Journal of Linear and Topological Algebra
Vol. 13*, No.* 04*,* 2024*,* 217- 224
DOR:
DOI:

Further inequalities for the numerical radii of Hilbert space operators

M. Hosseini[a], R. Nuraei[a], M. Shah Hosseini[b,*], M. R. Sorouhesh[a]

[a] *Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran.*
[b] *Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.*

Received 25 November 2024; Revised 22 January 2025; Accepted 23 January 2025.

Communicated by Ghasem Soleimani Rad

Abstract. This paper studies numerical radius inequalities in Hilbert space operators. We obtain some bounds for the accretive dissipative matrices, extending and improving earlier bounds. We also give results concerning block matrices.

Keywords: Numerical radii, norm, inequality.

2010 AMS Subject Classification: 47A30, 15A18, 47A63, 47TB10.

1. Introduction and preliminaries

Let $\mathcal{H}$ be an arbitrary Hilbert space, endowed with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The notation $\mathcal{B}(\mathcal{H})$ will be used to denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Upper case letters will be used to denote the element of $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, the adjoint operator T^* is the operator defined by $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for $x, y \in \mathcal{H}$, and the operator norm of T is defined by $\|T\| = \sup_{\|x\|=1} \|Tx\|$. If an operator $T \in \mathcal{B}(\mathcal{H})$ satisfies $\langle Tx, x \rangle \geqslant 0$ for all $x \in \mathcal{H}$, it will be called a positive operator. Related to the operator norm, the numerical radius of $T \in \mathcal{B}(\mathcal{H})$ is defined by $\omega(T) = \sup_{\|x\|=1} | \langle Tx, x \rangle |$. This latter quantity defines a norm on $\mathcal{B}(\mathcal{H})$ that is equivalent to the operator norm, where we have the equivalence

$$\frac{\|T\|}{2} \leqslant \omega(T) \leqslant \|T\| \tag{1}$$

*Corresponding author.
E-mail address: hoseinimahta804@gmail.com (M. Hosseini); rahele.nuraei@gmail.com (R. Nuraei); mohsen_shahhosseini@yahoo.com (M. Shah Hosseini); sorouhesh@azad.ac.ir (M. R. Sorouhesh).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

([6, Theorem 1.3-1]). The following estimates for the numerical radius are known [10, 11]

$$\omega(T) \leqslant \frac{1}{2}|||T| + |T^*|||, \tag{2}$$

$$\omega^2(T) \leqslant \frac{1}{2}||T^*T + TT^*||. \tag{3}$$

The inequalities (2) and (3) refine the second inequality in (1). Such inequalities are essential as one can have upper or lower bounds of one quantity in terms of the other. Consequently, sharper bounds are highly demanded in this field. We refer the reader to [2, 3, 8, 12, 14–16, 19–21] and [22] as a sample of treatments of this interest.

For $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, the operator T can be represented as an 2×2 operator matrix $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ with $T_{jk} \in \mathcal{B}(\mathcal{H}),\ j,k = 1,2$. For any $T \in \mathcal{B}(\mathcal{H})$, we can write

$$T = A + iB \tag{4}$$

in which $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$ are Hermitian operators. This is the so-called Cartesian decomposition of T. In this paper, we will represent the decomposition (4) by

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + i \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{5}$$

in which $T_{jk}, B_{jk}, A_{jk} \in \mathcal{B}(\mathcal{H}), j,k = 1,2$. Then $A_{12} = A_{21}^*$ and $B_{12} = B_{21}^*$. T is accretive (resp. dissipative) if in its Cartesian decomposition (4), A (resp. B) is positive, and T is accretive-dissipative if both A and B are positive.

Recently, several authors proved numerical radius inequalities for accretive-dissipative operator matrices [5, 13, 17]. In this paper, we prove inequalities which relate numerical radius for some components of the accretive-dissipative operator matrix $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$ with the norm of some components of its decomposition (5). To reach these results, we need the well-known lemmas, which are essential in our analysis.

Lemma 1.1 Let $a_i \geqslant 0$ for $i = 1, \ldots, n,\ r \geqslant 1$. Then $\sum\limits_{i=1}^{n} a_i^r \leqslant (\sum\limits_{i=1}^{n} a_i)^r \leqslant n^{r-1}\left(\sum\limits_{i=1}^{n} a_i^r\right)$. In particular, $a_1^r + a_2^r \leqslant (a_1 + a_2)^r \leqslant 2^{r-1}(a_1^r + a_2^r)$.

