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The Fractional-Order Differential Mof the Pollution for a System of Lakes

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Abstract

Pollution produced by human is a serious danger to the planet Earth in our time. In recent decades, a lot of efforts have been made to monitor and control the pollution to save the environment. In this paper, the fractional-order differential model of the pollution for a system of lakes has been introduced. There are three components; the amount of the pollution in lake 1, x, the amount of the pollution in lake 2, y, and the amount of the pollution in lake 3, z, at any time $t \geq 0$. The aim of this work is to get numerical solution of the proposed fractional-order model by Laplace Adomian decomposition method (LADM). The numerical solution has been obtained in a series form. The solution has been compared with the solutions of some other numerical approaches. The results illustrate the ability and accuracy of the present method. The Caputo form has been applied for fractional derivatives. All of computations have been done in Maple.

Keywords:
Numerical solution
Laplace Adomian decomposition method
System of fractional-order differential equations of the pollution
Caputo fractional derivative

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INTRODUCTION

Many natural phenomena in biology, medicine, physics, and other branches of science can be explained by a system of differential equations (Culshaw & Ruan, 2000; Hindmarsh & Rose, 1984; Chen et al., 1999; Lichae et al., 2019; Biazar et al., 2010; Merdan, 2010; Yüzbaşı et al., 2012). Hoggard (2007) presented a model of pollution for a system of lakes. Proposed model is simulated for three lakes with interconnecting channels. Each lake has been assumed to be a large compartment. First, a pollutant enters the first lake and then infects two other lakes (See Fig. 1). The function $p(t)$ denotes the rate of the pollutant that enters the lake 1 for $t \geq 0$. The rate of the pollutant may be vary or constant with any time.

It is important to know the amount of the pollutant in each lake at per time. The amount of the pollution in lake 1, 2, and 3 are defined $x(t), y(t)$, and $z(t)$, respectively. Constants F_{ji} denote the flow rate of water from lake i into lake j, V_i denote the volume of water in lake i, and $r_{ji}(t)$ denote the flow rate of contamination from lake i into lake j at any time t. If there is no flow of water between Lake i into Lake j, then $F_{ji} = 0$. The flux of pollution from lake i into lake j, called $r_{ji}(t)$, is defined as follows

$$\begin{cases} r_{j1}(t) = \dfrac{F_{j1}x(t),}{V_1} \\ r_{j2}(t) = \dfrac{F_{j2}y(t),}{V_2} \\ r_{j3}(t) = \dfrac{F_{j3}z(t).}{V_3} \end{cases}$$

In other words, $r_{ji}(t)$ determines the rate of concentration of contamination in Lake i flows into lake j.

The referred model is modeled as the following simple principle:

Rate of change of contamination = Input rate of contamination - output rate of contamination.

So, the proposed model will be obtained as the following form

$$\frac{dx(t)}{dt} = \frac{F_{13}}{V_3}z(t) + p(t) - \frac{F_{31}}{V_1}x(t) - \frac{F_{21}}{V_1}x(t),$$

$$\frac{dy(t)}{dt} = \frac{F_{21}}{V_1}x(t) - \frac{F_{32}}{V_2}y(t), \ t \geq 0$$

$$\frac{dz(t)}{dt} = \frac{F_{31}}{V_1}x(t) + \frac{F_{32}}{V_2}y(t) - \frac{F_{13}}{V_3}z(t), \qquad (1)$$

with initial conditions $x(0) = 0$, $y(0) = 0$, and $z(0) = 0$, which means the lakes are not contaminant from the beginning. In order to keep constant the volume of water in each lake, the following conditions have been assumed:

Lake 1: $F_{13} = F_{21} + F_{31}$,

Lake 2: $F_{21} = F_{32}$,

Lake 3: $F_{31} + F_{32} = F_{13}$, (see Biazar and Farrokhi, 2006).

Fig.1 shows system of three lakes with interconnecting channels. The source of pollutant and the constants F_{ji} have been marked.

