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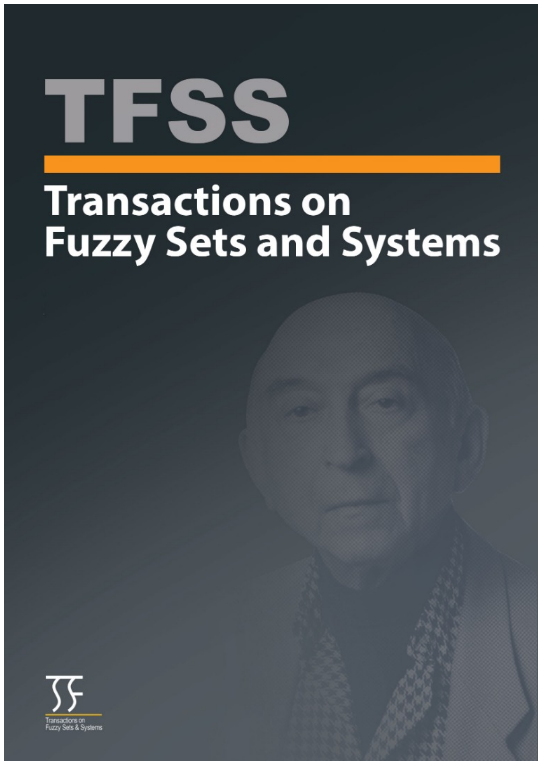

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Bipolar Fuzzy Sets in IUP-Algebras: Concepts and Analysis

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Abstract. This paper advances the theory of bipolar fuzzy sets by integrating them into the structure of IUP-algebras. It introduces and formalizes the concepts of bipolar fuzzy IUP-subalgebras, filters, ideals, and strong ideals, establishing their foundational properties through rigorous theorems and counterexamples. The interplay between these algebraic structures and their level cuts is examined to provide a nuanced understanding of dual-valued uncertainty in abstract logic. In addition to its theoretical contributions, this research holds significant educational value, particularly for science-oriented learning environments where mathematical abstraction and logical reasoning are cultivated. By translating complex fuzzy-algebraic concepts into structured and illustrative forms, the study offers a meaningful framework that can inspire student-led investigations and enrich curricula in research-focused schools. Furthermore, it encourages collaboration between educators and researchers in developing accessible tools and learning modules that bridge advanced mathematical logic with real-world inquiry, promoting a more inclusive and research-driven academic atmosphere.

AMS Subject Classification 2020: 03G25; 20K25
Keywords and Phrases: IUP-algebra, Bipolar fuzzy set, Bipolar fuzzy IUP-subalgebra, Bipolar fuzzy IUP-ideal, Bipolar fuzzy IUP-filter, Bipolar fuzzy strong IUP-ideal, Negative lower $\sqcup^-$-cut, Positive upper $\sqcup^+$-cut, $(\sqcup^-, \sqcup^+)$-cut.

1 Introduction

The concept of bipolar fuzzy sets (BFSs) was introduced as an extension of Zadehs seminal fuzzy set (FS) theory [1], which provided a groundbreaking framework for modeling uncertainty and imprecision. Zadehs theory utilized a single degree of membership within the range $[0, 1]$, enabling the representation of vagueness, but was limited in scenarios involving duality or conflict. To address these limitations, Zhang [2] introduced BFSs, incorporating two membership functions: one for positive degrees ($\digamma^+$) and another for negative degrees ($\digamma^-$), thus enabling the simultaneous modeling of satisfaction and dissatisfaction. Since its inception, the study of BFSs has significantly advanced the modeling of complex systems characterized by conflicting or dual-valued information. BFSs have become a cornerstone in the study of uncertainty and decision-making, with applications spanning decision support, sentiment analysis, and logical algebra. The continuous evolution of the field, fueled by the contributions of numerous mathematicians and researchers, underscores its relevance across diverse theoretical and applied domains. Over the years, researchers have extended BFSs into various algebraic and logical frameworks, yielding fruitful results in multiple fields. For instance, Abughazalah et al.

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[3] introduced bipolar fuzzy (closed) BCI-positive implicative ideals and (closed) BCI-commutative ideals in BCI-algebras. Further studies have examined the structural properties and applications of BFSs in various algebraic systems. Alolaiyan et al. [4] explored bipolar fuzzy isomorphisms in subrings, while Gaketem et al. [5] analyzed bipolar fuzzy comparative UP-filters. Significant progress has also been made in applying BFSs to logical and algebraic structures. Jun and Song [6] introduced bipolar-valued fuzzy subalgebras and closed ideals in BCH-algebras, providing foundational properties and conditions under which these structures are preserved. Their work laid important groundwork for integrating bipolar-valued uncertainty into algebraic systems. Jun and Park [7] extended this line of research by defining bipolar-valued fuzzy filters in BCH-algebras. They characterized the structural behavior of these filters and demonstrated their relationships with level sets, further enriching the theory of fuzzy algebra under bipolar-valued settings. Mahmood and Munir [8] introduced bipolar fuzzy subgroups and normal subgroups, while Malik et al. [9] investigated rough approximations in bipolar fuzzy subsemigroups. Recently, research has continued to expand the boundaries of BFS applications. For instance, Muhiddin et al. [10] characterized bipolar fuzzy positive implicative ideals in BCK-algebras, and Yaqoob et al. [11] extended the concept to bipolar (λ, δ)-fuzzy structures in τ-semihypergroups, adding further depth to the theory. Abughazalah et al. [3] applied BFS theory to the study of various ideals in BCI-algebras, introducing and analyzing bipolar fuzzy positive implicative, commutative, and closed ideals. Their work enriched the algebraic structure of BCI-algebras by incorporating bipolar-valued uncertainty, offering refined tools for modeling duality in logical reasoning systems. Al-Kadi and Muhiuddin [12] focused on the development of bipolar fuzzy BCI-implicative ideals, establishing key characterizations and algebraic properties within BCI-algebras. Their study contributed to the theoretical advancement of fuzzy logic by extending the notion of implicative ideals to a bipolar fuzzy context. The vast body of literature underscores the theoretical richness and practical versatility of BFSs. By combining dual membership perspectives, BFSs offer a sophisticated approach to capturing nuanced relationships in uncertain environments. As the field continues to grow, BFSs remain integral to advancing modern fuzzy logic and algebraic research.

The concept of IUP-algebras was first introduced by Iampan et al. in 2022 [13] as a novel algebraic structure characterized by four primary subsets: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. Their pioneering work not only laid the theoretical foundations of IUP-algebras but also highlighted their potential as a fertile ground for mathematical research. By establishing fundamental properties and exploring their initial applications, Iampan et al. opened new avenues for the study of algebraic structures. Building on this foundation, research in IUP-algebras has rapidly expanded, addressing both theoretical advancements and practical implications. Chanmanee et al. [14] contributed significantly to this growing body of knowledge in 2023 by exploring the direct product of infinite families of IUP-algebras. Their study introduced the notion of weak direct products and developed essential results concerning (anti-)IUP-homomorphisms, deepening the structural understanding of IUP-algebras and providing crucial tools for further exploration. In 2024, Kuntama et al. [15] integrated FS theory with IUP-algebras, introducing constructs such as fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals. Their meticulous analysis of these fuzzy subsets not only expanded the applicability of IUP-algebras but also bridged the gap between algebraic structures and fuzzy logic, enabling new interpretations and applications. Further enriching the framework, Suayngam et al. [16] extended the theory of IUP-algebras by incorporating intuitionistic FSs. Their introduction of intuitionistic fuzzy IUP-subalgebras, ideals, filters, and strong ideals created hybrid structures that offered fresh perspectives and avenues for research. This innovative integration demonstrated the flexibility of IUP-algebras in accommodating advanced fuzzy logic concepts, thereby broadening their potential applications. Additionally, Suayngam et al. [17] advanced the field by applying Fermatean FSs to IUP-algebras. Their work focused on Fermatean fuzzy IUP-subalgebras, IUP-ideals, IUP-filters, and strong IUP-ideals, further expanding the algebra's theoretical landscape and underscoring its versatility in integrating sophisticated fuzzy systems. In 2025, Suayngam et al. [18] explored the integration of neutrosophic set theory into IUP-algebras, introducing novel definitions and establishing structural properties within this hybrid framework. Their work broadens the theoretical scope of IUP-algebras by accommodating degrees of truth, indeterminacy, and falsity, offering a richer model for uncertainty. Suayngam et al. [19] applied Pythagorean fuzzy sets to IUP-algebras, presenting new characterizations of subalgebras, filters, and ideals. By leveraging the extended range of membership functions inherent in Pythagorean fuzzy logic, the authors provided refined insights into the algebraic behavior under high-order uncertainty. Suayngam et al. [20] advanced the theory of IUP-algebras by employing intuitionistic neutrosophic sets, analyzing complex structural relationships through multi-valued logic. Their study delivers a comprehensive perspective on how indeterminacy and hesitation interact with the algebraic constructs of the IUP framework. This ongoing progression in the study of IUP-algebras reflects their significance as a dynamic and evolving algebraic framework. With contributions ranging from direct product constructions to fuzzy and intuitionistic fuzzy extensions, IUP-algebras continue to inspire innovative research, offering a robust platform for mathematical exploration and interdisciplinary applications.

Despite the growing body of literature on fuzzy generalizations in IUP-algebras, such as those involving fuzzy, intuitionistic fuzzy, neutrosophic, and Pythagorean fuzzy sets, the bipolar perspective has yet to be explored. Existing studies do not account for the dual nature of uncertainty, which simultaneously captures both satisfaction and dissatisfaction, a crucial aspect in many real-world contexts. Motivated by this gap, the present study aims to incorporate bipolar fuzzy logic into IUP-algebras, providing a more expressive and flexible framework for modeling algebraic uncertainty.

Building upon these developments, this study extends the concept of BFSs to the framework of IUP-algebras. It introduces the notions of bipolar fuzzy IUP-subalgebras, bipolar fuzzy IUP-filters, bipolar fuzzy IUP-ideals, and bipolar fuzzy strong IUP-ideals, establishing their foundational properties and providing illustrative examples to validate and refine these constructs. The research further identifies conditions under which bipolar fuzzy IUP-filters can be characterized as bipolar fuzzy IUP-ideals or bipolar fuzzy IUP-subalgebras, thereby enriching the structural understanding of IUP-algebras. Additionally, it explores the relationships between bipolar fuzzy IUP-subalgebras (and their corresponding filters, ideals, and strong ideals) and their level cuts, as well as the interplay between IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals with their associated characteristic BFSs. These findings provide a robust foundation for integrating bipolar fuzzy logic into the study of IUP-algebras, highlighting the potential for further theoretical and practical applications.

The remainder of this paper is organized as follows. Section 2 reviews basic definitions and properties related to IUP-algebras. In Section 3, we introduce the concept of BFSs in IUP-algebras and establish key properties of bipolar fuzzy IUP-subalgebras, bipolar fuzzy IUP-filters, bipolar fuzzy IUP-ideals, and bipolar fuzzy strong IUP-ideals, accompanied by illustrative examples. Section 4 focuses on the analysis of level cuts and their connections to bipolar fuzzy structures. Finally, Section 5 concludes the paper and discusses potential directions for future research.

2 Preliminaries

IUP-algebra, a relatively recent addition to the landscape of algebraic structures, was first conceptualized to address specific gaps in the study of non-classical logic systems and fuzzy algebra. Introduced by Iampan et al. [13] in 2022, IUP-algebras are defined by their distinctive axiomatic framework, which includes four fundamental subsets: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These subsets form the cornerstone of this algebraic structure, offering a flexible yet robust platform for exploring logical relationships and algebraic operations.