Lemma 1.2 (McCarthy inequality) Let $A \in \mathcal{B}(\mathcal{H})$ be positive semidefinite and $x \in \mathcal{H}$ such that $\|x\| \leqslant 1$. Then
(i) $\langle Ax, x\rangle^r \leqslant \langle A^r x, x\rangle$ for $r \geqslant 1$.
(ii) $\langle A^r x, x\rangle \leqslant \langle Ax, x\rangle^r$ for $0 < r \leqslant 1$.

Lemma 1.3 [9, Theorem 1] Let $A \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(a)g(a) = a$ $(a \in [0, \infty))$, then $|\langle Ax, y\rangle|^2 \leqslant \langle |A|\, x, x\rangle \langle |A^*|\, y, y\rangle$ and more general

$$|\langle Ax, y\rangle|^2 \leqslant \left\langle f^2\left(|A|\right) x, x\right\rangle \left\langle g^2\left(|A^*|\right) y, y\right\rangle. \tag{6}$$

Lemma 1.4 [9, Lemma 1] Let $A, B, C \in \mathcal{B}(\mathcal{H})$ such that A and B are positive semidefinite. Then

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0 \Leftrightarrow |< Bx, y >|^2 \leqslant < Ax, x >< Cy, y > \ \forall x, y \in \mathcal{H}.$$

2. Main results

We start this section with the following result.

Theorem 2.1 Let $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$ be operator matrix with cartesian decomposition (5), $r \geqslant 2$. If f and g are non-negative continuous functions on $[0, \infty)$ satisfying the relation $f(a)g(a) = a$ $(a \in [0, \infty))$, then

$$\omega^r(T_{12}) \leqslant 2^{r-2} \left|\left| f^{2r}(|A_{12}|) + g^{2r}(|A_{12}^*|) + f^{2r}(|B_{12}|) + g^{2r}(|B_{12}^*|)\right|\right|. \tag{7}$$

In particular, if $r = 2$ and $f(t) = g(t) = \sqrt{t}$. We give

$$\omega^2(T_{12}) \leqslant \left|\left| \ |A_{12}|^2 + |A_{12}^*|^2 + |B_{12}|^2 + |B_{12}^*|^2 \ \right|\right|. \tag{8}$$

Let $r = 1$ and $f(t) = g(t) = \sqrt{t}$, we give

$$\omega(T_{12}) \leqslant \frac{1}{2} || \ |A_{12}| + |A_{12}^*| + |B_{12}| + |B_{12}^*| \ ||. \tag{9}$$

Proof. We have

$$\begin{aligned}
|\langle T_{12}x, x\rangle|^r &= |\langle (A_{12} + iB_{12})x, x\rangle|^r \\
&= |\langle A_{12}x, x\rangle + i \langle B_{12}x, x\rangle|^r \\
&\leqslant (|\langle A_{12}x, x\rangle| + |\langle B_{12}x, x\rangle|)^r \\
&\leqslant 2^{r-1} (|\langle A_{12}x, x\rangle|^r + |\langle B_{12}x, x\rangle|^r) \\
&\leqslant 2^{r-1} \left(\langle f^2(|A_{12}|)x, x\rangle^{\frac{r}{2}} \langle g^2(|A_{12}^*|)x, x\rangle^{\frac{r}{2}} + \langle f^2(|B_{12}|)x, x\rangle^{\frac{r}{2}} \langle g^2(|B_{12}^*|)x, x\rangle^{\frac{r}{2}} \right) \text{ (by (6))} \\
&\leqslant 2^{r-1} (\langle f^r(|A_{12}|)x, x\rangle \langle g^r(|A_{12}^*|)x, x\rangle + \langle f^r(|B_{12}|)x, x\rangle \langle g^r(|B_{12}^*|)x, x\rangle) \text{ (Lemma 1.2(i))} \\
&\leqslant 2^{r-2} \left(\langle f^r(|A_{12}|)x, x\rangle^2 + \langle g^r(|A_{12}^*|)x, x\rangle^2 + \langle f^r(|B_{12}|)x, x\rangle^2 + \langle g^r(|B_{12}^*|)x, x\rangle^2 \right) \\
&\quad \text{(since } ab \leqslant \frac{a^2 + b^2}{2} \text{ if } a, b \in (-\infty, \infty)) \\
&\leqslant 2^{r-2} \left(\langle f^{2r}(|A_{12}|)x, x\rangle + \langle g^{2r}(|A_{12}^*|)x, x\rangle + \langle f^{2r}(|B_{12}|)x, x\rangle \right. \\
&\quad \left. + \langle g^{2r}(|B_{12}^*|)x, x\rangle \right) \text{ (Lemma 1.2(i))} \\
&= 2^{r-2} \left(\langle f^{2r}(|A_{12}|) + g^{2r}(|A_{12}^*|) + f^{2r}(|B_{12}|) + g^{2r}(|B_{12}^*|)x, x\rangle \right).
\end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathbb{C}^n$, we give (7). Letting $f(t) = g(t) = \sqrt{t}$ and $r = 2$, we give (8). ∎