The system of differential Eqs. (1) has been solved by Adomian and RK4 methods (Biazar et al., 2006). In Biazar et al. (2010), the numerical solution of (1) has been obtained by the variational iteration method. In Merdan (2010), the modified differential transformation method has been used to achieve the numerical solution of (1). A collocation approach has been introduced to solve (1) in Yüzbaşı, Şahin, and Sezer (2012). The polluted lakes system (1) has been solved by PIM in Khalid et al. (2015).

In this work, we introduce a fractional-order of (1) and solve it by LADM. Generalizing system of differential equations (1) to a system of fractional-order differential Eqs. (2) indicates the novelty of the paper. Fractional-order differential equations are related to fractals (Tatom, 1995; Heymans & Bauwens, 1994; Giona & Roman, 1992), save memory on themselves (Arafa, Rida, & Khalil, 2013), have freedom on the degree of the derivative operator, and can explain many phenomena in sciences (Haq et al., 2017; Ertürk, Odibat, & Momani, 2011; Diethelm, 2010). We introduce a fractional-order of (1) as follows:

$$D^{\alpha_1}x(t) = \frac{F_{13}}{V_3}z(t) + p(t) - \frac{F_{31}}{V_1}x(t) - \frac{F_{21}}{V_1}x(t),$$

$$D^{\alpha_2}y(t) = \frac{F_{21}}{V_1}x(t) - \frac{F_{32}}{V_2}y(t), \ t \geq 0$$

$$D^{\alpha_3}z(t) = \frac{F_{31}}{V_1}x(t) + \frac{F_{32}}{V_2}y(t) - \frac{F_{13}}{V_3}z(t), \qquad (2)$$

with the same initial conditions, where $0 < \alpha_i \leq 1$, $i = 1,2,3$. When $\alpha \to 1$, $D^{\alpha}x(t) \to Dx(t)$, therefore system of fractional-order differential equations of the pollution (2) reduces to traditional model (1). The development of numerical methods, especially for the solution of fractional differential equations, has led to an increasing interest in fractional calculus (Kilbas, Srivastava, & Trujillo, 2006). Some numerical methods that can be implemented to solve a system of fractional-order differential equations are: optimal homotopy asymptotic (Marinca & Herişanu, 2008; Hashim et al., 2010; Jafari & Seifi, 2013; Khan et al., 2014), predictor-corrector (Diethelm, Ford, & Freed, 2002), homotopy analysis (Liao, 2003; Abbasbandy, 2007), variational iteration (Biazar et al., 2010), generalized Euler (Arafa, Rida, & Khalil, 2013), Laplace Adomian (Haq et al., 2017; Odibat, 2006), homotopy perturbation (He, 1999; He, 2006), differential transformation (Merdan, 2010), and Runge-Kutta method (Butcher, 2008). The rest of this paper is organized as follows: In Section 2, a brief review of fractional calculus has been presented. Section 3 will be devoted to solving (2) by LADM in three phases. In Section 4, the convergence of the method will be discussed. In the last section, we present the conclusion.

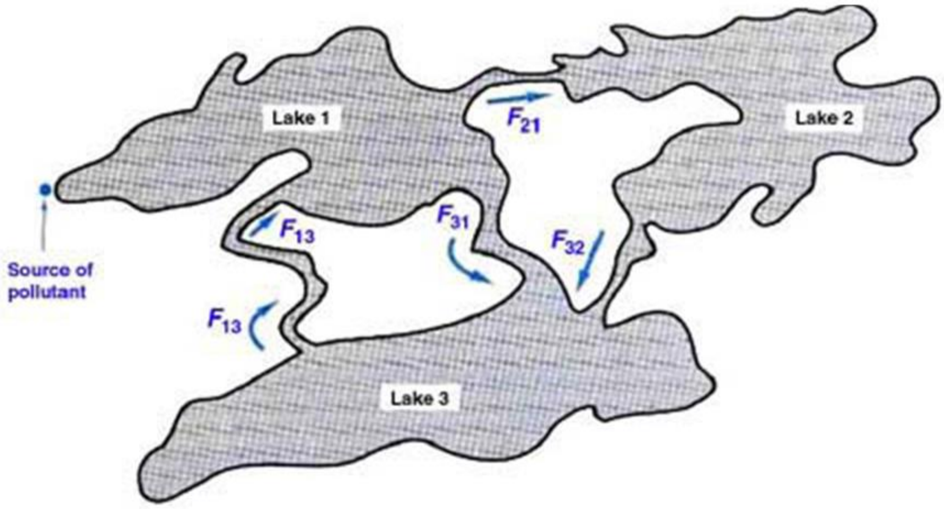

Fig. 1. System of three lakes with interconnecting channels (Biazar, Farrokhi, & Islam, 2006)

FRACTIONAL CALCULUS

The purpose of this section is to remind the reader of some fundamental preliminaries of fractional calculus.

