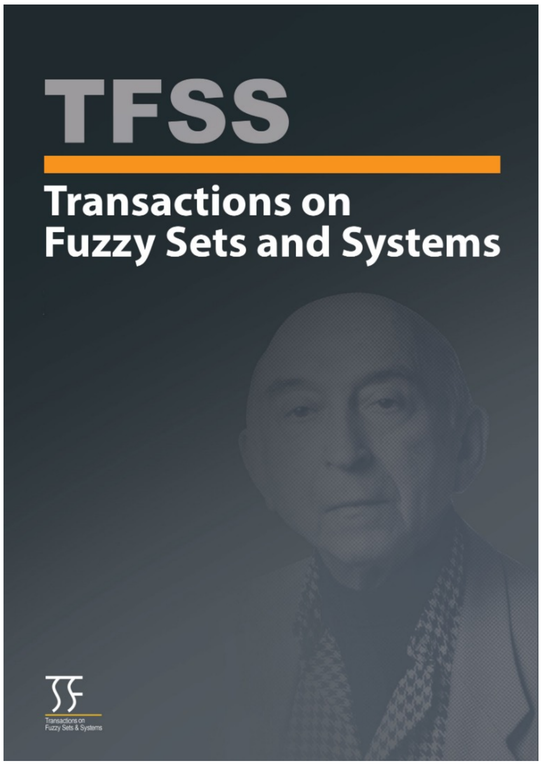
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

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Vol.5, No.2, (2026), 306-336. DOI: https://doi.org/10.71602/tfss.2026.1193350

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Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.2, (2026), 306-336
DOI: https://doi.org/10.71602/tfss.2026.1193350

Article Type: Original Research Article

Eye Lens Evaluation: Unpacking Performance, Comfort, and Cost through Fermatean Fuzzy Z-Numbers

Shahzaib Ashraf, **Sibgha Urooj**, **Muhammad Shazib Hameed**, **Chiranjibe Jana***

Abstract. In this research paper, a comprehensive evaluation of eye lenses is conducted, focusing on three key criteria: performance, comfort, and cost. However, to minimize the chances of large errors attributable to the vagueness and subjectivity of the approach, the Fermatean fuzzy Z-number approach should be employed because it enhances the accuracy of decision-making as it combines the level of uncertainty of the information with the credibility of the opinion of the expert. Moreover, second-order Fermatean fuzzy Z-number weighted averaging and geometric operations are used to fuse the experts' assessments, which arrive at different aspects. The present paper also defines some basic properties based on Fermat fuzzy Z-numbers and their proofs. To justify the credibility of the results that emerged from the proposed aggregation operators and those of the combined compromise solution method, similar and compromise-similar results were obtained by using the weighted averaging and geometric aggregation to pool the expertise evaluation over multiple attributes. Also, some fundamental properties dependent on Fermat fuzzy Z-numbers and their corresponding proofs are discussed. To overcome this issue and enhance the validity of this research, the results of the proposed aggregation operators have been checked with the combined compromise solution method and found similar outcomes. This paper demonstrates that these modern fuzzy-based methods are suitable for the required structural and balanced ranking of lenses while also providing information about the optimal eye lenses based on various criteria. Besides improving lens assessment capacity, this method also demonstrates the applicability of Fermatean fuzzy Z-numbers in decision-making, which are advantageous for consumers and optical industry professionals.

AMS Subject Classification 2020: 03E72; 94D05; 68T37; 91B06; 92C55
Keywords and Phrases: Fermatean fuzzy Z-number, Decision Model, Aggregation Information.

1 Introduction

Real-life systems have also become more complex than before, and as a result, it becomes challenging for the decision-maker to choose the best among several. When it comes down to it, it is not an easy thing to achieve something, but it can be done. Many companies struggle with the specifics of inspiration, objectives, and viewpoint setting. Indeed, when making a choice, an individual or a committee has to weigh one or more objectives against another at the same time. The notion of this theory means that the best solution for a given situation in detail cannot be arrived at by the decision maker, as the criteria are not constant. Therefore, the decision-maker is usually inclined and sensitive to devise strategies that are more suitable and most appropriate for coming to the right course of action. In decision-making situations, the classical or crisp model cannot be applied every time depending on confusing and ambiguous facts.