Remark 1 *By squaring both sides of (9), we give*

$$\omega^2(T_{12}) \leqslant \frac{1}{4} || \ |A_{12}| + |A_{12}^*| + |B_{12}| + |B_{12}^*| \ ||^2.$$

Remark 2 *If* $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$ *is positive semidefinite, then* $\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *in (4), and (7) reduces to* $\omega^r(T_{12}) \leqslant 2^{r-2} \left\| f^{2r}(|A_{12}|) + g^{2r}(|A_{12}^*|) \right\|$. *In particular, if* $r = 1$ *and* $f(t) = g(t) = \sqrt{t}$, *we give*

$$\omega(T_{12}) \leqslant \frac{1}{2} || \ |A_{12}| + |A_{12}^*| \ || = \frac{1}{2} || \ |T_{12}| + |T_{12}^*| \ ||. \tag{10}$$

Thus, (10) refines the second inequality of (7), which is exactly (8). From this point of view, we note that (7) generalizes (2).

Lemma 2.2 Let $A, B \geqslant 0$. Then

$$|| \ A + B \ ||^2 \leqslant 2|| \ A^2 + B^2 \ ||. \tag{11}$$

Equality holds iff $A = B$.

Proof. It is well known that $|| \ (A + B)^2 \ || = || \ A + B \ ||^2$ (since $A, B \geqslant 0$). To reach (11), it is enough to prove that $(A + B)^2 \leqslant 2(A^2 + B^2)$. We have

$$\begin{aligned} 2A^2 + 2B^2 - (A + B)^2 &= 2A^2 + 2B^2 - (A^2 + B^2 + AB + BA) \\ &= A^2 + B^2 - AB - BA \\ &= (A - B)^2 \geqslant 0 \text{ (as A-B is Hermitian)}. \end{aligned}$$

This implies that $(A + B)^2 \leqslant 2(A^2 + B^2)$, so we reach our claim. ■

Remark 3 *By squaring both sides of* (10), *we give* $\omega^2(T_{12}) \leqslant \frac{1}{4} || \ |T_{12}| + |T_{12}^*| \ ||^2$. *This inequality refines (3). To show this,*

$$\omega^2(T_{12}) \leqslant \frac{1}{4} || \ |T_{12}| + |T_{12}^*| \ ||^2 \leqslant \frac{1}{2} || \ |T_{12}|^2 + |T_{12}^*|^2 \ || \ \textit{(by (11))}.$$

Theorem 2.3 Let $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$ be accretive-dissipative operator matrix with cartesian decomposition (5), $r \geqslant 1$. Then

$$\omega^r(T_{11}) \leqslant 2^{r-1} || \ A_{11}^r + B_{11}^r \ || \ \textit{for all } r \in (-\infty, \infty). \tag{12}$$

Proof. We have

$$\begin{aligned} |\langle T_{11}x, x\rangle|^r &= |\langle (A_{11} + iB_{11})x, x\rangle|^r \\ &= |\langle A_{11}x, x\rangle + i \langle B_{11}x, x\rangle|^r \\ &\leqslant (|\langle A_{11}x, x\rangle| + |\langle B_{11}x, x\rangle|)^r \ \text{ (by triangle inequality)} \\ &= (\langle A_{11}x, x\rangle + \langle B_{11}x, x\rangle)^r \ \text{ (since } A_{11} \geqslant 0 \text{ and } B_{11} \geqslant 0) \\ &= \langle (A_{11} + B_{11})x, x\rangle^r \\ &= || \ A_{11} + B_{11} \ ||^r. \end{aligned}$$

