A hybrid approach for solving the Merton portfolio optimization problem with an infinite horizon

M. Pourmoradi[1], M. Soleimanivareki[2*], A. Nabavichashmi[3],

[1,3]*Dept. of Management, Islamic Azad University, Babol Branch, Babol, Iran.*

[2]*Dept. of Math, Islamic Azad University, Ayatollah Amoli Branch, Amol, Iran.*

[1]*pormoradim@yahoo.com* [2]*m.soleimanivareki@sutech.ac.ir* [3]*Anabavichashmi2003@gmail.com*

Abstract

In a world dominated by uncertainty, stochastic modeling is of the utmost importance, so that Many problems in economics, finance, and actuarial science naturally require decision makers to undertake choices in stochastic environments. Classical methods for solving infinite horizon stochastic optimal control problems primarily focus on deriving the solution by defining the value function through dynamic programming and the Hamilton-Jacobi-Bellman equation. However, obtaining a closed-form solution is generally challenging. To address this issue and identify an optimal trajectory and control, this article proposes a hybrid method for solving stochastic optimal control problems (SOCPs). This novel approach integrates the multi-step stochastic differential transform method with an approximation technique that solves infinite horizon problems by leveraging a finite horizon. An applicable example diagrams of the types of instances created from the simulation of the described approach, infinite horizon stochastic optimal control problem from management science is provided to show the method's effectiveness and efficiency, particularly in comparison with existing approaches.

Keywords: Stochastic optimal control problems; Dynamic programming; Hamilton-Jacobi-Bellman equation; Merton's Portfolio Problem

Mathematics Subject Classification: 93E20, 93E35

JEL Classification codes: C02, C15, C53.

1. Introduction

Banks, investment funds, and insurance companies are examples of investors that invest money in the financial markets. They want to make as much money as possible on their investments, but any serious investor also needs to consider the risk involved. An investor is, to a certain degree, risk-averse, i.e., the investor is reluctant to invest in an asset with high potential if this means that the risk of losing money is also high. The aim of such investors is to maximize the expected returns on their investments while at the same time limiting the risk involved. One way of modeling such behaviors is through the theory of stochastic control and the maximization of expected utility functions.

SOCPs frequently occur in many branches of science, especially in economics and finance. Stochastic differential equations have become the standard models for financial quantities such as asset prices, interest rates, and their derivatives (Oksendal, 2003; Steele, 2001). Three major approaches in SOCOs can be differentiated: dynamic programming, duality, and the maximum principle (Yong and Zhou, 1999). Dynamic programming obtains, by means of the optimality principle of Bellman, the Hamilton-Jacobi-Bellman equation, which characterizes the value function. In fact, in the case of continuous optimal control problems, the dynamic programming technique reduces the optimal control problem to solving a partial differential equation (the Hamilton-Jacobi-Bellman (HJB) equation). Almost all (just a few special cases) of these equations are difficult to solve analytically (Kushner and Dupuis, 2001). Under some smoothness and regularity assumptions on the solution, it is possible to obtain, at least implicitly, the optimal control. This is the content of the so-called verification theorems that appear in Fleming and Soner (2006). However, only a few classes of SOCPs admit analytical solutions for the value function in this manner. Besides, finite state Markov chain approximation and finite differences are two classical approaches in solving SOCPs numerically (Karatzas and Shreve, 2012; Krawczyk, 1999), but analytical and analytical-numerical methods are superior over the numerical methods due to the determination of closed-form solutions (Bousabaa, 2023).

The paper is organized as follows: First, we present the control problem and the preliminaries that will be used in the next sections. In section 3, we review the SDTM approach and its hypotheses. In section 4, we show how infinite horizon SOCOs can be solved by studying their finite horizon approximations, and the new hybrid method for solving infinite horizon SOCPs is explained. In order to demonstrate the application and efficiency of the new method, section 5 is devoted to describing and solving one of the famous and useful problems in management, namely the Merton problem. Finally, the conclusion is presented in section 6.

