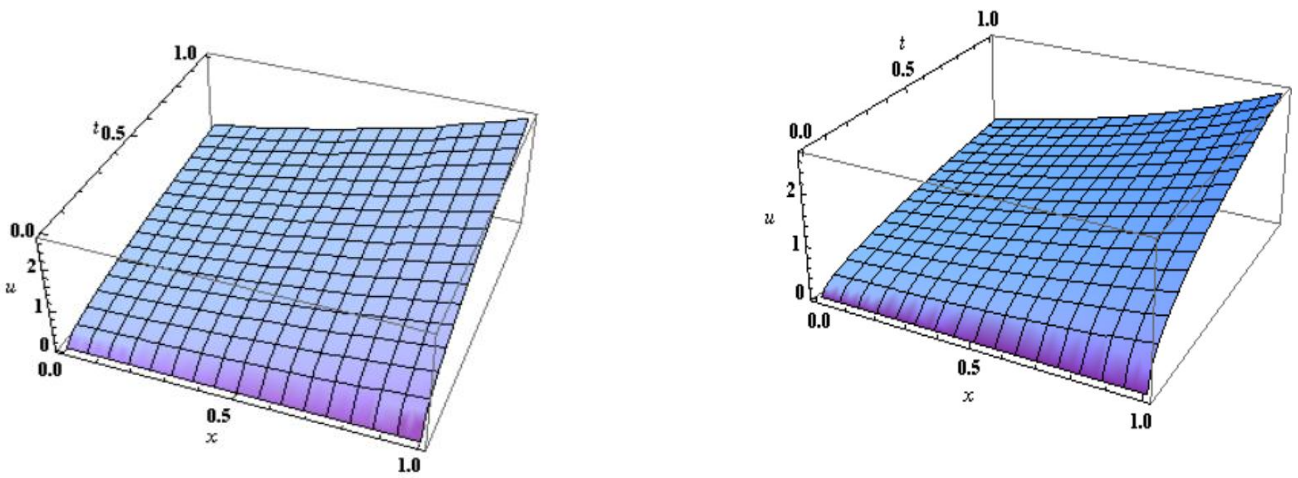

Figure2. The approximate (to the left) and the exact (to the right) solutions for example 4.2 with J=4 using Haar wavelet method.

Example4. 3

Notice the following 2-D linear fractional integro-differntial equation:

$$\begin{cases} D_t^{0.5}u(x,t) = 8\, t^{\frac{2}{3}}\, \dfrac{\sin(\pi x)}{\pi x} + 6(2-e) + \pi\sqrt{\pi} \displaystyle\int_0^1\int_0^1 y e^{\omega} u(y,\omega)\,dy\,d\omega \\ u(x,0) = 0 \end{cases} \tag{20}$$

The accurate solution of relation (20) is $u(x,y) = 3\sqrt{\pi}t^2\, \frac{\sin\pi x}{\pi x}$.

We use Haar wavelet method at J=4.

$$\frac{1}{\sqrt{\pi}}\sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij}h_i(x_m)\int_0^{t_n} h_j(\omega)(t_n-\omega)^{-0.5}\,d\omega = 8t_n^{\frac{3}{2}}\,\frac{\sin\pi x}{\pi x}x_m + 6(2-\mathrm{e})$$

$$+\pi\sqrt{\pi}\sum_{i=1}^{32}\sum_{j=1}^{32} b_{ij}\Big(\int_0^1 q_j(\omega)e^{\omega}\,d\omega\Big)\Big(\int_0^1 y h_i(y)\,dy\Big)$$

For m, n = 1, 2, 3, …, 32. And $t_n = \frac{n-0.5}{32}$, $x_m = \frac{m-0.5}{32}$.

Table4. Approximate, absolute error and exact for distinct of t_n and x_m in example 4.3 with J=4

x_m	t_n	approximate value	exact value	absolute error
0.984375		0.0054642	0.00463451	8.29692×10^{-4}
0.734375		0.0941176	0.0938081	$3..67342 \times 10^{-4}$
0.484375		0.191455	0.191718	2.63576×10^{-4}
0.359375	0.234375	0.233365	0.233875	5.0993×10^{-4}
0.234375		0.265705	0.266405	7.00023×10^{-4}
0.109375		0.285561	0.286377	8.16738×10^{-4}
0. 984375		0.0834443	0.0817527	1.69153×10^{-3}
0.734375		1.65523	1.65473	4.53913×10^{-4}
0.484375		3.38101	3.38101	9.04961×10^{-4}
0.359375	0.984375	4.12407	4.12556	1.49005×10^{-3}
0.234375		5.04948	5.0517	2.21871×10^{-3}
0.015625		5.14813	5.15042	2.29639×10^{-3}

The chart approximate and exact solution for example 4.3 with purpose J=4.

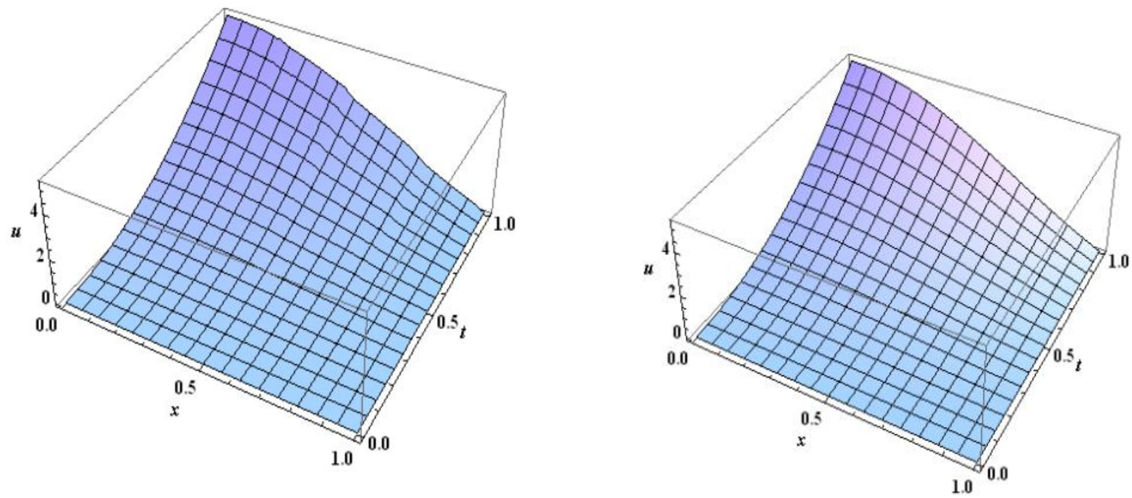

Figure3. The approximate (to the left) and the exact (to the right) solutions for example 4.2 with J=4 using Haar wavelet method.

5. Conclusions

In this article, Fredholm's two-dimensional partial differential integral equations were solved using the Haar wavelet. The proposed technique results in high accuracy. Theoretical discussions about convergence and approximation error estimation have also been presented, and the experimental results obtained from some illustrative examples prove this issue well. Finally, the reliability and simplicity of the method are shown using numerical examples, graphs, and tables.

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