Taking the supremum over all unit vectors x, we give (12). Let $r = 1$. We give $\omega(T_{11}) \leqslant || \ A_{11} + B_{11} \ ||$. ■

We conclude this paper by presenting an upper bound for the numerical radius of the off-diagonal operator matrix. It is well-known that for any $A, B \in \mathcal{B}(\mathcal{H})$

$$\omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\|A\| + \|B\|\right). \tag{13}$$

This follows from the following fact (see [7, (4.6)]) that

$$\omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|A + e^{\mathrm{i}\theta}B^*\right\|.$$

To obtain the following result, which contains a refinement of (13), we mimic some ideas from [18, Corollary 2.1].

Theorem 2.4 Let $A, B \in \mathcal{B}(\mathcal{H})$. Then, for any $t \in \mathbb{R}$,

$$\omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)B & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)A \\ (1+t)B & 0 \end{bmatrix}\right)\right).$$

Proof. It has been shown in [4, Corollary 2.4] that

$$\|A+B\| \leq \left\|tA + (1-t)\frac{A+B}{2}\right\| + \left\|tB + (1-t)\frac{A+B}{2}\right\| \leq \|A\| + \|B\|,$$

for any $t \in \mathbb{R}$. If we replace B by $e^{\mathrm{i}\theta}B$, we infer that

$$\begin{aligned}\left\|A + e^{\mathrm{i}\theta}B\right\| &\leq \left\|tA + (1-t)\frac{A+e^{\mathrm{i}\theta}B}{2}\right\| + \left\|te^{\mathrm{i}\theta}B + (1-t)\frac{A+e^{\mathrm{i}\theta}B}{2}\right\| \\ &= \frac{1}{2}\left(\left\|(1+t)A + (1-t)e^{\mathrm{i}\theta}B\right\| + \left\|(1-t)A + (1+t)e^{\mathrm{i}\theta}B\right\|\right).\end{aligned}$$

From this, we can write

$$\begin{aligned}\frac{1}{2}\left\|A + e^{\mathrm{i}\theta}B\right\| &\leq \frac{1}{4}\left(\left\|(1+t)A + (1-t)e^{\mathrm{i}\theta}B\right\| + \left\|(1-t)A + (1+t)e^{\mathrm{i}\theta}B\right\|\right) \\ &\leq \frac{1}{2}\left(\omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)B^* & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)A \\ (1+t)B^* & 0 \end{bmatrix}\right)\right),\end{aligned}$$

i.e.,

$$\frac{1}{2}\left\|A + e^{\mathrm{i}\theta}B\right\| \leq \frac{1}{2}\left(\omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)B^* & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)A \\ (1+t)B^* & 0 \end{bmatrix}\right)\right),$$

for any $t \in \mathbb{R}$. Now, if we take supremum over $\theta \in \mathbb{R}$, we obtain

$$\omega\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)B^* & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)A \\ (1+t)B^* & 0 \end{bmatrix}\right)\right).$$

We deduce the desired result by substituting B by B^*. ■

Assume that $0 \leq t \leq 1$. We can write from Theorem 2.4 that

$$\begin{aligned}
&\omega\left(\begin{bmatrix} 0 & (1+t)\,A \\ (1-t)\,B & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)\,A \\ (1+t)\,B & 0 \end{bmatrix}\right) \\
&\leq \frac{(1+t)\,\|A\| + (1-t)\,\|B\|}{2} + \frac{(1-t)\,\|A\| + (1+t)\,\|B\|}{2} \\
&= \|A\| + \|B\|,
\end{aligned}$$

due to (13). Consequently,

$$\begin{aligned}
\omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{1}{2}\left(\omega\left(\begin{bmatrix} 0 & (1+t)\,A \\ (1-t)\,B & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)\,A \\ (1+t)\,B & 0 \end{bmatrix}\right)\right) \\
&\leq \frac{1}{2}\left(\|A\| + \|B\|\right).
\end{aligned} \tag{14}$$