The unique properties of IUP-algebras, such as their closure under binary operations and their relationship to algebraic completeness, make them an invaluable tool for advancing theoretical mathematics. Researchers have leveraged these properties to investigate the interplay between algebraic constructs and fuzzy logic, thereby broadening the applicability of IUP-algebras across disciplines. This section aims to lay the groundwork for the study of BFSs within IUP-algebras by revisiting their foundational definitions, key examples, and properties.

Definition 2.1. [13] An algebra $\mathfrak{I} = (\mathfrak{I}; \cdot, \wp)$ of type $(2, 0)$ is defined as an IUP-algebra if it satisfies a specific set of axioms. Here, $\mathfrak{I}$ represents a nonempty set, $\cdot$ denotes a binary operation defined on $\mathfrak{I}$, and $\wp$ is the distinguished constant element of $\mathfrak{I}$. These foundational elements work in unison to establish the structural integrity and logical properties of IUP-algebras, as outlined by the following axioms:

$$(\forall \dot{\lambda} \in \mathfrak{I})(\wp \cdot \dot{\lambda} = \dot{\lambda}), \tag{IUP-1}$$

$$(\forall \dot{\lambda} \in \mathfrak{I})(\dot{\lambda} \cdot \dot{\lambda} = \wp), \tag{IUP-2}$$

$$(\forall \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I})((\dot{\lambda} \cdot \ddot{\gamma}) \cdot (\dot{\lambda} \cdot \breve{\beta}) = \ddot{\gamma} \cdot \breve{\beta}). \tag{IUP-3}$$

For clarity and ease of reference, we shall denote the IUP-algebra $\mathfrak{I}$ as $\mathfrak{I} = (\mathfrak{I}; \cdot, \wp)$ throughout this discussion, unless stated otherwise.

In $\mathfrak{I}$, the following assertions are valid (see [13]).

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})((\dot{\lambda} \cdot \wp) \cdot (\dot{\lambda} \cdot \ddot{\gamma}) = \ddot{\gamma}), \tag{1}$$

$$(\forall \dot{\lambda} \in \mathfrak{I})((\dot{\lambda} \cdot \wp) \cdot (\dot{\lambda} \cdot \wp) = \wp), \tag{2}$$

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})((\dot{\lambda} \cdot \ddot{\gamma}) \cdot \wp = \ddot{\gamma} \cdot \dot{\lambda}), \tag{3}$$

$$(\forall \dot{\lambda} \in \mathfrak{I})((\dot{\lambda} \cdot \wp) \cdot \wp = \dot{\lambda}), \tag{4}$$

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})(\dot{\lambda} \cdot ((\dot{\lambda} \cdot \wp) \cdot \ddot{\gamma}) = \ddot{\gamma}), \tag{5}$$

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})(((\dot{\lambda} \cdot \wp) \cdot \ddot{\gamma}) \cdot \dot{\lambda} = \ddot{\gamma} \cdot \wp), \tag{6}$$

$$(\forall \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I})(\dot{\lambda} \cdot \ddot{\gamma} = \dot{\lambda} \cdot \breve{\beta} \Leftrightarrow \ddot{\gamma} = \breve{\beta}), \tag{7}$$

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})(\dot{\lambda} \cdot \ddot{\gamma} = \wp \Leftrightarrow \dot{\lambda} = \ddot{\gamma}), \tag{8}$$

$$(\forall \dot{\lambda} \in \mathfrak{I})(\dot{\lambda} \cdot \wp = \wp \Leftrightarrow \dot{\lambda} = \wp), \tag{9}$$

$$(\forall \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I})(\ddot{\gamma} \cdot \dot{\lambda} = \breve{\beta} \cdot \dot{\lambda} \Leftrightarrow \ddot{\gamma} = \breve{\beta}), \tag{10}$$

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})(\dot{\lambda} \cdot \ddot{\gamma} = \ddot{\gamma} \Rightarrow \dot{\lambda} = \wp), \tag{11}$$

$$(\forall \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I})((\dot{\lambda} \cdot \ddot{\gamma}) \cdot \wp = (\breve{\beta} \cdot \ddot{\gamma}) \cdot (\breve{\beta} \cdot \dot{\lambda})), \tag{12}$$

$$(\forall \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I})(\dot{\lambda} \cdot \ddot{\gamma} = \wp \Leftrightarrow (\breve{\beta} \cdot \dot{\lambda}) \cdot (\breve{\beta} \cdot \ddot{\gamma}) = \wp), \tag{13}$$

$$(\forall \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I})(\dot{\lambda} \cdot \ddot{\gamma} = \wp \Leftrightarrow (\dot{\lambda} \cdot \breve{\beta}) \cdot (\ddot{\gamma} \cdot \breve{\beta}) = \wp), \tag{14}$$

$$\text{the right and the left cancellation laws hold.} \tag{15}$$

Definition 2.2. [13] A nonempty subset $\mathfrak{S}$ of $\mathfrak{I}$ is called

(*i*) an IUP-subalgebra (IUPS) of $\mathfrak{I}$ if it satisfies the following requirement:

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{S})(\dot{\lambda} \cdot \ddot{\gamma} \in \mathfrak{S}), \tag{16}$$

(*ii*) an IUP-filter (IUPF) of $\mathfrak{I}$ if it satisfies the following requirements:

$$\text{the constant element } \wp \text{ of } \mathfrak{I} \text{ is in } \mathfrak{S}, \tag{17}$$

$$(\forall \dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})(\dot{\lambda} \cdot \ddot{\gamma} \in \mathfrak{S}, \dot{\lambda} \in \mathfrak{S} \Rightarrow \ddot{\gamma} \in \mathfrak{S}), \tag{18}$$

(iii) an IUP-ideal (IUPI) of $\mathfrak{I}$ if it satisfies the requirement (17) and the following requirement:

$$(\forall\dot{\lambda}, \ddot{\gamma}, \check{\beta} \in \mathfrak{I})(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \check{\beta}) \in \mathfrak{S}, \ddot{\gamma} \in \mathfrak{S} \Rightarrow \dot{\lambda} \cdot \check{\beta} \in \mathfrak{S}), \tag{19}$$

(iv) a strong IUP-ideal (SIUPI) of $\mathfrak{I}$ if it satisfies the following requirement:

$$(\forall\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I})(\ddot{\gamma} \in \mathfrak{S} \Rightarrow \dot{\lambda} \cdot \ddot{\gamma} \in \mathfrak{S}). \tag{20}$$

As established in [13], the concept of IUPFs represents a significant generalization of IUPIs and IUPSs, broadening the theoretical scope of these subsets. Notably, both IUPIs and IUPSs serve as further generalizations of SIUPIs, illustrating a clear hierarchical relationship within the structure of IUP-algebras. In the specific case of $\mathfrak{I}$, the SIUPIs align precisely with $\mathfrak{I}$ itself, reflecting a unique symmetry in this algebraic framework.

To provide a comprehensive visualization of these interrelationships, the diagram of the special subsets of IUP-algebras is depicted in Figure 1, offering a concise representation of their structural hierarchy.

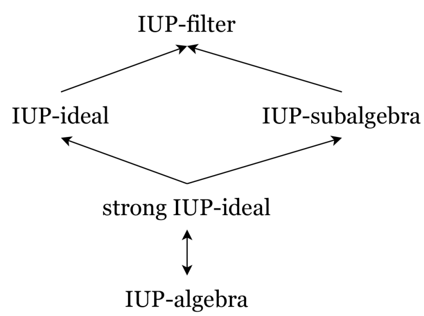

Figure 1: Special subsets of IUP-algebras

The following theorem is elementary, and its proof is straightforward.

Theorem 2.3. *Let $\{\mathfrak{B}_i\}_{i\in I}$ be a nonempty family of IUPSs (resp., IUPFs, IUPIs, SIUPIs) of $\mathfrak{I}$. Then $\bigcap_{i\in I} \mathfrak{B}_i$ is an IUPSs (resp., IUPFs, IUPIs, SIUPIs) of $\mathfrak{I}$.*

3 BFSs in IUP-Algebras

The concept of BFSs emerged as a groundbreaking extension of Zadehs classical FS theory, addressing the limitations of single-membership degrees in capturing dual or conflicting information. Introduced by Zhang, BFSs incorporate two distinct membership functions: one for positive membership (satisfaction) and another for negative membership (dissatisfaction). This dual structure allows BFSs to model complex systems characterized by simultaneous agreement and disagreement, offering a richer framework for representing uncertainty and imprecision.

Over the years, BFSs have evolved into a powerful tool in algebraic structures, decision-making, sentiment analysis, and logical systems. By capturing nuanced relationships in uncertain environments, BFSs provide a robust foundation for integrating advanced fuzzy logic into algebraic frameworks. This section introduces the concept of BFSs in the context of IUP-algebras, defining key properties and demonstrating their applicability through illustrative examples and rigorous analysis.

Let $\mathfrak{I}$ be the universe of discourse. A bipolar fuzzy set (BFS) $\digamma$ in $\mathfrak{I}$ is an object having the form

$$\digamma = \{(\dot{\lambda}, \digamma^{-}(\dot{\lambda}), \digamma^{+}(\dot{\lambda})) \mid \dot{\lambda} \in \mathfrak{I}\}$$

where $\digamma^{-} : \Im \to [-1,0]$ and $\digamma^{+} : \Im \to [0,1]$ are mappings. For convenience, we adopt the notation $\digamma = (\Im; \digamma^{-}, \digamma^{+})$ to represent the BFS $\digamma = \{(\dot{\lambda}, \digamma^{-}(\dot{\lambda}), \digamma^{+}(\dot{\lambda})) \mid \dot{\lambda} \in \Im\}$.

In the following, we introduce the notion of bipolar fuzzy IUP-subalgebras, along with their counterparts: bipolar fuzzy IUP-filters, bipolar fuzzy IUP-ideals, and bipolar fuzzy strong IUP-ideals, within the framework of the IUP-algebra $\Im$. To enhance understanding and demonstrate their practical relevance, we provide illustrative examples that highlight the fundamental properties and behaviors of these constructs.

Definition 3.1. A BFS $\digamma = (\Im; \digamma^{-}, \digamma^{+})$ in $\Im$ is called a bipolar fuzzy IUP-subalgebra (BFIUPS) of $\Im$ if it satisfies the following requirements: for any $\dot{\lambda}, \ddot{\gamma} \in \Im$,

$$\digamma^{-}(\dot{\lambda} \cdot \ddot{\gamma}) \leq \max\{\digamma^{-}(\dot{\lambda}), \digamma^{-}(\ddot{\gamma})\}, \tag{21}$$

$$\digamma^{+}(\dot{\lambda} \cdot \ddot{\gamma}) \geq \min\{\digamma^{+}(\dot{\lambda}), \digamma^{+}(\ddot{\gamma})\}. \tag{22}$$

Remark 3.2. If $\digamma = (\Im; \digamma^{-}, \digamma^{+})$ is a BFIUPS of $\Im$, then

$$\digamma^{-}(\wp) \leq \digamma^{-}(\dot{\lambda}) \text{ and } \digamma^{+}(\wp) \geq \digamma^{+}(\dot{\lambda}) \text{ for all } \dot{\lambda} \in \Im.$$

Indeed, for all $\dot{\lambda} \in \Im$,

$$\digamma^{-}(\wp) = \digamma^{-}(\dot{\lambda} \cdot \dot{\lambda}) \leq \digamma^{-}(\dot{\lambda}),$$

$$\digamma^{+}(\wp) = \digamma^{+}(\dot{\lambda} \cdot \dot{\lambda}) \geq \digamma^{+}(\dot{\lambda}).$$

Example 3.3. Let us consider the IUP-algebra $\Im = \{\wp, 1, 2, 3, 4, 5\}$, defined by the following table, which encapsulates its binary operation:

Table 1: The Cayley table for Example 3.3

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	5	$\wp$	3	1	2	4
2	2	4	$\wp$	5	1	3
3	4	5	1	$\wp$	3	2
4	3	2	5	4	$\wp$	1
5	1	3	4	2	5	$\wp$

Define $\digamma = (\Im; \digamma^{-}, \digamma^{+})$ in $\Im$ as follows:

$\Im$	$\wp$	1	2	3	4	5
$\digamma^{-}$	-0.9	-0.3	-0.6	-0.3	-0.3	-0.3
$\digamma^{+}$	0.8	0.1	0.5	0.1	0.1	0.1

Then $\digamma = (\Im; \digamma^{-}, \digamma^{+})$ is a BFIUPS of $\Im$.