2. Problem statement

In this section, we consider finite-time homogeneous SOCPs of the type

$$Sup_{u\in U}E(\int_0^\infty e^{-\beta t}F(X(t),u(t))dt),$$

$$S.to: dx(t) = f\big(X(t),u(t)\big) + \sigma\big(X(t),u(t)\big)dB(t);\ X(0) = x_0, \tag{1}$$

Where $u \in U \subseteq R^n$ denotes the set of admissible controls and $x \in \mathrm{X} \subseteq R^m$ is a state vector. Furthermore, $F : \mathrm{X} \times U \to R$ is the utility function and $\beta \geq 0$ is the discount rate. In our approach we have the following limitations, notations, and definitions:

(i) We restrict this problem to the one-dimensional SOCPs for simplicity of the exposition;

(ii) We denote (F_t) with the filtration generated by Brownian motion and assume that f, σ, and F are continuous functions on $S \times U$, that S and U are respectively closed intervals in $\mathcal{R}$ and refer to state and control space;

(iii) An (F_t) –adapted stochastic process is called a feasible control if almost surely for all t, we have $u(t) \in U$;

(iv) We let $D \in C^2(S)$ be all the smooth functions on S with bounded derivatives of all orders, so that it includes the value functions of (1).

Definition 1. A feasible control u(.) is admissible, if

(i) the governed equation of system (1), a stochastic differential equation, admits a unique solution;

(ii) For all $\emptyset \in D$ and for all $t > 0$, the Dynkin formula [1] holds and $E(\left|\emptyset\left(X^u(t)\right)\right| < \infty$;

(iii) $E\left(\int_0^\infty e^{-\beta}\left|F\left(X^u(s), u(s)\right)\right| ds\right) < \infty$.

3. Multi-step Stochastic Differential Transformation Method (MSDTM)

One of the effective methods for solving the differential equations is differential transformation method (DTM), which was originally introduced by Zhou (1986). This method obtains an analytical solution in the form of a polynomial based on the Taylor series for differential equations (Fakharzadeh and Hashemi, 2012) and has been applied for solving deterministic optimal control problems (see Gokdogan and Merdan (2010), and Hesameddini et al. (2012)). In multi-step DTM, a modification of DTM and the convergence of the obtained solution series are improved. In

Fakharzadeh et al. (2015), an efficient and fast approach for the multi-step DTM was proposed, a reliable modification of the DTM that improves the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions. For stochastic calculus, the DTM is developed to solve stochastic differential equations (Fakharzadeh et al., 2015). In order to introduce the tools for subsequent discussions, in this section, we remind some necessary basic definitions and properties, which can mostly be found in Oksendal (2003) and Villafuerte et al. (2010).

Definition 1 (stochastic calculus). For a given probability space (Ω, F, P), a real random variable X that is defined on this space and satisfying the condition $E(X^2) < +\infty$, is called a second-order random variable (2-r.v.); here E denotes the expectation operator. The space L_2 of all the 2-r.v.'s endowed with the norm $\|X\|_2 = \sqrt{E(X^2)}$, is a Banach space (Villafuerte et al., 2010). In the probability space (Ω, F, P), a stochastic process $\{X(t) : t \in I\}$ where I is a closed interval in the real line $\mathbb{R}$, is called a second-order stochastic process (2-s.p.) if for each $t \in I, X(t)$ is a 2-r.v. A sequence of 2-r.v.'s $\{X_n : n \geq 0\}$ is mean square convergent in L_2 to a 2-r.v. X as $n \to \infty$ if $\lim_{n\to\infty} \|X_n - X\|_2 = 0.$ Also, if we have a limiting condition such that $\lim_{\Delta t \to 0} \left\| \frac{X(t+\Delta t) - X(t)}{\Delta t} - \dot{X}(t) \right\|_2 = 0$ then, 2-s.p $\{\dot{X}(t) : t \in I\}$ is the mean square derivative of $\{X(t) : t \in I\}$.

Definition 2. Let $k \in \mathbb{N}$ and assume that the 4-s.p. $\{v(t) : t \in I\}$, has a mean fourth derivative of order k at $t \in I$ which is denoted by $v^{(k)}(t)$. The random differential transform of the process $v(t)$ is defined as:

$$V(k) = \frac{1}{k!} [\frac{d^k(v(t))}{dt^k}]_{t=t_0}, \tag{2}$$

Where V is the second process transformed and $\frac{d}{dt}$ denotes the mean square derivative. The inverse transform of V is defined as:

$$v(t) = \sum_{k=0}^{\infty} V(k)(t - t_0)^k, \qquad (3)$$

Here, it is formally assumed that the series (3) is uniformly mean fourth convergent in any closed interval into the domain of convergence, and as a result, (2) and (3) are well defined (Soong, 1973). For implementation purposes, the function *v(t)* is expressed by a finite series, and Eq. (3) can be written as:

$$v(t) \approx \sum_{k=0}^{N} V(k)(t - t_0)^k,$$

that *N* is decided by the convergence of natural frequency. For a stochastic differential equation, we assume that all the involved stochastic quantities take values in L_2-space and also that all the stochastic operations are in the mean square sense. We note that $B(t)$ is a Gaussian process with mean zero, and it is also a mean fourth continuous process (Merton, 1971). The Brownian motion $B(t)$ has trajectories belonging to $L^2([0,T])$ for almost all events. In this space, the Karhunen-Lèove expansion (Merton, 1971) for Brownian motion takes the form:

$$B(t) = B(t,\omega) = \sum_{i=0}^{\infty} z_i(\omega)\varphi_i(t), 0 \le t \le T,$$

With:

$$\varphi_i(t) = \frac{2\sqrt{2T}}{(2i+1)\pi} sin(\frac{(2i+1)\pi t}{2T})$$

Where the functions $\varphi_i(t)$'s form a basis of orthogonal functions and $\{z_i\}$ is a sequence of independent and identically distributed Gaussian random variables (Merton, 1971). This approach is also quite powerful for simulating paths of processes without independent increments of Brownian motion. By substituting finite terms of the Karhuneun-Loève expansion in (3.2), we have:

$$dx(t) = a(x,t)dt + \sigma(x,t)d(\sum_{i=0}^{M} z_i \varphi_i(t)), x(0) = x_0. \qquad (4)$$

Applying the random differential transformation method from (2) and using the properties (I)-(IV) in Theorem 1, one can transfer (4) into the following algebraic equation:

$$(k+1)X(k+1) = A(k) + \sum_{i=0}^{m} \sum_{j=0}^{k} z_i \, \Psi_i(k) \Sigma(k-j); X(0) = x_0, \qquad (5)$$

Where $X(k), A(k), \Sigma(k)$ and$\Psi(k)$ are the transformed processes of x, a, σ and the derivative of φ_i respectively. In order to simulate z_i, for instance, one can use the Maple random variable generator (*Random Tools Flavor: distribution)*.

We remind that *[0, T]* is the interval over which we want to find the solution of (4) over *t* and we apply a multi-step approach for ensuring the validity of the mentioned approximations for large T. In our new approach we assume that the interval *[0, T]* is divided into M subintervals $[t_{i-1}, t_i], i = 1,2,\ldots, M$ with equal step length. Thus, first, we use the SDTM to Eq. (4) over the interval $[0, t_1]$, and then at each subinterval $[t_{i-1}, t_i]$, for $i \geq 2$, the SDTM is applied to Eq. (6) over the interval $[t_{i-1}, t_i]$, where t_0 is replaced by t_{i-1}. It is necessary to refer to the initial conditions of the above approach as: $x_i(t_{i-1}) = x_{i-1}(t_{i-1})$ for $i = 2,3,\ldots,M$. Therefore, the process is repeated to generate a sequence of approximate solutions $x_i(t), i = 1,2,\ldots, M$. In fact, the MSDTM obtains the following solution as a piecewise polynomial on$[0,T]$:

$$x(t) = \begin{cases} x_1(t), t \in [0, t_1], \\ x_2(t), t \in [t_1, t_2], \\ \vdots \\ x_M(t), t \in [t_{M-1}, t_M], \end{cases} \tag{6}$$

Proposition 1. Consider the problem (4.4) and assume that all of the conditions of theorem 2 are satisfied. Also, by defining $F(X(t), t) = P_n(t)X(t) + Q(t)$, we assume $F: S \times T \to L_2$ is continuous and satisfies the m.s. Lipchitz condition. Then, there exists a unique m.s. solution for any initial condition $x_0 \in L_2$.

Proof: see Hesameddini et al. (2012).

The purpose of this approach is to extend the application of multi-step DTM for obtaining approximated analytical solution of SOCPs. For this aim, we use HJB equation and multi-step stochastic differential transform method (MSDTM) to determinate the approximated optimal trajectory and optimal control of SOCPs based on the Brownian motion properties in L^2-spaces and also the properties of the Karhunen-Loève expansion.