Remark 4 *We know that* $\omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right)$, *for any* $X, Y \in \mathcal{B}(\mathcal{H})$ *[7, Lemma 2.1 (c)]. So, we obtain from* (14) *that*

$$\omega(A) \leq \omega\left(\begin{bmatrix} 0 & (1+t)\,A \\ (1-t)\,A & 0 \end{bmatrix}\right) \leq \|A\|$$

for any $0 \leq t \leq 1$. *In particular, if* A *is a normal operator, then*

$$\omega\left(\begin{bmatrix} 0 & (1+t)\,A \\ (1-t)\,A & 0 \end{bmatrix}\right) = \|A\|; \;\; (0 \leq t \leq 1).$$

Remark 5 *It has been shown in Remark 4 that if* $A \in \mathcal{B}(\mathcal{H})$, *then for* $0 \leqslant t \leqslant 1$,

$$\omega(A) \leqslant \omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)A & 0 \end{bmatrix}\right) \leqslant \|A\|. \tag{15}$$

For simplicity, let us use the following notations. For $\theta \in \mathbb{R}$, *let*

$$f_\theta(t) = \frac{1}{2}\left\|(1+t)A + (1-t)e^{i\theta}A^*\right\| \;\; \text{and} \;\; f(t) = \omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)A & 0 \end{bmatrix}\right),$$

where $t \in \mathbb{R}$. *Since* $\omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta \in \mathbb{R}}\left\|X + e^{i\theta}Y^*\right\|$, *it follows that* $\sup_{\theta \in \mathbb{R}} f_\theta(t) = f(t)$.

It can be easily seen that the function f_θ *is a convex function of* t *for each* θ. *This is followed by a direct application of the triangle inequality. Since* $f = \sup_{\theta \in \mathbb{R}} f_\theta$, *it follows that* f *is a convex function, too. Further, due to the fact* $\omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right)$, *we see that* $f(t) = f(-t)$. *Therefore,* f *is a convex function on* $\mathbb{R}$, *which is symmetric about* $t = 0$. *Consequently,*

(i) f *is decreasing on* $(-\infty, 0)$ *and increasing on* $(0, \infty)$.
(ii) f *attains its minimum at* $t = 0$.

Noting that $f(0) = \omega(A)$ *and* $f(1) = \|A\|$ *may summarize as follows:*

(a) If $-1 \leqslant t \leqslant 1$, then

$$\omega(A) \leqslant \omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)A & 0 \end{bmatrix}\right) \leqslant \|A\|,$$

which is equivalent to the fact that $f(0) \leqslant f(t) \leqslant f(1) = f(-1)$, for $-1 \leqslant t \leqslant 1$.

(b) If $t \geqslant 1$ or $t \leqslant -1$, we have

$$\omega\left(\begin{bmatrix} 0 & (1+t)A \\ (1-t)A & 0 \end{bmatrix}\right) \geqslant \|A\|,$$

which is equivalent to the fact that $f(t) \geqslant f(1)$ for these values of t.

A concluding comment in this remark is that since f is continuous increasing on $[0,1]$, $f(0) = \omega(A)$ and $f(1) = \|A\|$, it means that f interpolates continuously between $\omega(A)$ and $\|A\|$. This implies that the function f can obtain explicit refinements of any given refinement of the inequality $\omega(A) \leqslant \|A\|$.

So, for example in (15), we have

$$\omega(A) \leqslant \frac{1}{2}\sqrt{\||A|^2 + |A^*|^2\| + \|\,|A||A^*| + |A^*||A|\,\|} \leqslant \|A\|.$$

Now using the above function f, there exists $t_0 \in [0,1]$ such that

$$f(t_0) = \frac{1}{2}\sqrt{\||A|^2 + |A^*|^2\| + \|\,|A||A^*| + |A^*||A|\,\|} = L.$$

When $0 < t_1 < t_0 < t_2 < 1$, we have $\omega(A) \leqslant f(t_1) \leqslant f(t_0) = L \leqslant f(t_2) \leqslant \|A\|$. Finding t_0 can be done using numerical calculations. This, of course, depends on A. Thus, we have a rigorous approach for finding refinements of any inequality that refines $\omega(A) \leqslant \|A\|$, as we have explained with (15).