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Received: 22 December 2024; **Revised**: 8 July 2025; **Accepted**: 18 September 2025; **Available Online**: 30 October 2025; **Published Online**: 7 November 2026.

How to cite: Ashraf S, Urooj S, Hameed MS, Jana C. Eye Lens Evaluation: Unpacking Performance, Comfort, and Cost Through Fermatean Fuzzy Z-Numbers. *Transactions on Fuzzy Sets and Systems.* 2026; 5(2): 306-336. DOI: https://doi.org/10.71602/tfss.2026.1193350

A Fermatean fuzzy set is one of the most modern mathematical tools used to describe risk in decision-making. It extends traditional fuzzy sets by incorporating three key components: the membership degree, the non-membership degree, and the hesitation degree. Such components provide a very accurate representation of uncertainties, especially when there are many forms of ambiguity or hesitations. The main advantage of Fermatean fuzzy sets is that the given approach enables one to navigate in situations where both of them are not clearly distinguishable, which increases the accuracy of the assessed uncertainty.

1.1 Review of the Literature

A classical fuzzy set has its measure of uncertainty accomplished with one membership ranging between 0 and 1. Every element has the measure of participation that indicates its degree of belonging to the set: the amount of 1 showing full membership and 0 showing no membership. Although this model is effective for general vagueness measurement, it is not suitable for hesitation or more detailed ambiguity. The radius of a Fermatean fuzzy set defines a degree of hesitation that combines the feature along with the membership and non-membership grades. The application of Fermatean fuzzy sets is most appropriate in operations like the multiple criteria decision-making processes and especially in the medical diagnosis, since the more complex vagueness leads to the formulation of better and more efficient decisions in case of hesitation.

In response to this fluctuation in 1965, Zadeh [1] proposed that every single constituent of a set get a membership grade of between zero and one. A number of properties that are characteristic of crisp scenarios also hold true regard in fuzzy sets. That is why the work that Zadeh has done in this field is amazing. Decision science, communication, medicine, intelligent science, advertisement, the field of engineering, the science of computers and various others can be utilized in various ways by FSs. Z-numbers, which are a type of uncertain numbers that take fuzzy numbers to further levels, were comprehensively discussed by Banerjee et al. [2] The author highlighted that Z-numbers are different from other uncertainty measures and that they can encompass a range of uncertainty and are able to combine both random and fuzziness within a single approach. The review also categorised the application of Z-numbers into control systems, pattern recognition [3], and decision-making. In total, it had informative details on the theory, probabilities, and uses of Z-numbers, making it a valuable asset to scholars and professionals in the uncertain modeling field.

While the fuzzy number is useful in the case of uncertainty, it loses the information's logical structure necessary for sorting, selecting, constructing algorithms or dealing with data. As such, Zadeh [4] applies these kinds of restrictions using a fuzzy Z-number that declares and refines the constraint. $Z = (T, V)$ is an ordered pair of fuzzy numbers where T represents a degree of assurance or some other pertinent elements of assessing such as likelihood, dependability, certainty etc., and V is the limit of fuzziness on the value of an S variable. From an intuitive sense, codifying the functionality of a decision-making system appears to be configured more impressively and elastically. Many researchers apply fuzzy Z-numbers in their research activities. Numerous important contributions have been made in the literature concerning the application and advancement of Z-numbers in decision-making and fuzzy modeling. Jiang et al. [5] proposed an improved ranking method for general fuzzy numbers to facilitate the ranking of Z-numbers, enhancing the interpretability and computational reliability of Z-number-based models in complex environments. In the same year, Jafari, Yu, and Li [6] introduced a numerical approach using neural networks to solve fuzzy equations involving Z-numbers, demonstrating the feasibility of machine learning methods in handling Z-number uncertainty. Subsequently, Jafari et al. [7] extended their work by developing a fuzzy modeling framework for uncertain nonlinear systems using Z-numbers, emphasizing its applicability in complex computational intelligence systems. Additional theoretical advancement was presented by Aliev, Pedrycz, and Huseynov [8], who defined the Hukuhara difference for Z-numbers, thereby addressing introductory aspect of arithmetic operations in Z-number theory. In terms of practical applications [9] developed a multi-criteria decision-making model that integrates Z-numbers with the FUCOM and MABAC methodologies, providing a robust framework for handling real-world decision problems involving uncertain and imprecise data. Earlier, Kang et al. [10] had already established the possibility of Z-numbers in decision-making processes under uncertain environments, laying the groundwork for many subsequent developments in this field.