4. Approximation Method for solving SOCPs with an infinite horizon

In this section, we review the method of Fleming and Soner (2006), which demonstrated how finite horizon approximations to infinite horizon stochastic optimal control problems can be used to obtain analytic solutions. This often leads to analytical solutions for the infinite horizon SOCPs by studying phase diagrams. First, we consider the truncated stochastic optimal control problem,

$$Sup_{u \in A} \ E(\int_0^T e^{-rt} F(X(s), u(s))ds),$$

$$S.to: dX(t) = f(X(t), u(t))dt + \sigma(X(t), u(t))dW(t), \tag{7}$$

For finite T. We denote its value function as follows:

$$V(t, T, x) = Sup_{u \in A_T} E(\int_t^T e^{-r(s-t)} F(X(s), u(s))ds), \tag{8}$$

And its HJB-equation has form shown in Eq. (9):

$$rV(t,T,x) - V_t(t,T,x) = Sup_{u\in U}\left\{F(x,u) + V_x(t,T,x)f(x,u) + \frac{1}{2}V_{xx}(t,T,x)\sigma^2(x,u)\right\};$$

$$V(T,T,x) = 0. \tag{9}$$

We include the parameter T, which indicates the length of the finite time horizon, explicitly in the notation and consider the value function as a function of three variables. For numerical and analytical results, a fundamental question is whether the value functions $V(t,T,x)$ represent a reasonable approximation for the value function $V(x)$ of the infinite horizon approximation problem.

Proposition 2. Assume that the value function *V (x)* of the infinite horizon problem exists and satisfies (TVC). Then

$$Lim_{T\to\infty}V(t,T,x) = V(x),$$

Independent of $t \in [0, \infty)$ and for all *x*.

Proposition 3. Assume that the value functions of the finite *T*-horizon problems can be written as

$$V(t,T,x) = A_1(t,T)g_1(x) + A_2(t,T)g_2(x) + \ldots + A_n(t,T)g_n(x),$$

Satisfying the following conditions:

1. The functions gi(x) are linearly independent (as functions)

2. For all I, the following limit exists

$$Lim_{T\to\infty}A_i(t,T) = A_i,$$

3. For all *i* and admissible controls

$$Lim_{t\to\infty}E(e^{-rt}|g_i(x(t))|) = 0,$$

Then the value function of the infinite horizon problem exists and is given by

$$V(x) = A_1g_1(x) + A_2g_2(x) + \ldots + A_ng_n(x) \tag{10}$$

5. A New Hybrid Method

In Hesameddini et al. (2012), MSDM is introduced and used for solving the finite horizon stochastic optimal control problems. In this section, we propose a new variation of this approach introduced in section 3, for solving infinite horizon stochastic optimal control problems. To be sure, solving governed equations and making a sophisticated guess for value function are major weaknesses of the method introduced in section 4. In this article, with a stochastic differential transformation, these objections are elevated to some extent. In this paper, the novelty comes from the fact that:

- We hybridize the methods for solving infinite horizon SOCPs which is described in section 4, with one of the effective methods for solving differential equations named SDTM.
- Similar to Villafuerte et al. (2010), the DTM is developed to solve stochastic differential equations but in our new method, SDTM is applied for definite horizon.
- Our new approach is in fact complement for approximation method to obtain analytic-numerical approximation of optimal control and trajectory.
- We assume that the parameters of the problem (1), belong to L4-space.
- Due to the special spaces in (4), the utility function is selected in HARA1 types. This topic eliminates the problem of constructing a value function in truncated definite problem.

However, we write all steps for determine of optimal control and trajectory as the step by step. The algorithm that will be used to obtain these purposes for problem (1) is as follows:

Algorithm

Step 1. For problem (1), write the truncated problem as (7);

Step 2. Set HJB-equation for (7) and suppose that for each T the function V (., T,.) defined on $[0, \mathrm{T}] \times \mathrm{S}$, is of class $\mathrm{C}^{1,2}$and satisfies in (9).

Step 3. Make a sophisticated guess for value function as (10) to obtain V(x);

Step 4. Compute the optimal control in feedback form from maximizing the right-hand side of (9);

Step 5. Solve SDE with SDTM by substituting optimal control in governed system equation.

We notice that the obtained answer with the above Algorithm is according to variable T, perhaps this topic is the most important advantage of our new method.