Definition 3.4. A BFS $\digamma = (\Im; \digamma^{-}, \digamma^{+})$ in $\Im$ is called a bipolar fuzzy IUP-filter (BFIUPF) of $\Im$ if it satisfies the following requirements: for any $\dot{\lambda}, \ddot{\gamma} \in \Im$,

$$\digamma^{-}(\wp) \leq \digamma^{-}(\dot{\lambda}), \tag{23}$$

$$\digamma^{+}(\wp) \geq \digamma^{+}(\dot{\lambda}), \tag{24}$$

$$\digamma^{-}(\ddot{\gamma}) \leq \max\{\digamma^{-}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^{-}(\dot{\lambda})\}, \tag{25}$$

$$\digamma^{+}(\ddot{\gamma}) \geq \min\{\digamma^{+}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^{+}(\dot{\lambda})\}. \tag{26}$$

Example 3.5. Take, for instance, the IUP-algebra $\mathfrak{I} = \{\wp, 1, 2, 3, 4, 5\}$, whose structure is defined by the following Cayley table:

Table 2: The Cayley table for Example 3.5

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	2	$\wp$	1	4	5	3
2	1	2	$\wp$	5	3	4
3	3	5	4	$\wp$	2	1
4	5	4	3	1	$\wp$	2
5	4	3	5	2	1	$\wp$

Define $F = (\mathfrak{I}; F^-, F^+)$ in $\mathfrak{I}$ as follows:

$\mathfrak{I}$	$\wp$	1	2	3	4	5
F^-	-0.8	-0.7	-0.7	-0.2	-0.2	-0.2
F^+	0.9	0.5	0.5	0.4	0.4	0.4

Then $F = (\mathfrak{I}; F^-, F^+)$ is a BFIUPF of $\mathfrak{I}$.

Definition 3.6. A BFS $F = (\mathfrak{I}; F^-, F^+)$ in $\mathfrak{I}$ is called a bipolar fuzzy IUP-ideal (BFIUPI) of $\mathfrak{I}$ if it satisfies (23), (24) and the following requirements: for any $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I}$,

$$F^-(\dot{\lambda} \cdot \breve{\beta}) \leq \max\{F^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), F^-(\ddot{\gamma})\}, \tag{27}$$

$$F^+(\dot{\lambda} \cdot \breve{\beta}) \geq \min\{F^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), F^+(\ddot{\gamma})\}. \tag{28}$$

Example 3.7. Let us consider the IUP-algebra $\mathfrak{I} = \{\wp, 1, 2, 3, 4, 5\}$, defined by the following table, which encapsulates its binary operation:

Table 3: The Cayley table for Example 3.7

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	3	$\wp$	1	4	5	2
2	4	3	$\wp$	5	2	1
3	1	2	5	$\wp$	3	4
4	2	5	4	1	$\wp$	3
5	5	4	3	2	1	$\wp$

Define $F = (\mathfrak{I}; F^-, F^+)$ in $\mathfrak{I}$ as follows:

$\mathfrak{I}$	$\wp$	1	2	3	4	5
F^-	-0.9	-0.8	-0.8	-0.4	-0.4	-0.4
F^+	0.7	0.5	0.5	0.1	0.1	0.1

Then $F = (\mathfrak{I}; F^-, F^+)$ is a BFIUPI of $\mathfrak{I}$.

Definition 3.8. A BFS $F = (\mathfrak{I}; F^-, F^+)$ in $\mathfrak{I}$ is called a bipolar fuzzy strong IUP-ideal (BFSIUPI) of $\mathfrak{I}$ if it satisfies the following requirements: for any $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$,

$$F^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq F^-(\ddot{\gamma}), \tag{29}$$

$$F^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq F^+(\ddot{\gamma}). \tag{30}$$

Theorem 3.9. *A BFS $\digamma = (\Im; \digamma^-, \digamma^+)$ in $\Im$ is constant if and only if it is a BFSIUPI of $\Im$.*

Proof. Assume that $\digamma = (\Im; \digamma^-, \digamma^+)$ is a constant BFS in $\Im$. Then there exists $(\nabla^-, \nabla^+) \in [-1, 0] \times [0, 1]$ such that $\digamma^-(\dot{\lambda}) = \nabla^-$ and $\digamma^+(\dot{\lambda}) = \nabla^+$ for all $\dot{\lambda} \in \Im$. Thus, $\digamma^-(\wp) = \nabla^- \leq \nabla^- = \digamma^-(\dot{\lambda})$ and $\digamma^+(\wp) = \nabla^+ \geq \nabla^+ = \digamma^+(\dot{\lambda})$ for all $\dot{\lambda} \in \Im$. So, $\digamma^-(\dot{\lambda} \cdot \ddot{\gamma}) = \nabla^- \leq \nabla^- = \digamma^-(\ddot{\gamma})$ and $\digamma^+(\dot{\lambda} \cdot \ddot{\gamma}) = \nabla^+ \geq \nabla^+ = \digamma^+(\dot{\lambda})$. Hence, $\digamma = (\Im; \digamma^-, \digamma^+)$ is a BFSIUPI of $\Im$.

Conversely, assume that $\digamma = (\Im; \digamma^-, \digamma^+)$ is a BFSIUPI of $\Im$. Then for all $\dot{\lambda}, \ddot{\gamma} \in \Im$,

$$\digamma^-(\wp) \leq \digamma^-(\dot{\lambda}) \text{ and } \digamma^+(\wp) \geq \digamma^+(\dot{\lambda}),$$

$$\digamma^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq \digamma^-(\ddot{\gamma}) \text{ and } \digamma^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq \digamma^+(\ddot{\gamma}).$$

Then

$$\digamma^-(\dot{\lambda}) = \digamma^-((\dot{\lambda} \cdot \wp) \cdot 0) = \digamma^-(\wp),$$

$$\digamma^+(\dot{\lambda}) = \digamma^+((\dot{\lambda} \cdot \wp) \cdot \wp) = \digamma^+(\wp).$$

Hence, $\digamma^-(\dot{\lambda}) = \digamma^-(\wp)$ and $\digamma^+(\dot{\lambda}) = \digamma^+(\wp)$. Therefore, $\digamma = (\Im; \digamma^-, \digamma^+)$ is a constant BFS in $\Im$. □

Theorem 3.10. *Every BFSIUPI of $\Im$ is a BFIUPI.*

Proof. Let $\digamma = (\Im; \digamma^-, \digamma^+)$ be a BFSIUPI of $\Im$. By Theorem 3.9, there exists $(\nabla^-, \nabla^+) \in [-1, 0] \times [0, 1]$ such that $\digamma^-(\dot{\lambda}) = \nabla^-$ and $\digamma^+(\dot{\lambda}) = \nabla^+$ for all $\dot{\lambda} \in \Im$. For all $\dot{\lambda}, \ddot{\gamma}, \check{\beta} \in \Im$,

$$\digamma^-(\dot{\lambda} \cdot \ddot{\gamma}) = \nabla^- \leq \nabla^- = \max\{\nabla^-, \nabla^-\} = \max\{\digamma^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \check{\beta})), \digamma^-(\ddot{\gamma})\},$$

$$\digamma^+(\dot{\lambda} \cdot \ddot{\gamma}) = \nabla^+ \geq \nabla^+ = \min\{\nabla^+, \nabla^+\} = \min\{\digamma^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \check{\beta})), \digamma^+(\ddot{\gamma})\}.$$

Hence, $\digamma = (\Im; \digamma^-, \digamma^+)$ is a BFIUPI of $\Im$. □

Example 3.11. Let us examine an IUP-algebra $\Im = \{\wp, 1, 2, 3, 4, 5\}$, characterized by the following table:

Table 4: The Cayley table for Example 3.11

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	3	$\wp$	1	4	5	2
2	4	3	$\wp$	5	2	1
3	1	2	5	$\wp$	3	4
4	2	5	4	1	$\wp$	3
5	5	4	3	2	1	$\wp$

We define a BFS $\digamma = (\Im; \digamma^-, \digamma^+)$ in $\Im$ as follows:

$\Im$	$\wp$	1	2	3	4	5
$\digamma^-$	-0.8	-0.4	-0.8	-0.4	-0.8	-0.4
$\digamma^+$	0.3	0.1	0.3	0.1	0.3	0.1

Then $\digamma$ is a BFIUPI of $\Im$, but it is not a BFSIUPI of $\Im$. Indeed,

$$\digamma^-(1 \cdot \wp) = \digamma^-(3) = -0.4 \nleq -0.8 = \digamma^-(\wp),$$

$$\digamma^+(5 \cdot \wp) = \digamma^+(5) = 0.1 \ngeq 0.3 = \digamma^+(\wp).$$

Theorem 3.12. *Every BFSIUPI of $\mathfrak{I}$ is a BFIUPS.*

Proof. Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFSIUPI of $\mathfrak{I}$. By Theorem 3.9, there exists $(\nabla^-, \nabla^+) \in [-1, 0] \times [0, 1]$ such that $F^-(\dot{\lambda}) = \nabla^-$ and $F^+(\dot{\lambda}) = \nabla^+$ for all $\dot{\lambda} \in \mathfrak{I}$. For all $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$,

$$F^-(\dot{\lambda} \cdot \ddot{\gamma}) = \nabla^- \leq \nabla^- = \max\{\nabla^-, \nabla^-\} = \max\{F^-(\dot{\lambda}), F^-(\ddot{\gamma})\},$$

$$F^+(\dot{\lambda} \cdot \ddot{\gamma}) = \nabla^+ \geq \nabla^+ = \min\{\nabla^+, \nabla^+\} = \min\{F^+(\dot{\lambda}), F^+(\ddot{\gamma})\}.$$

Hence, $F = (\mathfrak{I}; F^-, F^+)$ is a BFIUPS of $\mathfrak{I}$. $\square$

Example 3.13. Let us examine an IUP-algebra $\mathfrak{I} = \{\wp, 1, 2, 3, 4, 5\}$, characterized by the following table:

Table 5: The Cayley table for Example 3.13

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	5	$\wp$	3	1	2	4
2	2	4	$\wp$	5	1	3
3	4	5	1	$\wp$	3	2
4	3	2	5	4	$\wp$	1
5	1	3	4	2	5	$\wp$

We define a BFS $F = (\mathfrak{I}; F^-, F^+)$ in $\mathfrak{I}$ as follows:

$\mathfrak{I}$	$\wp$	1	2	3	4	5
F^-	-0.9	-0.4	-0.4	-0.5	-0.5	-0.4
F^+	0.8	0.6	0.6	0.7	0.7	0.6