Among these, the remarkably worthy of note was Atanassov's [11] on intuitionistic fuzzy sets (IFS). He amplified FS by associating with it membership grades that are $\mu(s)$ and non-membership grades that are $\nu(s)$ and the condition $0 \leq \mu(s) + \nu(s) \leq 1$. Although decision-makers have somewhat restricted options over degrees, since it depends on $\mu(s)$ and $\nu(n)$, much appreciation goes to Atanassov for the design of IFSs. More often than not, there is more than one of them combined as membership degrees. In this case, it becomes impossible to have IFS produce a desirable result. To address the problem of this situation the what Pythagorean was proposed. Moreover, the idea of PFSs was presented in 2013 by Yager in [12] and Yager and Abbasov in [13] as further generalization of IFSs. In situations where there cannot be defined IFSs but only ambiguous or defective ones, it is better to use PFSs. For instance, let an expert express a degree of non-membership (DNM) = 0.5 and degree of membership (DM) = 0.8 for an alternative related to a criterion. When the two degrees are summed up it gives 1.3 which is below the degree requirement that has been stated in IFS. Also, if instead of addition, the squares of the two degrees are summed, $0.82^2 + 0.52^2$ we have the fact that $0.89 < 1$. Because of this, mounted frame data can be best represented in PFS rather than IFS. For this reason, the PFS theory has effectively grown fast to serve as a reservoir of solutions to almost every real-life issue in the world today. Moreover, Zhang and Xu [14] extended the TOPSIS approach in a Pythagorean fuzzy environment. Yager [15] has recently proposed some new AOs for aggregating Pythagorean fuzzy numbers (PFNs). Wei and Lu [16] examined the power AOs in a fuzzy Pythagorean setting. As stated in [17], Peng et al. presented MCDM-based Pythagorean fuzzy CoCoSo and CRITIC technique. Based on Pythagorean fuzzy information, Akram et al. [18] presented a two-phase group decision making process using ELECTRE III method. Ejegwa [19] came up with a modified distance measure of Zhang and Xu by solving the pattern recognition difficulties using Pythagorean fuzzy information [20]. The PROMETHEE approach was extended by Molla et al. [21] through the incorporation of PFSs and was applied to medical diagnostics. In addition, a comprehensive study by Bakioglu and Atahan [22] considered risk prioritization in self-driving cars utilising hybrid techniques and Pythagorean fuzzy information.

Let's assume once more that in the example above the DM is 0.9 and the DNM 0.6. It can then be seen that we do not use IFS and PFS to represent this information. Senapati and Yager developed the solution of this problem by proposing the concept of Fermatean fuzzy set (FFS) [23]. In this set, the DM and DNM are real numbers that are bounded above by the value, 1 and bound below by the value 0, and satisfy the constraint that: $0 \leq (DM)^3 + (DNM)^3 \leq 1$. The main advantage of the FFS is that in comparison to IFS and PFS which might be adequate when there is a high level of data precision and low level of uncertainty it will have a greater application whilst decoding more uncertainty existing in many practical decision making situations. Hence, since the result is $0.93 + 0.63 < 1$, FFS is right for capturing this grey data. Further, Senapati and Yager discussed decision making using FFSs and defined some operations on them [24]. Senapati and Yager developed several weighted averaging / geometric AOs for aggregating different Fermatean fuzzy numbers (FFNs) and applied them in the context of multiple criteria, decision making [25]. Aydemir and Yilmaz Gunduz used the TOPSIS approach with Dombi AOs to solve the decision making problems under Fermatean fuzzy environment [26]. Mishra et al. proposed a Fermatean fuzzy CRITIC-EDAS method for the selection of sustainable logistics suppliers where the Fermatean fuzzy set has significant importance in the decision-making process [27].