Definition 1. The fractional integral of Riemann-Liouville type of order α for a function $f:(0,\infty)\to R$ is defined by

$$J^{\alpha}f(s)=\frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}f(t)dt, \qquad (3)$$

where $\alpha\epsilon(0,\infty)$, (See Diethelm, 2010)

Definition 2. The Caputo fractional derivative of a function $f:(0,\infty)\to R$ on the closed interval $[0,T]$ is defined as

$$D^{\alpha}f(s)=\frac{1}{\Gamma(m-\alpha)}\int_0^s (s-t)^{m-\alpha-1}f^{(m)}(t)dt, \quad m=\lfloor\alpha\rfloor+1. \qquad (4)$$

where $\lfloor\alpha\rfloor$ is the integer part of α.

Definition 3. The Caputo fractional derivative has another presentation that can be shown as follows

$$D^{\alpha}f(s)=J^{m-\alpha}\big(D^m f(s)\big), \qquad (5)$$

(See Diethelm, 2010).

Lemma 1. If $\alpha\epsilon(0,\infty)$, then the following result holds for fractional calculus

$$J^{\alpha}[D^{\alpha}f](s)=f(s)+\sum_{j=0}^{m-1}\frac{f^{(j)}(0)}{j!}s^j, \qquad (6)$$

where $m=\lfloor\alpha\rfloor+1$.

Proof. (See Diethelm, 2010; Kilbas, 2006).

Definition 4. We remind that the Laplace transform of Caputo fractional derivative is defined as follows

$$\mathcal{L}\{D^{\alpha}f(t)\}=s^{\alpha}F(s)-\sum_{j=0}^{m-1}s^{\alpha-k-1}f^{(j)}(0),$$
$$m-1<\alpha<m, m\epsilon\mathbb{N}, \qquad (7)$$

SOLUTION OF SYSTEM OF FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS (2)

In this section, the system of fractional-order differential equations (2) will be solved by LADM, and the results will be compared with the results of some other numerical methods. Because the proposed model (2) better describes the system of polluted lakes, three types of input models such as impulse, step, and sinusoidal have been considered (Aguirre & Tully, 1999).

Impulse input

The impulse input model describes pollutants that are released very quickly into the lake. The impulse input functions are zero everywhere except when contamination enters the lake. Impulse input functions have a spike. The spike indicates the time at which the pollution has been evacuated.

For example, suppose a barrel of oil drains into the lake suddenly; therefore, we assume that the input function is equal to 100 at the interval of 0 to 10. The values of parameters in (2) are reported in Aguirre and Tully (1999).

$V_1 = 2900\, mi^3, V_2 = 850\, mi^3, V_3 = 1180\, mi^3$, $F_{21} = 18\, mi^3/year, F_{32} = 18\, mi^3/year,\ F_{31} = 20\, mi^3/year,\ F_{13} = 38\, mi^3/year$.

So, model (2) will be obtained as the following form

$$D^{\alpha_1}x(t) = \frac{38}{1180}z(t) + 100 - \frac{20}{2900}x(t) - \frac{18}{2900}x(t),$$
$$D^{\alpha_2}y(t) = \frac{18}{2900}x(t) - \frac{18}{850}y(t), \quad t \geq 0$$
$$D^{\alpha_3}z(t) = \frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t), \quad (8)$$

with the same initial conditions, where $0 < \alpha_i \leq 1,\ i = 1,2,3$.