Then F is a BFIUPS of $\mathfrak{I}$ but it is not a BFSIUPI of $\mathfrak{I}$. Indeed,

$$F^-(1 \cdot 3) = F^-(1) = -0.4 \nleq -0.5 = F^-(3),$$

$$F^+(2 \cdot 3) = F^+(5) = 0.6 \ngeq 0.7 = F^+(3).$$

Theorem 3.14. *Every BFIUPI of $\mathfrak{I}$ is a BFIUPF.*

Proof. Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFIUPI of $\mathfrak{I}$. Then for all $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}, F^-(\wp) \leq F^-(\dot{\lambda})$ and $F^+(\wp) \geq F^+(\dot{\lambda})$ and

$$\begin{aligned} F^-(\ddot{\gamma}) &= F^-(\wp \cdot \ddot{\gamma}) \\ &\leq \max\{F^-(\wp \cdot (\dot{\lambda} \cdot \ddot{\gamma})), F^-(\dot{\lambda})\} \\ &\leq \max\{F^-(\dot{\lambda} \cdot \ddot{\gamma}), F^-(\dot{\lambda})\}, \end{aligned}$$

$$\begin{aligned} F^+(\ddot{\gamma}) &= F^+(\wp \cdot \ddot{\gamma}) \\ &\geq \min\{F^+(\wp \cdot (\dot{\lambda} \cdot \ddot{\gamma})), F^+(\dot{\lambda})\} \\ &\geq \min\{F^+(\dot{\lambda} \cdot \ddot{\gamma}), F^+(\dot{\lambda})\}. \end{aligned}$$

Hence, $F = (\mathfrak{I}; F^-, F^+)$ is a BFIUPF of $\mathfrak{I}$. $\square$

Example 3.15. Let us examine an IUP-algebra $\mathfrak{I} = \{\wp, 1, 2, 3, 4, 5\}$, characterized by the following table:

Table 6: The Cayley table for Example 3.15

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	1	$\wp$	4	5	2	3
2	2	3	$\wp$	1	5	4
3	4	5	1	$\wp$	3	2
4	3	2	5	4	$\wp$	1
5	5	4	3	2	1	$\wp$

We define a BFS $F = (\mathfrak{I}; F^-, F^+)$ in $\mathfrak{I}$ as follows:

$\mathfrak{I}$	$\wp$	1	2	3	4	5
F^-	-0.9	-0.3	-0.6	-0.3	-0.3	-0.3
F^+	0.8	0.1	0.7	0.1	0.1	0.1

Then F is a BFIUPF of $\mathfrak{I}$, but it is not a BFIUPI of $\mathfrak{I}$. Indeed,

$$F^-(3 \cdot 1) = F^-(5) = -0.3 \not\leq -0.6 = F^-(2) = \max\{F^-(\wp), F^-(2)\} = \max\{F^-(3 \cdot (2 \cdot 1)), F^-(2)\},$$

$$F^+(1 \cdot 5) = F^+(3) = 0.1 \not\geq 0.7 = F^+(2) = \min\{F^+(2), F^+(2)\} = \min\{F^+(1 \cdot (2 \cdot 5)), F^+(2)\}.$$

Theorem 3.16. *Every BFIUPS of $\mathfrak{I}$ is a BFIUPF.*

Proof. Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFIUPS of $\mathfrak{I}$. Then for all $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}, F^-(0) \leq F^-(\dot{\lambda})$ and $F^+(0) \geq F^+(\dot{\lambda})$ and

$$\begin{aligned}
F^-(\dot{\lambda}) &= F^-(\wp \cdot \dot{\lambda}) \\
&= F^-((\ddot{\gamma} \cdot \wp) \cdot (\ddot{\gamma} \cdot \dot{\lambda})) \\
&\leq \max\{F^-(\ddot{\gamma} \cdot \wp), F^-(\ddot{\gamma} \cdot \dot{\lambda})\} \\
&\leq \max\{\max\{F^-(\ddot{\gamma}), F^-(\wp)\}, F^-(\ddot{\gamma} \cdot \dot{\lambda})\} \\
&\leq \max\{F^-(\ddot{\gamma}), F^-(\ddot{\gamma} \cdot \dot{\lambda})\} \\
&= \max\{F^-(\ddot{\gamma} \cdot \dot{\lambda}), F^-(\ddot{\gamma})\},
\end{aligned}$$

$$\begin{aligned}
F^+(\dot{\lambda}) &= F^+(\wp \cdot \dot{\lambda}) \\
&= F^+((\ddot{\gamma} \cdot \wp) \cdot (\ddot{\gamma} \cdot \dot{\lambda})) \\
&\geq \min\{F^+(\ddot{\gamma} \cdot \wp), F^+(\ddot{\gamma} \cdot \dot{\lambda})\} \\
&\geq \min\{\min\{F^+(\ddot{\gamma}), F^+(\wp)\}, F^+(\ddot{\gamma} \cdot \dot{\lambda})\} \\
&\geq \min\{F^+(\ddot{\gamma}), F^+(\ddot{\gamma} \cdot \dot{\lambda})\} \\
&= \min\{F^+(\ddot{\gamma} \cdot \dot{\lambda}), F^+(\ddot{\gamma})\}.
\end{aligned}$$

Hence, $F = (\mathfrak{I}; F^-, F^+)$ is a BFIUPF of $\mathfrak{I}$. $\square$

If $\mathfrak{A}$ is a subset of a nonempty set $\mathfrak{I}$, the functions $\digamma^-_{\mathfrak{A}} : \mathfrak{I} \to [-1, 0]$ and $\digamma^+_{\mathfrak{A}} : \mathfrak{I} \to [0, 1]$ defined as follows:

$$\digamma^-_{\mathfrak{A}}(\dot\lambda) = \begin{cases} -1 & \text{if } \dot\lambda \in \mathfrak{A} \\ 0 & \text{otherwise} \end{cases}$$

$$\digamma^+_{\mathfrak{A}}(\dot\lambda) = \begin{cases} 1 & \text{if } \dot\lambda \in \mathfrak{A} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the BFS $\digamma_{\mathfrak{A}} = (\mathfrak{I}; \digamma^-_{\mathfrak{A}}, \digamma^+_{\mathfrak{A}})$ is defined as the characteristic BFS of $\mathfrak{A}$ in $\mathfrak{I}$.

Lemma 3.17. *Let $\mathfrak{A}$ be a nonempty subset of $\mathfrak{I}$. Then the constant $\wp$ of $\mathfrak{I}$ is in $\mathfrak{A}$ if and only if $\digamma_{\mathfrak{A}}$ satisfies* (23) *and* (24).

Proof. If $\wp \in \mathfrak{A}$, then $\digamma^-_{\mathfrak{A}}(\wp) = -1$. Thus, $\digamma^-_{\mathfrak{A}}(\wp) = -1 \leq \digamma^-_{\mathfrak{A}}(\dot\lambda)$ for all $\dot\lambda \in \mathfrak{I}$, that is, it satisfies (23). Since $\wp \in \mathfrak{A}$, we have $\digamma^+_{\mathfrak{A}}(\wp) = 1$. Thus, $\digamma^+_{\mathfrak{A}}(\wp) = 1 \geq \digamma^+_{\mathfrak{A}}(\dot\lambda)$ for all $\dot\lambda \in \mathfrak{I}$, that is, it satisfies (24).

Conversely, assume that $\digamma^-_{\mathfrak{A}}$ satisfies (23) and (24). Then $\digamma^-_{\mathfrak{A}}(\wp) \leq \digamma^-_{\mathfrak{A}}(\dot\lambda)$ for all $\dot\lambda \in \mathfrak{I}$. Since $\mathfrak{A}$ is a nonempty subset of $\mathfrak{I}$, we let $\dot\lambda \in \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\wp) \leq \digamma^-_{\mathfrak{A}}(\dot\lambda) = -1$, so $\digamma^-_{\mathfrak{A}}(\wp) = -1$. Hence, $\wp \in \mathfrak{A}$. $\square$

Theorem 3.18. *A nonempty subset $\mathfrak{A}$ of $\mathfrak{I}$ is an IUPS of $\mathfrak{I}$ if and only if the characteristic BFS $\digamma_{\mathfrak{A}}$ is a BFIUPS of $\mathfrak{I}$.*

Proof. Assume that $\mathfrak{A}$ is an IUPS of $\mathfrak{I}$. Let $\dot\lambda, \ddot\gamma \in \mathfrak{I}$.

Case 1: Suppose $\dot\lambda, \ddot\gamma \in \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\dot\lambda) = -1$, $\digamma^-_{\mathfrak{A}}(\ddot\gamma) = -1$, $\digamma^+_{\mathfrak{A}}(\dot\lambda) = 1$ and $\digamma^+_{\mathfrak{A}}(\ddot\gamma) = 1$. Since $\mathfrak{A}$ is an IUPS of $\mathfrak{I}$, we have $\dot\lambda \cdot \ddot\gamma \in \mathfrak{A}$. Thus, $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) = -1 \leq -1 = \max\{-1, -1\} = \max\{\digamma^-_{\mathfrak{A}}(\dot\lambda), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\}$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) = 1 \geq 1 = \min\{1, 1\} = \min\{\digamma^+_{\mathfrak{A}}(\dot\lambda), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\}$.

Case 2: Suppose $\dot\lambda \notin \mathfrak{A}$ or $\ddot\gamma \notin \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\dot\lambda) = 0$ or $\digamma^-_{\mathfrak{A}}(\ddot\gamma) = 0$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda) = 0$ or $\digamma^+_{\mathfrak{A}}(\ddot\gamma) = 0$. Thus, $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) \leq 0 = \max\{\digamma^-_{\mathfrak{A}}(\dot\lambda), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\}$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) \geq 0 = \min\{\digamma^+_{\mathfrak{A}}(\dot\lambda), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\}$.
Hence, $\digamma_{\mathfrak{A}}$ is a BFIUPS of $\mathfrak{I}$.

Conversely, assume that $\digamma_{\mathfrak{A}}$ is a BFIUPS of $\mathfrak{I}$. Let $\dot\lambda, \ddot\gamma \in \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\dot\lambda) = -1$, $\digamma^-_{\mathfrak{A}}(\ddot\gamma) = -1$, $\digamma^+_{\mathfrak{A}}(\dot\lambda) = 1$ and $\digamma^+_{\mathfrak{A}}(\ddot\gamma) = 1$. By (21) and (22), we have $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) \leq \max\{\digamma^-_{\mathfrak{A}}(\dot\lambda), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\} = \max\{-1, -1\} = -1$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) \geq \min\{\digamma^+_{\mathfrak{A}}(\dot\lambda), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\} = \min\{1, 1\} = 1$. Thus, $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) = -1$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \ddot\gamma) = 1$, that is, $\dot\lambda \cdot \ddot\gamma \in \mathfrak{A}$. Hence, $\mathfrak{A}$ is an IUPS of $\mathfrak{I}$. $\square$

Theorem 3.19. *A nonempty subset $\mathfrak{A}$ of $\mathfrak{I}$ is an IUPI of $\mathfrak{I}$ if and only if the characteristic BFS $\digamma_{\mathfrak{A}}$ is a BFIUPI of $\mathfrak{I}$.*

Proof. Assume that $\mathfrak{A}$ is an IUPI of $\mathfrak{I}$. Since $\wp \in \mathfrak{A}$, it follows from Lemma 3.17 that $\digamma_{\mathfrak{A}}$ satisfies (23) and (24). Next, let $\dot\lambda, \ddot\gamma, \breve\beta \in \mathfrak{I}$.