6. A Case-study and simulation: Merton's portfolio problem

In order to explain the application of the mentioned method for solving SOCPs, as an example, we solve the well-known economic problem which is worked out by Merton in Fleming and Soner (2006). Merton's portfolio problem is a famous problem in continuous-time finance. This problem was formulated and solved by Robert C. Merton in 1969, for both finite lifetimes and infinite time horizons. The optimization problem of a portfolio mainly involves describing the investment choices of an individual whose degree of risk aversion is known, which is described by the utility function. The model in its first version, assumes a market with a certain number of risky securities and a risk-free security. The individual has an initial wealth to invest freely in the available assets and the objective function to maximize is given by the function of expected utility of the wealth to a certain future moment that represents the temporal horizon of the optimization problem. In fact, in Merton's portfolio problem the investor has the possibility to invest only in stock assets. In the discussion section of this paper, we have tried to understand what changes should be made to the solution in the event that the investor has the opportunity to access both the stock market and the derivatives market (Apollinaire and Amanda, 2022; Ji and Zhang, 2024).

In Merton's model, we have the option to invest part of our wealth in either a risk-free bond or a risky stock, while also planning to consume a portion of our wealth as time progresses. This problem assumes that an investor holds a portfolio consisting of two assets, a risk-free bond and risky stock. The price $b(t)$ per share of the bond evolves according to $db = rbdt,$ whereas the price of the stock follows the stochastic differential equation $dS = S(Rdt + \sigma dB)$, where r, R, and σ are constants, and $R > r > 0$, $\sigma \neq 0$. Let $x(t)$ denote the investor's wealth at time t, $u_1(t)$ be the fraction of wealth allocated to the risky asset, and $u_2(t)$ represent the consumption rate. Thus, $u(t) = (u_1(t), u_2(t))$ is control variable and attains its values in $U = [0,1] \times [0, +\infty)$. In this manner, the total wealth evolves as:

$$dx(t) = (1 - u_1(t))x(t)\frac{db}{b} + u_1(t)x(t)\frac{dS}{S} - u_2(t)dt.$$

Therefore,

$$dx(t) = r\big(1 - u_1(t)\big)x(t)dt + u_1(t)x(t)\{Rdt + \sigma dB(t)\} - u_2(t)dt, x(0) = x_0. \quad (11)$$

We stop the process if the wealth reaches zero (bankruptcy). In addition, we assume that the running cost is $F(x(t), u(t)) = l(u_2(t))$, where $l(u_2(t))$ represents the utility of consuming at rate $u_2(t) > 0$. Of note, the problem is to maximize the total expected utility, discounted at rate $r > 0$:

$$E(\int_0^{\tau} e^{-rt} l(u_2(s))ds),$$

Where τ denotes the random first time $x(.)$ leaves $Q = \{(x,t): 0 \leq t \leq T, x \geq 0\}$.

For this problem, we apply algorithm described in the previous section. In step 1, we have the truncated problem, and we choose the following utility function, which is of the hyperbolic absolute risk aversion type (Abazari and Abazari, 2010; Steele, 2001). In next step, the HJB-equation takes the form in Eq. (11).

In Krawczyk (1999), the HJB equation for this SOCP is given as:

$$v_t + max_{\substack{u_2 \geq 0 \\ 0 \leq u_1 \leq 1}} \{\frac{(u_1 x\sigma)^2}{2} v_{xx} + (r(1 - u_1)x + Rxu_1 - u_2)v_x + e^{-\beta t} l(u_2)\} = 0, \quad (12)$$

With boundary conditions $v(0,t) = 0$and$v(x,T) = 0$.

The chosen utility function, which is of the hyperbolic absolute risk aversion type (Abazari and Abazari, 2010; Steele, 2001), is:

$$l(c) = \frac{1}{\gamma} c^{\gamma};\ \ 0 < \gamma < 1.$$

For next step, the optimal control is computed by maximizing the expression given in the HJB equation. This yields the following optimal control form:

$$U^* = (U_1^*, U_2^*) = (-\frac{(R-r)V_x}{\sigma^2 x V_{xx}}, V_x^{\frac{1}{\gamma-1}})$$