Using Laplace transform on both sides of each equation of (8) gives

which implies that

$$\begin{cases} s^{\alpha_1}\mathcal{L}\{x(t)\} - s^{\alpha_1-1}x(0) = \frac{38}{1180}\mathcal{L}\{z(t)\} + \frac{100}{s} \\ \quad -\frac{38}{2900}\mathcal{L}\{x(t)\}, \\ s^{\alpha_2}\mathcal{L}\{y(t)\} - s^{\alpha_2-1}y(0) = \frac{18}{2900}\mathcal{L}\{x(t)\} - \frac{18}{850}\mathcal{L}\{y(t)\}, \\ s^{\alpha_3}\mathcal{L}\{z(t)\} - s^{\alpha_3-1}z(0) = \frac{20}{2900}\mathcal{L}\{x(t)\} + \frac{18}{850}\mathcal{L}\{y(t)\} \\ \quad -\frac{38}{1180}\mathcal{L}\{z(t)\}, \end{cases} \quad (10)$$

Substitution of initial conditions in (10) results in

$$\begin{cases} \mathcal{L}\{x(t)\} = \frac{38}{1180}\frac{1}{s^{\alpha_1}}\mathcal{L}\{z(t)\} + \frac{100}{s^{\alpha_1+1}} \\ \quad -\frac{38}{2900}\frac{1}{s^{\alpha_1}}\mathcal{L}\{x(t)\}, \\ \mathcal{L}\{y(t)\} = \frac{18}{2900}\frac{1}{s^{\alpha_2}}\mathcal{L}\{x(t)\} - \frac{18}{850}\frac{1}{s^{\alpha_2}}\mathcal{L}\{y(t)\}, \\ \mathcal{L}\{z(t)\} = \frac{20}{2900}\frac{1}{s^{\alpha_3}}\mathcal{L}\{x(t)\} + \frac{18}{850}\frac{1}{s^{\alpha_3}}\mathcal{L}\{y(t)\} \\ \quad -\frac{38}{1180}\frac{1}{s^{\alpha_3}}\mathcal{L}\{z(t)\}, \end{cases} \quad (11)$$

Applying inverse Laplace transform reads to

$$\begin{cases} x(t) = \frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{z(t)\}\right] \\ +\frac{100t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \frac{38}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{x(t)\}\right], \\ y(t) = \frac{18}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{x(t)\}\right] \\ -\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{y(t)\}\right], \\ z(t) = \frac{20}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{x(t)\}\right] \\ +\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{y(t)\}\right] \\ -\frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{z(t)\}\right]. \end{cases}$$

Let's consider x, y, and z, are as the following series

$$x = \sum_{i=0}^{\infty} x_i,\ y = \sum_{i=0}^{\infty} y_i,\ z = \sum_{i=0}^{\infty} z_i. \quad (12)$$

To compute the Adomian polynomials, using an alternate algorithm (Biazar et al., 2003), the following recursive sequence would be derived:

$$\begin{cases} x_0(t) = \frac{100t^{\alpha_1}}{\Gamma(\alpha_1+1)}, \\ y_0(t) = 0, \\ z_0(t) = 0, \end{cases} \quad (13)$$

$$x_{n+1}(t) = \frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{z_n(t)\}\right] - \frac{38}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{x_n(t)\}\right],$$
$$y_{n+1}(t) = \frac{18}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{x_n(t)\}\right] - \frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{y_n(t)\}\right],$$
$$z_{n+1}(t) = \frac{20}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{x_n(t)\}\right] + \frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{y_n(t)\}\right] - \frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{z_n(t)\}\right]. \quad (14)$$

We will calculate four terms of infinite series of x, y, and z, as an approximate solution.

$$\begin{cases} x(t)=\sum_{i=0}^{\infty}x_i(t)\approx x_0(t)+x_1(t)+x_2(t)+x_3(t),\\ y(t)=\sum_{i=0}^{\infty}y_i(t)\approx y_0(t)+y_1(t)+y_2(t)+y_3(t),\\ z(t)=\sum_{i=0}^{\infty}z_i(t)\approx z_0(t)+z_1(t)+z_2(t)+z_3(t).\end{cases} \quad (15)$$