Remark 6 *It is well-known from [1, Corollary 3] that if $X, Y \in \mathcal{B}(\mathcal{H})$ are positive operators, then*

$$\omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = \frac{1}{2}\left(\|X\| + \|Y\|\right).$$

Thus, from (14), we infer that

$$\omega\left(\begin{bmatrix} 0 & (1+t)\,A \\ (1-t)\,B & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & (1-t)\,A \\ (1+t)\,B & 0 \end{bmatrix}\right) = \|A\| + \|B\|\,; \;\; (0 \leq t \leq 1)$$

provided that A, B are positive operators.

Acknowledgments

The authors thank the Editorial Board and the referees for their valuable comments that helped to improve the article.

References

[1] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for $n \times n$ operator matrices, Linear Algebra Appl. 468 (2015), 18-26.
[2] M. W. Alomari, M. Sababheh, C. Conde, H. R. Moradi, Generalized Euclidean operator radius, Georgian Math. J. 31 (3) (2023), 369-380.
[3] C. Conde, H. R. Moradi, M. Sababheh, A family of semi-norms between the numerical radius and the operator norm, Results Math. 79 (2024), 79:36.
[4] S. Furuichi, H. R. Moradi, M. Sababheh, New inequalities for interpolational operator means, J. Math. Inequal. 15 (1) (2021), 107-116.
[5] I. H. Gumus, H. R. Moradi, M. Sababheh, Operator inequalities via accretive transforms, Hacet. J. Math. Stat. 53 (1) (2024), 40-52.
[6] K. E. Gustafsun, D. K. M. Rao, Numerical range, Springer-Verlag, New York, 1997.
[7] O. Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices, Integr. Equ. Oper. Theory. 71 (2011), 129-147.
[8] E. Jaafari, M. S. Asgari, M. Shah Hosseini, B. Moosavi, On the Jensen's inequality and its variants, AIMS Math. 5 (2) (2021), 1177-1185.
[9] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. Res. Inst. Math. Sci. 24 (1988), 283-293.
[10] F. Kittaneh, Numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Stud. Math. 158 (2003), 11-17.
[11] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Stud. Math. 168 (2003), 73-80.
[12] F. Kittaneh, H. R. Moradi, M. Sababheh, Mean inequalities for the numerical radius, Numer. Funct. Anal. Optim. 44 (14) (2023), 1523-1537.
[13] F. Kittaneh, H. R. Moradi, M. Sababheh, Norm and numerical radius inequalities for operator matrices, Linear Multilinear Algebra. (2023), 2023:2297393.
[14] B. Moosavi, M. Shah Hosseini, New lower bound for numerical radius for off-diagonal 2×2 matrices, J. Linear. Topol. Algebra. 13 (2024), 13-18.
[15] B. Moosavi, M. Shah Hosseini, Some inequalities for the numerical radius for operators in Hilbert C^*-modules space, J. Inequal. Spec. Func. 10 (2019), 77-84.
[16] H. R. Moradi, M. E. Omidvar, S. S. Dragomir, M. S. Khan, Sesquilinear version of numerical range and numerical radius, Acta Univ. Sapientiae Math. 9 (2) (2017), 324-335.
[17] H. R. Moradi, M Sababheh, New estimates for the numerical radius, Filomat. 35 (14) (2021), 4957-4962.
[18] M. Sababheh, C. Conde, H. R. Moradi, A convex-block approach for numerical radius inequalities, Funct. Anal. Appl. 57 (2023), 26-30.
[19] M. Sababheh, H. R. Moradi, S. Sahoo, Inner product inequalities with applications, Linear Multilinear Algebra. (2024), 1-14.
[20] S. Sahoo, H. R. Moradi, M. Sababheh, Some numerical radius bounds, Acta Sci. Math. (2024), 1-19.
[21] M. Shah Hosseini, B. Moosavi, H. R. Moradi, An alternative estimate for the numerical radius of Hilbert space operators, Mathematica Slovaca. 70 (1) (2020), 233-237.
[22] M. Shah Hosseini, M. E. Omidvar, B. Moosavi, H. R. Moradi, Some inequalities for the numerical radius for Hilbert C^*-modules space operators, Georgian Math. J. 28 (2) (2020), 255-260.