It is assumed that an investor with an initial wealth of $x_0 = 10^5$ unit aims to maximize his/her satisfaction over the next ten years. The parameters considered in this study are $r = 0.05, R = 0.11, \sigma = 0.4,$ and $\gamma = 0.5.$ In this paper, it is assumed that the investor can invest using the above data over an infinite horizon. Substituting the derived optimal control into equation (12), we obtain the following equation:

$$\beta V - V_t = \frac{1}{V_x} + rxV_x - \frac{1}{2}\frac{(R-r)^2 V_x^2}{\sigma^2 V_{xx}};\ \ V(T,T,x) = 0. \qquad (13)$$

To solve equation (13), we define the following value function as:

$V(t,T,x) = \ A_1(t,T)\sqrt{x},$

Given the boundary condition of (13), this holds in state space for all x, implying that:

$$A_1(T,T) = 0. \qquad (14)$$

By substituting $A_1(T,T)=0$ from equation (14) into equation (13), we obtain the following differential equation, which is of Bernoulli equation type:

$$\dot{A_1}(t,T) - \left(\beta - 2r - \frac{(R-r)^2}{2\sigma^2}\right) A_1(t,T) = \frac{-2}{A_1(t,T)};\ A_1(T,T) = 0.$$

By solving the differential equation, we obtain the following solution for $A_1(t,T)$:

$$A_1(t,T) = e^{\theta t}\sqrt{\frac{2}{\theta}(e^{-2\theta t} - e^{-2\theta T})}$$

And the corresponding value function is:

$$V(t,T,x) = e^{\theta t}\sqrt{\frac{2x}{\theta}(e^{-2\theta t} - e^{-2\theta T})}$$

Where:

$$\theta = \beta - 2r - \frac{(R-r)^2}{2\sigma^2}$$

It can then be included from above result that

$$\lim_{T\to\infty} A_1(t,T) = \sqrt{\frac{2}{\theta}}$$

Therefore, the suitable candidate for the value function is:

$$V(x) = \sqrt{\frac{2x}{\theta}}$$

We therefore obtain that the optimal control is

$$U^* = (U_1^*, U_2^*) = \left(\frac{2(R-r)}{\sigma^2}, 2\theta x(t)\right)$$

By substitute this form of optimal control into governed equation (11), we obtain (15):

$$dx(t) = \left(5r + 3\frac{(R-r)}{\sigma^2} - 2\beta\right)x(t)dt + \frac{2(R-r)}{\sigma^2}x(t)dB(t) \qquad (15)$$

Which describes the dynamics of the investor's wealth over time.

By solving the above equation using SDTM for a definite horizon in the mean square space, we can obtain simulations of the optimal control and the optimal trajectory as function of T. Figure 1 presents the approximated optimal controls for 5 random runs, assuming T=1o. In fact, we examine the solution obtained with the hybrid approach and substitute one instead T. In the second stage, the new method was applied to this problem fifty random runs, assuming T=10. The average wealth from these simulations is plotted in Figure 2. For this choice, the obtained solution is compared with the results from Kirk (2004). Our results have been satisfactory for several reasons. Frist, the obtained optimal trajectory matches the result presented in Kirk (2004), indicating that the introduced approach is valid and reliable. Second, we introduce a time continues trajectory and optimal control strategy. Third, the selection of the endpoint in this problem implies determining the optimal investment strategy that aligns with the moment the investment period concludes. Accordingly, the value of x reaches zero at this time. This fact is consistent with the results obtained in Gokdogan and Merdan (2010) using the new method. The endpoint or selection of a definite horizon can represent the equilibrium point of the system. Therefore, accurately estimating this point demonstrates the efficiency and effectiveness of the new approach.

Conclusions

In this work, we have presented a new method for solving infinite horizon SOCPs to determine an approximate analytical strategy. The obtained solution is a series in which the coefficients can be computed very accurately. Our hybrid approach has some advantages over previous methods, such as the variational iteration method, Markov chain techniques, and others. First, it provides a continuous curve for the solution; second, the convergence of the new approach is guaranteed and the solution is presented as a polynomial with suitable approximation. An advantage of this approach is that it provides solutions for SOCPs while accounting for noise in the system. This topic is particularly useful in addressing financial and economic problems, such as nominal shocks, real shocks, and productivity shocks. It is designed to outline and evaluate various strategies to manage and mitigate such challenges effectively. We would like to emphasize that the novelty presented in this paper lies in the hybridization of two well-known methods, effectively creating a comprehensive approximation approach.