Case 1: Suppose $\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta) \in \mathfrak{A}$ and $\ddot\gamma \in \mathfrak{A}$. Then $\dot\lambda \cdot \breve\beta \in \mathfrak{A}$. Thus, $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) = -1$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) = 1$. Hence, $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) = 1 \leq \max\{\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\}$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) = 1 \geq \min\{\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\}$.

Case 2: Suppose $\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta) \notin \mathfrak{A}$ or $\ddot\gamma \notin \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)) = 0$ or $\digamma^-_{\mathfrak{A}}(\ddot\gamma) = 0$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)) = 0$ or $\digamma^+_{\mathfrak{A}}(\ddot\gamma) = 0$. Thus, $\max\{\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\} = 0$ and $\min\{\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\} = 0$. Hence, $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) \leq 0 = \max\{\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\}$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) \geq 0 = \min\{\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\}$.
Therefore, $\digamma_{\mathfrak{A}}$ is a BFIUPI of $\mathfrak{I}$.

Conversely, assume that $\digamma_{\mathfrak{A}}$ is a BFIUPI of $\mathfrak{I}$. Since $\digamma_{\mathfrak{A}}$ satisfies (23) and (24), it follows from Lemma 3.17 that $\wp \in \mathfrak{A}$. Next, let $\dot\lambda, \ddot\gamma, \breve\beta \in \mathfrak{A}$ be such that $\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta) \in \mathfrak{A}$ and $\ddot\gamma \in \mathfrak{A}$. Assume that $\dot\lambda \cdot \breve\beta \notin \mathfrak{A}$. By (27) and (28), we have $0 = \digamma^-_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) \leq \max\{\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\}$ and $0 = \digamma^+_{\mathfrak{A}}(\dot\lambda \cdot \breve\beta) \geq \min\{\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\}$. Thus, $\max\{\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)), \digamma^-_{\mathfrak{A}}(\ddot\gamma)\} = 0$ and $\min\{\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta), \digamma^+_{\mathfrak{A}}(\ddot\gamma)\} = 0$. It means that $\digamma^-_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)) = 0$ or $\digamma^-_{\mathfrak{A}}(\ddot\gamma) = 0$ and $\digamma^+_{\mathfrak{A}}(\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta)) = 0$ or $\digamma^+_{\mathfrak{A}}(\ddot\gamma) = 0$. Thus, $\dot\lambda \cdot (\ddot\gamma \cdot \breve\beta) \notin \mathfrak{A}$ or $\ddot\gamma \notin \mathfrak{A}$, a contradiction. Hence, $\dot\lambda \cdot \breve\beta \in \mathfrak{A}$, so $\mathfrak{A}$ is an IUPI of $\mathfrak{I}$. $\square$

Theorem 3.20. *A nonempty subset $\mathfrak{A}$ of $\mathfrak{I}$ is an IUPF of $\mathfrak{I}$ if and only if the characteristic BFS $\digamma_{\mathfrak{A}}$ is a BFIUPF of $\mathfrak{I}$.*

Proof. Assume that $\mathfrak{A}$ is an IUPF of $\mathfrak{I}$. Since $\wp \in \mathfrak{I}$, it follows from Lemma 3.17 that $\digamma_{\mathfrak{A}}$ satisfies (23) and (24). Next, let $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$.

Case 1: Suppose $\ddot{\gamma} \in \mathfrak{A}$. Thus, $\digamma^-_{\mathfrak{A}}(\ddot{\gamma}) = -1$ and $\digamma^+_{\mathfrak{A}}(\ddot{\gamma}) = 1$. Hence $\digamma^-_{\mathfrak{A}}(\ddot{\gamma}) = -1 \leq \max\{\digamma^-_{\mathfrak{A}}(\dot{\lambda}\cdot\ddot{\gamma}), \digamma^-_{\mathfrak{A}}(\dot{\lambda})\}$ and $\digamma^+_{\mathfrak{A}}(\ddot{\gamma}) = 1 \geq \min\{\digamma^+_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^+_{\mathfrak{A}}(\dot{\lambda})\}$.

Case 2: Suppose $\ddot{\gamma} \notin \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\ddot{\gamma}) = 0$ and $\digamma^+_{\mathfrak{A}}(\ddot{\gamma}) = 0$. Since $\mathfrak{A}$ is an IUPF of $\mathfrak{I}$, we have $\dot{\lambda} \notin \mathfrak{A}$ or $\dot{\lambda} \cdot \ddot{\gamma} \notin \mathfrak{A}$. Thus, $\digamma^-_{\mathfrak{A}}(\dot{\lambda}) = 0$ or $\digamma^-_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}) = 0$ and $\digamma^+_{\mathfrak{A}}(\dot{\lambda}) = 0$ or $\digamma^+_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}) = 0$. Hence, $\digamma^-_{\mathfrak{A}}(\ddot{\gamma}) = 0 \leq 0 = \max\{\digamma^-_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^-_{\mathfrak{A}}(\dot{\lambda})\}$ and $\digamma^+_{\mathfrak{A}}(\ddot{\gamma}) = 0 \geq 0 = \min\{\digamma^+_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^+_{\mathfrak{A}}(\dot{\lambda})\}$.
Therefore, $\digamma_{\mathfrak{A}}$ is a BFIUPF of $\mathfrak{I}$.

Conversely, assume that $\digamma_{\mathfrak{A}}$ is a BFIUPF of $\mathfrak{I}$. Since $\digamma_{\mathfrak{A}}$ satisfies (23) and (24), it follows from Lemma 3.17 that $\wp \in \mathfrak{A}$. Next, let $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{A}$ be such that $\dot{\lambda} \cdot \ddot{\gamma} \in \mathfrak{A}$ and $\dot{\lambda} \in \mathfrak{A}$. Then $\digamma^-_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}) = -1$, $\digamma^-_{\mathfrak{A}}(\dot{\lambda}) = -1$, $\digamma^+_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}) = 1$ and $\digamma^+_{\mathfrak{A}}(\dot{\lambda}) = 1$. Assume that $\ddot{\gamma} \notin \mathfrak{A}$. By (25) and (26), we have $0 = \digamma^-_{\mathfrak{A}}(\ddot{\gamma}) \leq \max\{\digamma^-_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^-_{\mathfrak{A}}(\dot{\lambda})\} = \max\{-1, -1\} = -1$ and $0 = \digamma^+_{\mathfrak{A}}(\ddot{\gamma}) \geq \min\{\digamma^+_{\mathfrak{A}}(\dot{\lambda} \cdot \ddot{\gamma}), \digamma^+_{\mathfrak{A}}(\dot{\lambda})\} = \min\{1, 1\} = 1$, a contradiction. Hence, $\ddot{\gamma} \in \mathfrak{A}$, so $\mathfrak{A}$ is an IUPF of $\mathfrak{I}$. □

Theorem 3.21. *A nonempty subset $\mathfrak{A}$ of $\mathfrak{I}$ is an SIUPI of $\mathfrak{I}$ if and only if the characteristic BFS $\digamma_{\mathfrak{A}}$ is a BFSIUPI of $\mathfrak{I}$.*

Proof. This result can be directly inferred from Theorem 3.9, providing a clear and concise validation. □

Example 3.22. Let $\mathbb{R}^*$ denote the set of all nonzero real numbers. The structure $(\mathbb{R}^*; \cdot, 1)$ forms an IUP-algebra, where the binary operation $\cdot$ on $\mathbb{R}^*$ is defined as $\dot{\lambda} \cdot \ddot{\gamma} = \frac{\ddot{\gamma}}{\dot{\lambda}}$ for all $\dot{\lambda}, \ddot{\gamma} \in \mathbb{R}^*$. Consider the subset $\mathfrak{S} = \{\dot{\lambda} \in \mathbb{R}^* \mid \dot{\lambda} \geq 1\}$. This subset qualifies as both an IUPI and an IUPF of $\mathbb{R}^*$, but it does not satisfy the criteria for being an IUPS.

Furthermore, by Theorems 3.18, 3.19, and 3.20, the characteristic BFS $\digamma_{\mathfrak{S}} = (\mathbb{R}^*; \digamma^-_{\mathfrak{S}}, \digamma^+_{\mathfrak{S}})$ is classified as both a BFIUPI and a BFIUPF of $\mathbb{R}^*$, yet it does not meet the requirements to be a BFIUPS.

Example 3.23. Consider an IUP-algebra $\mathfrak{I} = \{\wp, 1, 2, 3, 4, 5\}$ with the Cayley table:

Table 7: The Cayley table for Example 3.23

$\cdot$	$\wp$	1	2	3	4	5
$\wp$	$\wp$	1	2	3	4	5
1	4	$\wp$	5	2	1	3
2	2	5	$\wp$	4	3	1
3	3	2	1	$\wp$	5	4
4	1	4	3	5	$\wp$	2
5	5	3	4	1	2	$\wp$

We define a BFS $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ in $\mathfrak{I}$ as follows:

$\mathfrak{I}$	$\wp$	1	2	3	4	5
$\digamma^-$	-0.8	-0.2	-0.2	-0.2	-0.2	-0.4
$\digamma^+$	0.9	0.5	0.5	0.5	0.5	0.7

Then $\digamma$ is a BFIUPS of $\mathfrak{I}$, but it is not a BFIUPI of $\mathfrak{I}$. Indeed,

$$\digamma^-(3 \cdot 1) = \digamma^-(2) = -0.2 \not\leq -0.4 = \digamma^-(5) = \max\{\digamma^-(\wp), \digamma^-(5)\} = \max\{\digamma^-(3 \cdot (5 \cdot 1)), \digamma^-(5)\},$$

$$\digamma^+(4 \cdot 1) = \digamma^+(4) = 0.5 \not\geq 0.7 = \digamma^+(5) = \min\{\digamma^+(5), \digamma^+(5)\} = \min\{\digamma^+(4 \cdot (5 \cdot 1)), \digamma^+(5)\}.$$

4 Level Cuts of a BFS

Level cuts serve as a fundamental analytical tool in the study of BFSs, providing a structured approach to examining their inherent properties. By partitioning a BFS into subsets defined by specific membership thresholds, level cuts allow for a more granular investigation of both the positive and negative membership functions. These cuts, typically categorized as negative lower cuts and positive upper cuts, capture critical information about the satisfaction and dissatisfaction levels within a given BFS.

In the context of bipolar fuzzy IUP-algebras, level cuts play a pivotal role in characterizing and analyzing various types of BFSs, such as subalgebras, filters, and ideals. By connecting these subsets to their respective level cuts, it becomes possible to derive deeper insights into their structural behavior and algebraic relationships. This section introduces the formal definitions of level cuts, establishes their connection to different types of bipolar fuzzy subsets, and illustrates their utility through rigorous analysis and examples.