1.2 Motivation of the study

Decision making is a fundamental part of human life since it is present in every day life. Decision making is making a choice from a number of choices that are available. The preparation for during the confronting stage, and this final stage of preparation is important choices made in Personal and Professional endeavors of your life define your productivity level. If you have only one choice, there won't be much of a dilemma when making your decision. It becomes challenging particularly when you are in a position to make a choice from promising alternatives. For the theoretical gaps such as the ones mentioned above to be filled up, it will require concerted efforts from the research community and the practitioners to apply the latest advancements to solve practical problems to achieve high performance and high quality outcomes in practice [28]. To benefit from the current developments and fresh ideas in fuzzy sets, systems and decision-making key business applications, fermatean fuzzy Z -numbers have to be defined. The main objective of this work is to lay the basis for a novel model that is very good at assigning ambiguous information: The fuzzy Z-number model that has been developed by the researcher is the fermatean fuzzy Z-number model. As such, it may serve as the device applicable for expressing the reflected decisions' vagueness for flexibility and applicability and for making the real uncertain choices and increasing the reliability of the data employed in making decisions.

It has some application in medical sciences, risk management and decision making IT as well. FFZNs have better features for the discretized uncertainty assessments in the medical sciences based on probabilistic and possibilistic uncertainties. For example, FFZNs can combine unclear patient information with hesitating or hazy professional decisions in diagnostics that can be useful in patient prognosis, treatment strategy, and disease probability. Especially in clinical cases, where data sometimes is ambiguous or missing in some aspect, it is more possible to measure the degree of tetrad and falsity but taking into account the factor of reluctance, it is possible to receive more accurate and valid conclusions. Consequently, for the evaluation of eye lenses, we employ the FFZN. Since High-Index Plastic Lenses, Polycarbonate Lenses, Photochromic Lenses, and Bifocal Lenses are some of the key choices between which preferences have to be made, Fermatean fuzzy Z-numbers (FFZNs) are useful in that they provide a way to evaluate several characteristics of eye lenses at once, namely, comfort, performance, and cost. The characteristics above are somewhat relative and are as a rule translated subjectively and with a certain degree of uncertainity every type of lens has its strong and weak sides in this respect. FFZNs complement the synthesise of these features by considering the uncertainty associated with the expert judgment or experience and patient feedback. For example, High-Index Plastic Lenses are valued for their ultra-sleek designs as well as high clarity; however, wearer comfort could vary with regard to the lens thickness or weight. Even though polycarbonate lenses are incredibly lightweight and impact-resistant, they could be a shade less better than other lens materials as far as vision is concerned. While there are many benefits to using photochromatic lenses they are very comfortable and the fact that they darken and lighten dependening on the lighting makes them very convenient for patients. However their efficacy can be suspect and this could lower patient satisfaction. Bifocal lenses are an option for both the near and distant vision although often, certain aspects of comfort or ease of viewing may be compromised. FFZNs can model these complexities by gathering subjective and objective uncertainties about each attribute. In this way, decision-makers may integrate the variations in price, functionality, and comfort for each of these four lens varieties, ensuring a more comprehensive and accurate assessment that is appropriate for each patient's unique requirements even when there is doubt.