References

Abazari R, Abazari R (2010): *Numerical Study of Some Coupled PDEs by using differential Transformation Method.* Word Academy of science, Engineering and Technology.

Apollinaire NM, Amanda PN (2022): Stochastic Optimal Control Theory Applied in Finance. *Mathematics and Computer Science*, 7(4):59-67.

Bousabaa A (2023): *Numerical Methods for Finance and Programming Module.* University of Evry-Paris Saclay (M2 Risk and Assets Management - GRA).

Ewald CO, Wang WK (2011): Analytic Solutions for Infinite Horizon Stochastic Control Problems via Finite Horizon Approximation: a Practical Guide. *Mathematical Social Sciences*, 61:146-151.

Fakharzadeh AJ, Hashemi S (2012): Solving a Class of Nonlinear Optimal Control Problems by Differential Transformation Method. *The Journal of Mathematics and Computer Science*, 5(3):146-152.

Fakharzadeh AJ, Hesameddini E, Soleimanivareki M (2015): Multi-step Stochastic Differential Transformation Method for Solving Some Class of Random differential equations. *Applied mathematics in Engineering, Management and Technology*, 3(3):115-123.

Fleming WH, Soner HM (2006): *Controlled Markov Processes and Viscosity Solutions.* 2nd ed. Springer.

Gokdogan A, Merdan M: A Numeric-Analytic Method for Approximating the Holling Tanner Model. *Studies in Nonlinear Sciences*, 1(3):77-81.

Hesameddini E, Fakharzadeh A, Soleimanivareki M (2012): Differential Transformation Method for Solving a Class of Nonlinear Optimal Control Problems. *The Journal of Mathematics and Computer Science*, 5(1): 67-74.

Ji Y, Zhang Q (2024): Infinite Horizon Backward Stochastic Difference Equations and Related Stochastic Recursive Control Problems. *SIAM Journal on Control and Optimization*, 62:5.

Karatzas I, Shreve S (2012): *Brownian motion and stochastic calculus.* Vol. 113. Springer Science & Business Media.

Kirk DE (2004): *Optimal Control Theory: An Introduction.* Dover Publication, Inc. Mineola, New York.

Krawczyk JB (1999): *Approximated Numerical Solutions to Portfolio Management Problem.* Workshop on Computational Economic and Economic Theory, Stanford.

Kushner HJ, Dupuis P (2001): *Numerical Methods for Solving Stochastic Control Problems in Continuous time.* Springer-Verlag.

Merton RC (1971): Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Economic Theory*, 3:373-413.

Oksendal B (2003): *Stochastic Differential Equations: An Introduction with Applications.* 6th ed. Springer, Berlin Heidelberg, New York.

Soong TT (1973): *Random Differential Equations in Science and Engineering.* Academic Press, New York.

Steele JM (2001): *Stochastic Calculus and Financial Applications.* Springer, New York.

Stefano MI (2008): *Simulation and Inference for Stochastic Differential Equations.* Springer, New York.

Villafuerte L, Braumann CA, Cortes JC, Jodar L (2010): Random Differential Operational calculus: Theory and Application. *Computers & Mathematics with Applications*, 59:115-125.

Yong J, Zhou XY (1999): *Stochastic controls, Applications of Mathematics.* Hamiltonian systems and HJB equations, Springer-Verlag, New York.

Zhou JK (1986): *Differential Transformation and Application for Electrical Circuits.* Huazong University Press, Wuhan, China.