Definition 4.1. Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFS in $\mathfrak{I}$. For $(\sqcup^-, \sqcup^+) \in [-1, 0] \times [0, 1]$, the sets

$$\mathcal{L}_{\text{neg}}(F, \sqcup^-) = \{\dot{\lambda} \in \mathfrak{I} \mid F^-(\dot{\lambda}) \leq \sqcup^-\},$$

$$\mathcal{U}_{\text{pos}}(F, \sqcup^+) = \{\dot{\lambda} \in \mathfrak{I} \mid F^+(\dot{\lambda}) \geq \sqcup^+\}.$$

are called the negative lower $\sqcup^-$-cut and the positive upper $\sqcup^+$-cut of $F = (\mathfrak{I}; F^-, F^+)$, respectively. The set

$$\mathcal{C}(F; (\sqcup^-, \sqcup^+)) = \mathcal{L}_L(F, \sqcup^-) \cap \mathcal{U}_{\text{pos}}(F, \sqcup^+).$$

is called the $(\sqcup^-, \sqcup^+)$-cut of $F = (\mathfrak{I}; F^-, F^+)$. For any $\nabla^+ \in [0, 1]$, we denote the set

$$\mathcal{C}(F; \nabla^+) = \mathcal{C}(F; (-\nabla^+, \nabla^+)) = \mathcal{L}_L(F, -\nabla^+) \cap \mathcal{U}_{\text{pos}}(F, \nabla^+).$$

is called the ∇^+-cut of $F = (\mathfrak{I}; F^-, F^+)$.

Example 4.2. From Example 3.15, we have $\mathcal{L}_{\text{neg}}(F; -0.5) = \{\wp, 2\}, \mathcal{U}_{\text{pos}}(F; 0.1) = \{\wp, 1, 2, 3, 4, 5\}$, and $\mathcal{C}(F; (-0.5, 0.1)) = \{\wp, 2\}$.

Theorem 4.3. *Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFS in $\mathfrak{I}$. Then $F = (\mathfrak{I}; F^-, F^+)$ is a BFIUPS of $\mathfrak{I}$ if and only if the following statements are vaild:*

(1) *for all $\sqcup^- \in [-1, 0], \mathcal{L}_{\text{neg}}(F; \sqcup^-)$ is an IUPS of $\mathfrak{I}$ if $\mathcal{L}_{\text{neg}}(F; \sqcup^-)$ is nonempty, and*

(2) *for all $\sqcup^+ \in [0, 1], \mathcal{U}_{\text{pos}}(F; \sqcup^+)$ is an IUPS of $\mathfrak{I}$ if $\mathcal{U}_{\text{pos}}(F; \sqcup^+)$ is nonempty.*

Proof. Assume that F is a BFIUPS of $\mathfrak{I}$. Let $\sqcup^- \in [-1, 0]$ be such that $\mathcal{L}_{\text{neg}}(F; \sqcup^-) \neq \emptyset$ and $\dot{\lambda}, \ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F; \sqcup^-)$. Then $F^-(\ddot{\gamma}) \leq \sqcup^-$ and $F^-(\dot{\lambda}) \leq \sqcup^-$. Since F is a BFIUPS of $\mathfrak{I}$, we have $F^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq \max\{F^-(\dot{\lambda}), F^-(\ddot{\gamma})\} \leq \sqcup^-$. Thus, $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F; \sqcup^-)$. Hence, $\mathcal{L}_{\text{neg}}(F; \sqcup^-)$ is an IUPS of $\mathfrak{I}$. Next, let $\sqcup^+ \in [0, 1]$ be such that $\mathcal{U}_{\text{pos}}(F; \sqcup^+) \neq \emptyset$ and let $\dot{\lambda}, \ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F; \sqcup^+)$. Then $F^+(\ddot{\gamma}) \geq \sqcup^+$ and $F^+(\dot{\lambda}) \geq \sqcup^+$. Since F is a BFIUPS of $\mathfrak{I}$, we have $F^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq \min\{F^+(\dot{\lambda}), F^+(\ddot{\gamma})\} \geq \sqcup^+$. Thus, $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F; \sqcup^+)$. Hence, $\mathcal{U}_{\text{pos}}(F; \sqcup^+)$ is an IUPS of $\mathfrak{I}$.

Conversely, assume that for all $\sqcup^- \in [-1, 0], \mathcal{L}_{\text{neg}}(F; \sqcup^-)$ is an IUPS of $\mathfrak{I}$ if $\mathcal{L}_{\text{neg}}(F, \sqcup^-)$ is nonempty and for all $\sqcup^+ \in [0, 1], \mathcal{U}_{\text{pos}}(F; \sqcup^+)$ is an IUPS of $\mathfrak{I}$ if $\mathcal{U}_{\text{pos}}(F; t^+)$ is nonempty. Let $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$. Then $F^-(\dot{\lambda}), F^-(\ddot{\gamma}) \in [-1, 0]$. Choose $\sqcup^- = \max\{F^-(\dot{\lambda}), F^-(\ddot{\gamma})\}$. Then $F^-(\dot{\lambda}) \leq \sqcup^-$ and $F^-(\ddot{\gamma}) \leq \sqcup^-$, that is, $\dot{\lambda}, \ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F; \sqcup^-) \neq \emptyset$. By hypothesis, we get $\mathcal{L}_{\text{neg}}(F; \sqcup^-)$ is an IUPS of $\mathfrak{I}$. So, $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F; \sqcup^-)$. Hence, $F^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq \sqcup^- = \max\{F^-(\dot{\lambda}), F^-(\ddot{\gamma})\}$. Let $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$. Then $F^+(\dot{\lambda}), F^+(\ddot{\gamma}) \in [0, 1]$. Choose $\sqcup^+ = \max\{F^+(\dot{\lambda}), F^+(\ddot{\gamma})\}$. Then $F^+(\dot{\lambda}) \geq \sqcup^+$ and $F^+(\ddot{\gamma}) \geq \sqcup^+$, that is, $\dot{\lambda}, \ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F; \sqcup^+) \neq \emptyset$. By hypothesis, we get $\mathcal{U}_{\text{pos}}(F; \sqcup^+)$ is an IUPS of $\mathfrak{I}$. So, $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F; \sqcup^+)$. Hence, $F^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq \sqcup^+ = \min\{F^+(\dot{\lambda}), F^+(\ddot{\gamma})\}$. Therefore, $F = (\mathfrak{I}; F^+, F^-)$ is a BFIUPS of $\mathfrak{I}$. $\square$

Corollary 4.4. *If $F = (\Im; F^-, F^+)$ is a BFIUPS of $\Im$, then for all $\nabla^+ \in [0,1], \mathcal{C}(F;\nabla^+)$ is an IUPS of $\Im$ while $\mathcal{C}(F;\nabla^+)$ is nonempty.*

Proof. This conclusion follows seamlessly from Theorems 2.3 and 4.3, reinforcing its logical clarity. □

Theorem 4.5. *Let $F = (\Im; F^-, F^+)$ be a BFS in $\Im$. Then $F = (\Im; F^-, F^+)$ is BFIUPF of $\Im$ if and only if the following statements are vaild:*

(1) *for all $\sqcup^- \in [-1,0], \mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is an IUPF of $\Im$ if $\mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is nonempty, and*

(2) *for all $\sqcup^+ \in [0,1], \mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is an IUPF of $\Im$ if $\mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is nonempty.*

Proof. Assume that F is a BFIUPF of $\Im$. Let $\sqcup^- \in [-1,0]$ be such that $\mathcal{L}_{\text{neg}}(F;\sqcup^-) \neq \emptyset$ and let $\tilde{\delta} \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Then $F^-(\tilde{\delta}) \leq \sqcup^-$. Since F is a BFIUPF of $\Im$, we have $F^-(\wp) \leq F^-(\tilde{\delta}) \leq \sqcup^-$. Thus, $\wp \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Next, let $\dot{\lambda}, \ddot{\gamma} \in \Im$ be such that $\dot{\lambda}\cdot\ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Then $F^-(\dot{\lambda}\cdot\ddot{\gamma})$ and $F^-(\dot{\lambda}) \leq \sqcup^-$. Since F is a BFIUPF of $\Im$, we have $F^-(\ddot{\gamma}) \leq \max\{F^-(\dot{\lambda}\cdot\ddot{\gamma}), F^-(\dot{\lambda})\} \leq \sqcup^-$. So, $\ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F,\sqcup^-)$. Hence, $\mathcal{L}_{\text{neg}}$ is an IUPF of $\Im$. Let $\sqcup^+ \in [0,1]$ be such that $\mathcal{U}_{\text{pos}}(F;\sqcup^+) \neq \emptyset$ and let $\tilde{\delta} \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$. Then $F^+(\tilde{\delta}) \geq t^+$. Since F is a BFIUPF of $\Im$, we have $F^+(\wp) \geq F^+(\tilde{\delta}) \geq \sqcup^+$. Thus, $\wp \in \mathcal{U}_{\text{pos}}(F,\sqcup^+)$. Next, let $\dot{\lambda}, \ddot{\gamma} \in \Im$ be such that $\dot{\lambda}\cdot\ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F,\sqcup^+)$. Then $F^+(\dot{\lambda}\cdot\ddot{\gamma}) \geq \sqcup^+$ and $F^+(\dot{\lambda}) \geq \sqcup^+$. Since F is a BFIUPF of $\Im$, we have $F^+(\ddot{\gamma}) \geq \min\{F^+(\dot{\lambda}\cdot\ddot{\gamma}), F^+(\dot{\lambda})\} \geq t^+$. So, $\ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$. Hence, $\mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is an IUPF of $\Im$.

Conversely, assume that for all $\sqcup^- \in [-1,0], \mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is an IUPF of $\Im$ if $\mathcal{L}_{\text{neg}}(F,\sqcup^-)$ is nonempty and for all $\sqcup^+ \in [0,1], \mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is an IUPF of $\Im$ if $\mathcal{U}_{\text{pos}}(F;t^+)$ is nonempty. Let $\dot{\lambda} \in \Im$. Then $F^-(\dot{\lambda}) \in [-1,0]$. Choose $t^- = F^-(\dot{\lambda})$. Then $F^-(\dot{\lambda}) \leq \sqcup^-$, that is, $\dot{\lambda} \in \mathcal{L}_{\text{neg}}(F;\sqcup^-) \neq \emptyset$. By hypothesis, we get $\mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is an IUPF of $\Im$. So, $\wp \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Hence, $F^-(\wp) \neq \sqcup^- = F^-(\dot{\lambda})$. Next, let $\dot{\lambda}, \ddot{\gamma} \in \Im$. Then $F^-(\dot{\lambda}\cdot\ddot{\gamma}), F^-(\dot{\lambda}) \in [-1,0]$. Choose $\sqcup^- = \max\{F^-(\dot{\lambda}\cdot\ddot{\gamma}), F^-(\dot{\lambda})\}$. Then $F^-(\dot{\lambda}\cdot\ddot{\gamma}) \leq \sqcup^-$ and $F^-(\dot{\lambda}) \leq t^-$, that is, $\dot{\lambda}\cdot\ddot{\gamma}, \dot{\lambda} \in \mathcal{L}_{\text{neg}}(F;t^-) \neq \emptyset$. By hypothesis, we get $\mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is an IUPF of $\Im$. So, $\ddot{\gamma} \in \mathcal{L}_{\text{neg}}$. Hence, $F^-(\ddot{\gamma}) \leq \sqcup^- = \max\{F^-(\dot{\lambda}), F^-(\dot{\lambda}\cdot\ddot{\gamma})\}$. Let $\dot{\lambda} \in \Im$. Then $F^+(\dot{\lambda}) \in [0,1]$. Choose $\sqcup^+ = F^+(\dot{\lambda})$. Then $F^+(\dot{\lambda}) \geq \sqcup^+$, that is, $\dot{\lambda} \in \mathcal{U}_{\text{pos}}(F;\sqcup^+) \neq \emptyset$. By hypothesis, we get $\mathcal{U}_{\text{pos}}(F;t^+)$ is an IUPF of $\Im$. So, $\wp \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$. Hence, $F^+(\wp) \geq \sqcup^+ = F^+(\dot{\lambda})$. Next, let $\dot{\lambda}, \ddot{\gamma} \in \Im$. Then $F^+(\dot{\lambda}\cdot\ddot{\gamma}), F^+(\dot{\lambda}) \in [0,1]$. Choose $\sqcup^+ = \min\{F^+(\dot{\lambda}\cdot\ddot{\gamma}), F^+(\dot{\lambda})\}$. Then $F^+(\dot{\lambda}\cdot\ddot{\gamma}) \geq \sqcup^+$ and $F^+(\dot{\lambda}) \geq \sqcup^+$, that is, $\dot{\lambda}\cdot\ddot{\gamma}, x \in \mathcal{U}_{\text{pos}}(F,\sqcup^+) \neq \emptyset$. By hypothesis, we get $\mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is an IUPF of $\Im$. So, $\ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$. Hence, $F^+(\ddot{\gamma}) \geq \sqcup^+ = \min\{F^+(\dot{\lambda}), F^+(\dot{\lambda}\cdot\ddot{\gamma})$. Therefore, F is a BFIUPF of $\Im$. □