Let's take α_1, α_2, and α_3 equal to α, so the approximate solution of system (3) would be derived as follows

$$\begin{cases} x(t)=\frac{100t^{\alpha}}{\Gamma(\alpha+1)}-\frac{38}{29}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{48849}{1240475}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\ \qquad -\frac{1982603583}{1804084816250}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},\\ y(t)=\frac{18}{29}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{1521}{71485}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\ \qquad +\frac{361275039}{519821048750}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},\\ z(t)=\frac{20}{29}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{381738}{21088075}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\ +\frac{1238903361}{3066944187625}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}.\end{cases} \quad (16)$$

When $\alpha=1$, the solution of (3) will be obtained as the following form

$$\begin{cases} x(t)=\frac{100t}{\Gamma(2)}-\frac{38}{29}\frac{t^2}{\Gamma(3)}+\frac{48849}{1240475}\frac{t^3}{\Gamma(4)}\\ \qquad -\frac{1982603583}{1804084816250}\frac{t^4}{\Gamma(5)},\\ y(t)=\frac{18}{29}\frac{t^2}{\Gamma(3)}-\frac{1521}{71485}\frac{t^3}{\Gamma(4)}+\frac{361275039}{519821048750}\frac{t^4}{\Gamma(5)},\\ z(t)=\frac{20}{29}\frac{t^2}{\Gamma(3)}-\frac{381738}{21088075}\frac{t^3}{\Gamma(4)}\\ \qquad +\frac{1238903361}{3066944187625}\frac{t^4}{\Gamma(5)}.\end{cases} \quad (17)$$

Step input

The step input model describes pollutants that are added to the lake at steady concentration. Before time zero, the pollutant concentration is zero. After time zero, the pollutant enters into the lake suddenly and input contaminant increases with constant rate. For an example, suppose a manufacturing plant begins to produce at time zero and dumps raw sewage on a constant rate, therefore, we assume input function is equal to 100t. So, model (3) with parameters that given in subsection 3.1 will be obtained as the following form

$$D^{\alpha_1}x(t)=\frac{38}{1180}z(t)+100t-\frac{20}{2900}x(t)-\frac{18}{2900}x(t),$$
$$D^{\alpha_2}y(t)=\frac{18}{2900}x(t)-\frac{18}{850}y(t), \quad t\geq 0$$
$$D^{\alpha_3}z(t)=\frac{20}{2900}x(t)+\frac{18}{850}y(t)-\frac{38}{1180}z(t),$$
$$x(0)=0, y(0)=0, z(0)=0.$$

According to previous subsection, we derive

$$\begin{cases} x(t)=\frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{z(t)\}\right]+\frac{100t^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\\ \qquad -\frac{38}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{x(t)\}\right],\\ y(t)=\frac{18}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{x(t)\}\right]-\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{y(t)\}\right],\\ z(t)=\frac{20}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{x(t)\}\right]+\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{y(t)\}\right]\\ \qquad -\frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{z(t)\}\right].\end{cases} \quad (18)$$

To calculate the approximate solution, using an alternate algorithm for Adomian polynomials (Biazar et al., 2003), the following recursive sequence would be derived:

$$\begin{cases} x_0(t)=\frac{100t^{\alpha_1+1}}{\Gamma(\alpha_1+2)},\\ y_0(t)=0,\\ z_0(t)=0,\end{cases} \quad (19)$$

$$\begin{cases} x_{n+1}(t)=\frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{z_n(t)\}\right]-\frac{38}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{x_n(t)\}\right],\\ y_{n+1}(t)=\frac{18}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{x_n(t)\}\right]-\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{y_n(t)\}\right],\\ z_{n+1}(t)=\frac{20}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{x_n(t)\}\right]+\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{y_n(t)\}\right].\\ \qquad -\frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{z_n(t)\}\right]\end{cases} \quad (20)$$

We take α_1, α_2, and α_3 equal to α. We will calculate four terms of infinite series of $x, y,$ and z, as an approximate solution as the following form:

$$\begin{cases} x(t)=\frac{100\,t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{38}{29}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{48849}{1240475}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\\ \qquad -\frac{1982603583}{1804084816250}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)},\\ y(t)=\frac{18}{29}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{1521}{71485}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\\ \qquad +\frac{361275039}{519821048750}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)},\\ z(t)=\frac{20}{29}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{381738}{21088075}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\\ +\frac{1238903361}{3066944187625}\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}.\end{cases} \quad (21)$$