Figures

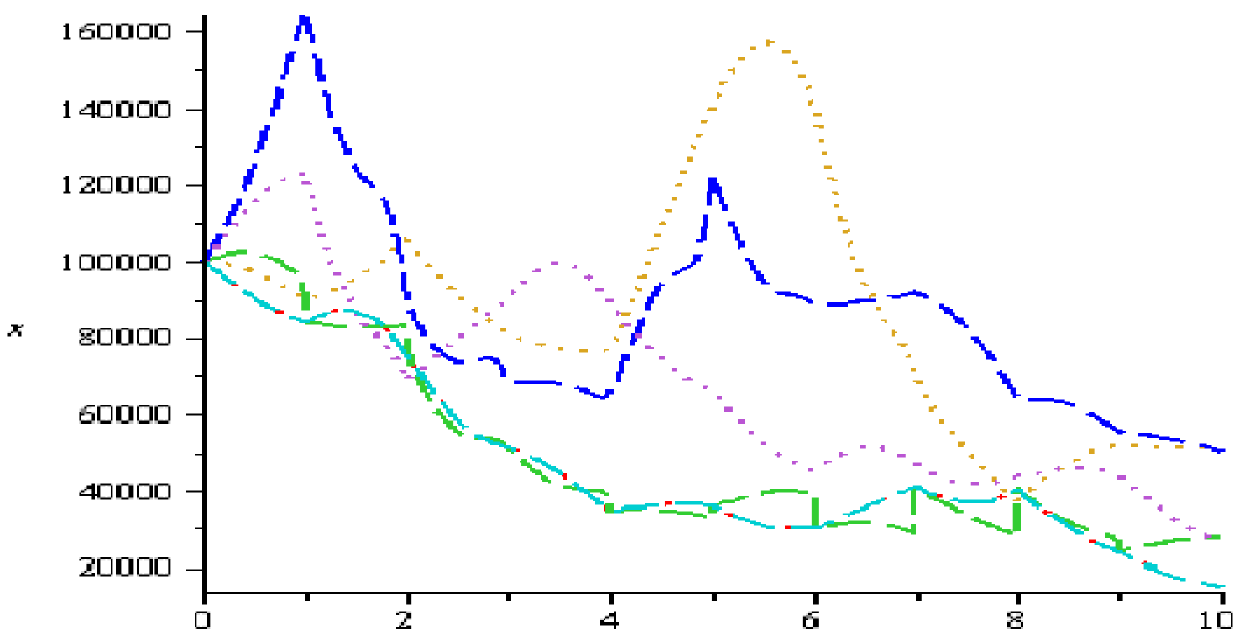

Figure 1. Five samples of trajectory.

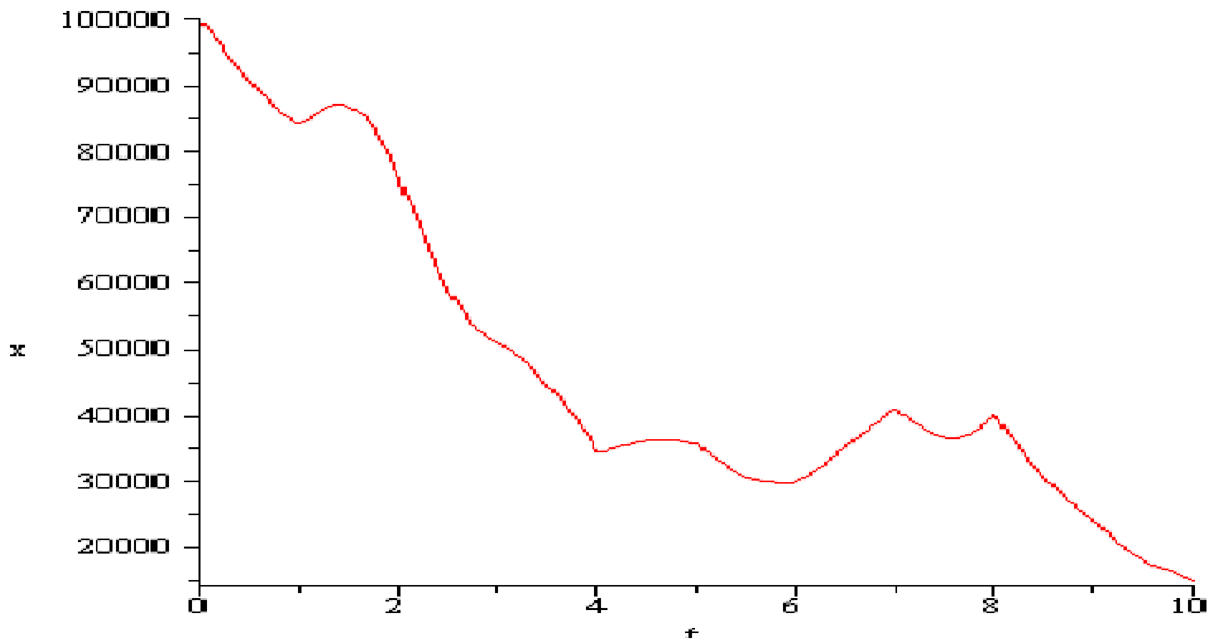

Figure 2. The average wealth of running fifty times randomly.