Corollary 4.6. *If $F = (\Im; F^-, F^+)$ is a BFIUPF of $\Im$, then for all $\nabla^+ \in [0,1], \mathcal{C}(F;\nabla^+)$ is an IUPF of $\Im$ while $\mathcal{C}(F;\nabla^+)$ is nonempty.*

Proof. The conclusion derives naturally and clearly from Theorems 2.3 and 4.5. □

Theorem 4.7. *Let $F = (\Im; F^-, F^+)$ be a BFS in $\Im$. Then $F = (\Im; F^-, F^+)$ is BFIUPI of $\Im$ if and only if the following statements are vaild:*

(1) *for all $\sqcup^- \in [-1,0], \mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is an IUPI of $\Im$ if $\mathcal{L}_{\text{neg}}(F;\sqcup^-)$ is nonempty, and*

(2) *for all $\sqcup^+ \in [0,1], \mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is an IUPI of $\Im$ if $\mathcal{U}_{\text{pos}}(F;\sqcup^+)$ is nonempty.*

Proof. Assume that F is a BFIUPI of $\Im$. Let $\sqcup^- \in [-1,0]$ be such that $\mathcal{L}_{\text{neg}}(F;\sqcup^-) \neq \emptyset$ and let $\tilde{\delta} \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Then $F^-(\tilde{\delta}) \leq \sqcup^-$. Since F is a BFIUPI of $\Im$, we have $F^-(\wp) \leq F^-(\tilde{\delta}) \leq \sqcup^-$. Thus, $\wp \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Next, let $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \Im$ be such that $\dot{\lambda}\cdot(\ddot{\gamma}\cdot\breve{\beta}) \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$ and $\ddot{\gamma} \in \mathcal{L}_{\text{neg}}(F;\sqcup^-)$. Then $F^-(\dot{\lambda}\cdot(\ddot{\gamma}\cdot\breve{\beta}))$ and $F^-(\ddot{\gamma}) \leq \sqcup^-$. Since F is a BFIUPI of $\Im$, we have $F^-(\dot{\lambda}\cdot\breve{\beta}) \leq \max\{F^-(\dot{\lambda}\cdot(\ddot{\gamma}\cdot\breve{\beta})), F^-(\ddot{\gamma})\} \leq \sqcup^-$. So, $\dot{\lambda}\cdot\breve{\beta} \in \mathcal{L}_{\text{neg}}(F;t^-)$. Hence, $\mathcal{L}_{\text{neg}}(F,t^-)$ is an IUPI of $\Im$. Let $t^+ \in [0,1]$ be such that $\mathcal{U}_{\text{pos}}(F;\sqcup^+) \neq \emptyset$ and let $\tilde{\delta} \in \mathcal{U}_{\text{pos}}$. Then $F^+(\tilde{\delta}) \geq t^+$. Since F is a BFIUPI of $\Im$, we have $F^+(\wp) \geq F^+(\tilde{\delta}) \geq t^+$. Thus, $\wp \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$. Next, let $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \Im$ be such that $\dot{\lambda}\cdot(\ddot{\gamma}\cdot\breve{\beta}) \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$ and $\ddot{\gamma} \in \mathcal{U}_{\text{pos}}(F;\sqcup^+)$. Then $F^+(\dot{\lambda}\cdot(\ddot{\gamma}\cdot\breve{\beta})) \geq \sqcup^+$

and $\digamma^+(\ddot{\gamma}) \geq \sqcup^+$. Since $\digamma$ is a BFIUPI of $\mathfrak{I}$, we have $\digamma^+(\dot{\lambda} \cdot \breve{\beta}) \geq \min\{\digamma^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), \digamma^+(\ddot{\gamma})\} \geq \sqcup^+$. So, $\dot{\lambda} \cdot \breve{\beta} \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$. Hence, $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an IUPI of $\mathfrak{I}$.

Conversely, assume that for all $\sqcup^- \in [-1, 0], \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an IUPI of $\mathfrak{I}$ if $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is nonempty and for all $\sqcup^+ \in [0, 1], \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an IUPI of $\mathfrak{I}$ if $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is nonempty. Let $\dot{\lambda} \in \mathfrak{I}$. Then $\digamma^-(\dot{\lambda}) \in [-1, 0]$. Choose $\sqcup^- = \digamma^-(\dot{\lambda})$. Then $\digamma^-(\dot{\lambda}) \leq \sqcup^-$, that is, $\dot{\lambda} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-) \neq \emptyset$. By hypothesis, we get $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an IUPI of $\mathfrak{I}$. So, $\wp \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$. Hence, $\digamma^-(\wp) \leq \sqcup^- = \digamma^-(\dot{\lambda})$. Next, let $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I}$. Then $\digamma^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), \digamma^-(\ddot{\gamma}) \in [-1, 0]$. Choose $\sqcup^- = \max\{\digamma^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), \digamma^-(\ddot{\gamma})\}$. Then $\digamma^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta}))$ and $\digamma^-(\ddot{\gamma}) \leq t^-$, that is, $\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta}), \ddot{\gamma} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-) \neq \emptyset$. By hypothesis, we get $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an IUPI of $\mathfrak{I}$. So, $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$. Hence, $\digamma^-(\dot{\lambda} \cdot \breve{\beta}) \leq t^- = \max\{\digamma^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta}), \digamma^-(\ddot{\gamma})\}$. Let $\dot{\lambda} \in \mathfrak{I}$. Then $\digamma^+(\dot{\lambda}) \in [0, 1]$. Choose $\sqcup^+ = \digamma^+(\dot{\lambda})$. Then $\digamma^+(\dot{\lambda}) \geq \sqcup^+$, that is, $\dot{\lambda} \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+) \neq \emptyset$. By hypothesis, we get $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an IUPI of $\mathfrak{I}$. So, $\wp \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$. Hence, $\digamma^+(\wp) \geq \sqcup^+ = \digamma^+(\dot{\lambda})$. Next, let $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I}$. Then $\digamma^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), \digamma^+(\ddot{\gamma}) \in [0, 1]$. Choose $\sqcup^+ = \min\{\digamma^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), \digamma^+(\ddot{\gamma})\}$. Then $\digamma^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta}))$ and $\digamma^+(\ddot{\gamma}) \geq \sqcup^+$, that is, $\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta}), \ddot{\gamma} \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+) \neq \emptyset$. By hypothesis, we get $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an IUPI of $\mathfrak{I}$. So, $\dot{\lambda} \cdot \breve{\beta} \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$. Hence, $\digamma^+(\dot{\lambda} \cdot \breve{\beta}) \geq \sqcup^+ = \min\{\digamma^+(\dot{\lambda} \cdot \breve{\beta}), \digamma^+(\ddot{\gamma})\}$. Therefore, $\digamma$ is a BFIUPI of $\mathfrak{I}$. □

Corollary 4.8. *If $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ is a BFIUPI of $\mathfrak{I}$, then for all $\nabla^+ \in [0, 1], \mathcal{C}(\digamma; \nabla^+)$ is an IUPI of $\mathfrak{I}$ while $\mathcal{C}(\digamma; \nabla^+)$ is nonempty.*

Proof. This result follows directly and intuitively from Theorems 2.3 and 4.7. □

Theorem 4.9. *Let $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ be a BFS in $\mathfrak{I}$. Then $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ is a BFSIUPI of $\mathfrak{I}$ if and only if the following statements are vaild:*

(1) *for all $\sqcup^- \in [-1, 0], \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an SIUPI of $\mathfrak{I}$ if $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is nonempty, and*

(2) *for all $\sqcup^+ \in [0, 1], \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an SIUPI of $\mathfrak{I}$ if $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is nonempty.*

Proof. Assume that $\digamma$ is a BFSIUPI of $\mathfrak{I}$. Let $\sqcup^- \in [-1, 0]$ be such that $\mathcal{L}_{\mathrm{neg}} \neq \emptyset$ let $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$ be such that $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ and $\ddot{\gamma} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$. Then $\digamma^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq \sqcup^-$ and $\digamma^-(\ddot{\gamma}) \leq \sqcup^-$. Since $\digamma$ is a BFSIUPI of $\mathfrak{I}$, we have $\digamma^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq \digamma^-(\ddot{\gamma}) \leq t^-$. Hence, $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an SIUPI of $\mathfrak{I}$. Let $\sqcup^+ \in [0, 1]$ be such that $\mathcal{U}_{\mathrm{pos}} \neq \emptyset$ and let, $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$ be such that $\dot{\lambda} \cdot \ddot{\gamma} \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ and $\ddot{\gamma} \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$. Then $\digamma^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq \sqcup^+$ and $\digamma^+(\ddot{\gamma}) \geq \sqcup^+$. Since $\digamma$ is a BFSIUPI of $\mathfrak{I}$, we have $\digamma^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq \digamma^+(\ddot{\gamma}) \geq t^+$. Hence, $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an SIUPI of $\mathfrak{I}$.

Conversely, assume that for all $\sqcup^- \in [-1, 0], \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an SIUPI of $\mathfrak{I}$ if $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is nonempty and for all $\sqcup^+ \in [0, 1], \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an SIUPI of $\mathfrak{I}$ if $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is nonempty. Let $\ddot{\gamma} \in \mathfrak{I}$. Then $\digamma^-(\ddot{\gamma}) \in [-1, 0]$. Choose $\sqcup^- = \digamma^-(\ddot{\gamma})$. Then $\digamma^-(\ddot{\gamma}) \leq \sqcup^-$, that is, $\ddot{\gamma} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-) \neq \emptyset$. By hypothesis, we get $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an SIUPI of $\mathfrak{I}$. Since $\mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$ is an SIUPI of $\mathfrak{I}$ and $(\ddot{\gamma}) \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^-)$, we have $\digamma^-(\dot{\lambda} \cdot \ddot{\gamma}) \leq \sqcup^- = \digamma^-(\ddot{\gamma})$. Let $\ddot{\gamma} \in \mathfrak{I}$. Then $\digamma^+(\ddot{\gamma}) \in [0, 1]$. Choose $\sqcup^+ = \digamma^+(\ddot{\gamma})$. Then $\digamma^+(\ddot{\gamma}) \geq \sqcup^+$, that is, $\ddot{\gamma} \in \mathcal{L}_{\mathrm{neg}}(\digamma; \sqcup^+) \neq \emptyset$. By hypothesis, we get $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an SIUPI of $\mathfrak{I}$. Since $\mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$ is an SIUPI of $\mathfrak{I}$ and $(\ddot{\gamma}) \in \mathcal{U}_{\mathrm{pos}}(\digamma; \sqcup^+)$, we have $\digamma^+(\dot{\lambda} \cdot \ddot{\gamma}) \geq \sqcup^+ = \digamma^+(\ddot{\gamma})$. Therefore, $\digamma$ is a BFSIUPI of $\mathfrak{I}$. □

Corollary 4.10. *Let $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ be a BFS in $\mathfrak{I}$. Then $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ is a BFSIUPI of $\mathfrak{I}$ if and only if for all $\nabla^+ \in [0, 1], \mathcal{C}(\digamma; \nabla^+)$ is an SIUPI of $\mathfrak{I}$ while $\mathcal{C}(\digamma; \nabla^+)$ is nonempty.*

Proof. The conclusion follows directly from Theorems 2.3, 3.9, and 4.9, establishing that $\mathfrak{I}$ is uniquely identified as the sole SIUPI of itself. □

Theorem 4.11. *Let $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ be a BFIUPF of $\mathfrak{I}$ satisfies the following assertion:*

$$\digamma^-(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \breve{\beta})) \leq \digamma^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})) \text{ and } \digamma^+(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \breve{\beta})) \geq \digamma^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})) \text{ for all } \dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I}.$$

Then $\digamma = (\mathfrak{I}; \digamma^-, \digamma^+)$ is a BFIUPI of $\mathfrak{I}$.