When $\alpha = 1$, we get the solution of (17) as follows

$$\begin{cases} x(t) = \frac{100\, t^2}{\Gamma(3)} - \frac{38}{29}\frac{t^{2\alpha+1}}{\Gamma(4)} + \frac{48849}{1240475}\frac{t^4}{\Gamma(5)} \\ \qquad - \frac{1982603583}{1804084816250}\frac{t^5}{\Gamma(6)}, \\ y(t) = \frac{18}{29}\frac{t^3}{\Gamma(4)} - \frac{1521}{71485}\frac{t^4}{\Gamma(5)} + \frac{361275039}{519821048750}\frac{t^5}{\Gamma(6)}, \\ z(t) = \frac{20}{29}\frac{t^3}{\Gamma(4)} - \frac{381738}{21088075}\frac{t^4}{\Gamma(5)} + \frac{1238903361}{3066944187625}\frac{t^5}{\Gamma(6)}. \end{cases} \quad (22)$$

Sinusoidal input

The step input model describes pollutants that are entered to the lake periodically. For an example, we assume that $p(t) = \alpha + \beta \sin\frac{2\pi t}{T}$, where α is the average input concentration of pollutant and β is the amplitude of fluctuations. Let's consider $\alpha = \beta$, and $T = 2\pi$, therefore, we have $p(t) = 1 + \sin t$. So, model (3) with parameters that given in subsection Impulse input will be obtained as follows

$$D^{\alpha_1}x(t) = \frac{38}{1180}z(t) + 1 + \sin t - \frac{20}{2900}x(t) - \frac{18}{2900}x(t),$$

$$D^{\alpha_2}y(t) = \frac{18}{2900}x(t) - \frac{18}{850}y(t) \quad t \geq 0$$

$$D^{\alpha_3}z(t) = \frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t), \quad (23)$$

with the same initial conditions as previous subsections.

According to Impulse input subsection,

$$\begin{aligned} x(t) &= \frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{z(t)\}\right] + \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1+1)} \\ &+ \sum_{k=0}^{\infty}(-1)^k\frac{t^{\alpha_1+2k+1}}{\Gamma(\alpha_1+2k+1)} - \frac{38}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{x(t)\}\right], \\ y(t) &= \frac{18}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{x(t)\}\right] - \frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{y(t)\}\right], \\ z(t) &= \frac{20}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{x(t)\}\right] + \frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{y(t)\}\right] \\ &- \frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{z(t)\}\right]. \end{aligned} \quad (24)$$

The recursive sequence would be derived:

$$\begin{cases} x_0(t) = \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{\alpha_1+2k+1}}{\Gamma(\alpha_1+2k+1)}, \\ y_0(t) = 0, \\ z_0(t) = 0, \end{cases} \quad (24)$$

$$\begin{cases} x_{n+1}(t) = \frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{z_n(t)\}\right] \\ -\frac{38}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_1}}\mathcal{L}\{x_n(t)\}\right], \\ y_{n+1}(t) = \frac{18}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{x_n(t)\}\right] \\ -\frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_2}}\mathcal{L}\{y_n(t)\}\right], \\ z_{n+1}(t) = \frac{20}{2900}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{x_n(t)\}\right] \\ \qquad + \frac{18}{850}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{y_n(t)\}\right] \\ -\frac{38}{1180}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha_3}}\mathcal{L}\{z_n(t)\}\right]. \end{cases} \quad (25)$$

Let's take α_1, α_2, and α_3 equal to α.

for $n = 0$,

$$\begin{cases} x_1(t) = -\frac{38}{2900}\left\{\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)}\right\}, \\ y_1(t) = \frac{18}{2900}\left\{\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)}\right\}, \\ z_1(t) = \frac{20}{2900}\left\{\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)}\right\}, \end{cases} \quad (26)$$

for $n = 1$,

$$\begin{cases} x_2(t) = \frac{48849}{124047500}\left\{\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)}\right\}, \\ y_2(t) = -\frac{1521}{7148500}\left\{\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)}\right\}, \\ z_2(t) = -\frac{190869}{1054403750}\left\{\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)}\right\}, \end{cases} \quad (27)$$

for $n = 2$,

$$\begin{cases} x_3(t) = -\frac{1982603583}{180408481625000}\left\{\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)}\right\}, \\ y_3(t) = \frac{361275039}{51982104875000}\left\{\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)}\right\}, \\ z_3(t) = \frac{1238903361}{306694418762500}\left\{\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \sum_{k=0}^{\infty}(-1)^k\frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)}\right\}, \end{cases} \quad (28)$$

so the approximate solution of (22), by calculating four terms of infinite series of $x, y,$ and z, is obtained as the following form:

$$\begin{cases} x(t)=\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{\alpha+2k+1}}{\Gamma(\alpha+2k+1)} \\ -\frac{38}{2900}\left\{\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)}\right\}, \\ +\frac{48849}{124047500}\left\{\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)}\right\} \\ -\frac{1982603583}{180408481625000}\left\{\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)}\right\} \\ y(t)=\frac{18}{2900}\left\{\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)}\right\} \\ -\frac{1521}{7148500}\left\{\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)}\right\} \\ +\frac{361275039}{51982104875000}\left\{\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)}\right\} \\ z(t)=\frac{20}{2900}\left\{\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)}\right\} \\ -\frac{190869}{1054403750}\left\{\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)}\right\} \\ +\frac{1238903361}{306694418762500}\left\{\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)}\right\} \end{cases} \tag{29}$$

When $\alpha = 1$, the solution of (22) is as follows

$$\begin{cases} x(t)=\frac{t^{2}}{\Gamma(2)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+2}}{\Gamma(2k+2)} \\ -\frac{38}{2900}\left\{\frac{t^{2}}{\Gamma(3)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+3}}{\Gamma(2k+4)}\right\}, \\ +\frac{48849}{124047500}\left\{\frac{t^{3}}{\Gamma(4)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+4}}{\Gamma(2k+5)}\right\} \\ -\frac{1982603583}{180408481625000}\left\{\frac{t^{4}}{\Gamma(5)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+5}}{\Gamma(2k+6)}\right\} \\ y(t)=\frac{18}{2900}\left\{\frac{t^{2}}{\Gamma(3)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+3}}{\Gamma(2k+4)}\right\} \\ -\frac{1521}{7148500}\left\{\frac{t^{3}}{\Gamma(4)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+4}}{\Gamma(2k+5)}\right\} \\ +\frac{361275039}{51982104875000}\left\{\frac{t^{4}}{\Gamma(5)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+5}}{\Gamma(2k+6)}\right\} \\ z(t)=\frac{20}{2900}\left\{\frac{t^{2}}{\Gamma(3)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+3}}{\Gamma(2k+4)}\right\} \\ -\frac{190869}{1054403750}\left\{\frac{t^{3}}{\Gamma(4)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+4}}{\Gamma(2k+5)}\right\} \\ +\frac{1238903361}{306694418762500}\left\{\frac{t^{4}}{\Gamma(5)}+\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+5}}{\Gamma(2k+6)}\right\} \end{cases} \tag{30}$$

CONVERGENCE ANALYSIS OF THE METHOD

In this section, the convergence of the proposed method, using the idea presented in Ayati and Biazar (2015), is studied.

CONCLUSION

In this paper, a fractional-order model of HIV-1 with three components has been introduced. By applying Laplace transform and Adomian decomposition method (LADM) which is a strong approach to compute numerical solution of fractional differential equations, we gain an approximate solution of the proposed model. The accuracy of the proposed approach has been made it a reliable method. The result of LADM has been compared with the results of some other methods such as GEM, HAM, RK4 (Arafa, Rida, & Khalil, 2013), and HPM (Merdan & Khan, 2010). The results are presented in Tables1-3. When $\alpha \to 1$, then $D^{\alpha}x(t) \to Dx(t)$, therefore the fractional-order of presented model reduces to traditional model. Because of the fact that obtaining the exact solution for system of fractional equation is difficult or impossible, our suggestion for future research is solving them by such numerical methods.

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