Proof. For all $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \Im$,

$$F^{-}(\wp) \leq F^{-}(\dot{\lambda}),$$

$$F^{-}(\dot{\lambda} \cdot \breve{\beta}) \leq \max\{F^{-}(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \breve{\beta})), F^{-}(\ddot{\gamma})\} \leq \max\{F^{-}(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), F^{-}(\ddot{\gamma})\},$$

$$F^{+}(\wp) \geq F^{+}(\dot{\lambda}),$$

$$F^{+}(\dot{\lambda} \cdot \breve{\beta}) \geq \min\{F^{+}(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \breve{\beta})), F^{+}(\ddot{\gamma})\} \geq \min\{F^{+}(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), F^{+}(\ddot{\gamma})\}.$$

Hence, $F = (\Im; F^{-}, F^{+})$ is a BFIUPI of $\Im$. □

Theorem 4.12. *Let $F = (\Im; F^{-}, F^{+})$ be a BFIUPF of $\Im$ satisfies the following assertion:*

$F^{-}(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \breve{\beta})) \leq F^{-}(\dot{\lambda}), F^{-}(\ddot{\gamma}) \leq F^{-}(\breve{\beta})$ and $F^{+}(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \breve{\beta})) \geq F^{+}(\dot{\lambda}), F^{+}(\ddot{\gamma}) \geq F^{+}(\breve{\beta})$ for all $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \Im$.

Then $F = (\Im; F^{-}, F^{+})$ is a BFIUPS of $\Im$.

Proof. For all $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \Im$,

$$F^{-}(\wp) \leq F^{-}(\dot{\lambda}),$$

$$F^{-}(\dot{\lambda} \cdot \breve{\beta}) \leq \max\{F^{-}(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \ddot{\gamma})), F^{-}(\ddot{\gamma})\} \leq \max\{F^{-}(\dot{\lambda}), F^{-}(\breve{\beta})\},$$

$$F^{-}(\wp) \geq F^{-}(\dot{\lambda}),$$

$$F^{+}(\dot{\lambda} \cdot \breve{\beta}) \geq \min\{F^{+}(\ddot{\gamma} \cdot (\dot{\lambda} \cdot \ddot{\gamma})), F^{+}(\ddot{\gamma})\} \geq \min\{F^{+}(\dot{\lambda}), F^{+}(\breve{\beta})\}.$$

Hence, $F = (\Im; F^{-}, F^{+})$ is a BFIUPS of $\Im$. □

Definition 4.13. Let $F = (\Im; F^{-}, F^{+})$ be a BFS $\Im$. We define the subset $F^{-}(\wp, \wp)$ of $\Im$ by

$$F^{-1}(\wp, \wp) = \{\dot{\lambda} \in \Im \mid F^{-}(\dot{\lambda}) = F^{-}(\wp) \text{ and } F^{+}(\dot{\lambda}) = F^{+}(\wp)\}. \tag{31}$$

Theorem 4.14. *Let $F = (\Im; F^{-}, F^{+})$ be a BFIUPS of $\Im$. Then $F^{-1}(\wp, \wp)$ is an IUPS of $\Im$.*

Proof. Clearly, $\wp \in F^{-1}(\wp, \wp)$. Let $\dot{\lambda}, \ddot{\gamma} \in F^{-1}(\wp, \wp)$. Then $F^{-}(\dot{\lambda}) = F^{-}(\wp), F^{+}(\dot{\lambda}) = F^{+}(\wp), F^{-}(\ddot{\gamma}) = F^{-}(\wp)$, and $F^{+}(\ddot{\gamma}) = F^{+}(\wp)$. Thus,

$$\begin{aligned} F^{-}(\wp) &\leq F^{-}(\dot{\lambda} \cdot \ddot{\gamma}) \\ &\leq \max\{F^{-}(\dot{\lambda}), F^{-}(\ddot{\gamma})\} \\ &= \max\{F^{-}(\wp), F^{-}(\wp)\} \\ &= F^{-}(\wp), \end{aligned}$$

$$\begin{aligned} F^{+}(\wp) &\geq F^{+}(\dot{\lambda} \cdot \ddot{\gamma}) \\ &\geq \min\{F^{+}(\dot{\lambda}), F^{+}(\ddot{\gamma})\} \\ &= \min\{F^{+}(\wp), F^{-}(\wp)\} \\ &= F^{+}(\wp). \end{aligned}$$

So, $F^{-}(\dot{\lambda} \cdot \ddot{\gamma}) = F^{-}(\wp)$ and $F^{+}(\dot{\lambda} \cdot \ddot{\gamma}) = F^{+}(\wp)$, that is, $\dot{\lambda} \cdot \ddot{\gamma} \in F^{-1}(\wp, \wp)$. Therefore, $F^{-1}(\wp, \wp)$ is an IUPS of $\Im$. □

Theorem 4.15. *Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFIUPF of $\mathfrak{I}$. Then $F^{-1}(\wp, \wp)$ is an IUPF of $\mathfrak{I}$.*

Proof. Clearly, $\wp \in F^{-1}(\wp, \wp)$. Let $\dot{\lambda}, \ddot{\gamma} \in \mathfrak{I}$ be such that $\dot{\lambda} \cdot \ddot{\gamma} \in F^{-1}(\wp, \wp)$ and $\dot{\lambda} \in F^{-1}(\wp, \wp)$. Then $F^-(\dot{\lambda}) = F^-(\wp), F^+(\dot{\lambda}) = F^+(\wp), F^-(\dot{\lambda} \cdot \ddot{\gamma}) = F^-(\wp)$, and $F^+(\dot{\lambda} \cdot \ddot{\gamma}) = F^+(\wp)$. Thus,

$$\begin{aligned} F^-(\wp) &\leq F^-(\ddot{\gamma}) \\ &\leq \max\{F^-(\dot{\lambda} \cdot \ddot{\gamma}), F^-(\dot{\lambda})\} \\ &= \max\{F^-(\wp), F^-(\wp)\} \\ &= F^-(\wp), \end{aligned}$$

$$\begin{aligned} F^+(\wp) &\geq F^+(\ddot{\gamma}) \\ &\geq \min\{F^+(\dot{\lambda} \cdot \ddot{\gamma}), F^+(\dot{\lambda})\} \\ &= \min\{F^+(\wp), F^+(\wp)\} \\ &= F^+(\wp). \end{aligned}$$

So, $F^-(\ddot{\gamma}) = F^-(\wp)$ and $F^+(\ddot{\gamma}) = F^+(\wp)$, that is, $\ddot{\gamma} \in F^{-1}(\wp, \wp)$. Therefore, $F^{-1}(\wp, \wp)$ is an IUPF of $\mathfrak{I}$. □

Theorem 4.16. *Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFIUPI of $\mathfrak{I}$. Then $F^{-1}(\wp, \wp)$ is an IUPI of $\mathfrak{I}$.*

Proof. Clearly, $\wp \in F^{-1}(\wp, \wp)$. Let $\dot{\lambda}, \ddot{\gamma}, \breve{\beta} \in \mathfrak{I}$ be such that $\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta}) \in F^{-1}(\wp, \wp)$ and $\ddot{\gamma} \in F^{-1}(\wp, \wp)$. Then $F^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})) = F^-(\wp), F^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})) = F^+(\wp), F^-(\ddot{\gamma}) = F^-(\wp)$, and $F^+(\ddot{\gamma}) = F^+(\wp)$. Thus,

$$\begin{aligned} F^-(\wp) &\leq F^-(\dot{\lambda} \cdot \breve{\beta}) \\ &\leq \max\{F^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), F^-(\ddot{\gamma})\} \\ &= \max\{F^-(\wp), F^-(\wp)\} \\ &= F^-(\wp), \end{aligned}$$

$$\begin{aligned} F^+(\wp) &\geq F^+(\dot{\lambda} \cdot \breve{\beta}) \\ &\geq \min\{F^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})), F^+(\ddot{\gamma})\} \\ &= \min\{F^+(\wp), F^-(\wp)\} \\ &= F^+(\wp). \end{aligned}$$

So, $F^-(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})) = F^-(\wp)$ and $F^+(\dot{\lambda} \cdot (\ddot{\gamma} \cdot \breve{\beta})) = F^+(\wp)$, that is, $\ddot{\gamma} \in F^{-1}(\wp, \wp)$. Therefore, $F^{-1}(\wp, \wp)$ is an IUPI of $\mathfrak{I}$. □

Theorem 4.17. *Let $F = (\mathfrak{I}; F^-, F^+)$ be a BFS in $\mathfrak{I}$. Then $F = (\mathfrak{I}; F^-, F^+)$ is a BFSIUPI of $\mathfrak{I}$ if and only if $F^{-1}(\wp, \wp)$ is an SIUPI of $\mathfrak{I}$.*

Proof. This result follows directly from Theorem 3.9, affirming that $\mathfrak{I}$ is uniquely characterized as the sole SIUPI of itself. □

5 Conclusion

This study has yielded several significant findings that advance the theoretical understanding of BFSs within the framework of IUP-algebras:

(1) Introduction of Core Concepts: We introduced and formalized four key types of BFSs in IUP-algebras: BFIUPSs, BFIUPFs, BFIUPIs, and BFSIUPIs, laying the groundwork for future exploration of these structures.

(2) Characterization of Relationships: Conditions under which BFIUPFs can be classified as BFIUPIs or BFIUPSs were established, as detailed in Theorems 4.11 and 4.12.

(3) Relationship Between BFS and Subsets: A clear and concise relationship between a BFS $\mathcal{F}$ and its associated subset $\mathcal{F}^{-1}(\wp, \wp)$ was identified and proved; see Theorem 4.14.

(4) Interplay with Characteristic BFSs: We demonstrated a straightforward relationship between BFIUPSs (and their counterparts: BFIUPFs, BFIUPI, and BFSIUPIs) and their characteristic BFSs, providing an important connection for further algebraic investigation; see Theorem 3.18.

These findings significantly contribute to the study of IUP-algebras, enriching their theoretical framework and providing a foundation for future research on the integration of bipolar fuzzy logic with algebraic structures.

Future work may focus on extending the developed concepts to other generalized fuzzy frameworks, such as hesitant or picture fuzzy environments, or investigating applications in decision theory, rough set approximations, and knowledge representation. Additionally, the categorical properties and morphisms between bipolar fuzzy IUP-algebras merit deeper exploration.

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