We will also cover the method used to rank the alternatives that will be applied in this paper known as CoCoSo. MADM problems have been solved through many solution techniques that include the MOORA method [29], TOPSIS method [30], VIKOR method [31], GRA technique [32], EDAS technique [33], MABAC technique [34] and so on. Jazdani et al. [35] formalized the relatively novel CoCoSo approach (Combined Compromise Solution) based on combining MEP (Exponentially Weighted Product Model) and SAW (Simple Additive Weighting). The integration of compromise opinions is the core of this model since it brings together prejudiced evaluation criteria in the long run. This approach of CoCoSo provides a brief of compromise choices to the decision-maker. Researchers including Bagal et al. (2021a), Peng et al. (2021), Peng & Luo (2021), Ulutas et al. (2021), Deveci et al. (2021), Ecer (2021), Torkayesh et al. (2021a, b), and others are solving various problems. The following are some benefits of the CoCoSo (Combined Compromise Solution) method:
1. It is applicable in multi-criteria decision-making and other related problems that call for a number of

standards or criteria because all the criteria are integrated and solved to give the solution.
2. CoCoSo uses both compromise and consensus approaches and guarantees that the final solution will be a compromise between the identified criteria.
3. Flexible approach in decision analysis since it can handle both qualitative and quantitative information.
4. The method is more efficient in terms of computation and less complex compared to other MCDM techniques, thus it can be useful in practice.
5. It offers a clear hierarchy of decision options, making it easy to compare as well as make a selection.
6. CoCoSo encompasses the strengths of other successful MCDM approaches, including SAW (Simple Additive Weighting) and TOPSIS, and, therefore, has a strong theoretical background.
7. It is versatile for usage and can be used in a broad specialty area, which includes civil engineering, engineering economy, operations research, system engineering, and even business administration.

As a result, CoCoSo is selected to be the main development organization for this research. However, there are three major problems with the classical CoCoSo, namely a lack of ability to work with uncertainty and incomplete information while making decisions [36]. The shortcoming of the CoCoSo method is that it does not have much practice, may have a low quality, has a hard time dealing with complicated cross-correlations between criteria, and lastly, does not have a lot of transparency when there are many criteria and/or alternatives, which makes it challenging for the decision makers to comprehend. To address this, the CoCoSo method will be paired with Fermatean fuzzy sets and fuzzy Z-numbers.

1.3 Contributions of the Study:

The following fundamental objectives are aimed to be addressed by the research proposal:

- To look into productivity and flexibility in *FFZNs* figures according to Algebraic's norm.
- To make algebraic norms fundamental steps in the *FFZNs* framework more understandable while maintaining an unwavering grip of their core concepts.
- Describe and critically analyze the Fermatean aggregation processes in *FFZNs*, analyzing critical features to obtain meaningful information.
- Emphasizethe benefits and drawbacks of the particular Fermatean fuzzy operators and different *FFZNs* combination operators at the present time.
- In Fermatean fuzzy Z-numbers logic research, Algebric norms and aggregation operators are applied to increase the potential applications and ability to make decisions of fuzzy logic.

The article's remaining sections are arranged as follows: Section 2 addresses the general preprocessing and the operators that correspond to the covered basic preliminary actions. In Section 3, we introduced the FFZNs, and both their basic and advanced operators as well as the accuracy and scoring functions, and the distance formula. Some of the aggregation operators developed in Section 4 are the FFZNs weighted averaging operator, FFZNs ordered weighted averaging operator, FFZNs hybrid averaging operator, FFZNs weighted geometric operator, FFZNs ordered weighted geometric operator as well as the FFZNs hybrid geometric operator. Their properties were also built with proofs, as well as their theorems. In Section 5, the MADM algorithm is described and numerical example of using FFZNs information to determine in which province the greatest impact on the evaluation of the eye lenses will occur. In Section 6, we have mentioned about the CoCoSo methodology and mathematical examples for the understanding and verification of our proposed work. The concept of relative comparison was discussed in Section 7, and in Section 8 has been concluded.

2 Preliminaries

This section outlines some definitions and process that led to the formulation of the recommended work.

Definition 2.1. [1] *If S is universal set then the fuzzy set is defined as follow:*

$$B = \{\langle s, \mu_B(s)\rangle | s \in S\}$$

where μ_B is a membership grade of s in B and $\mu_B : S \to [0,1]$.

Definition 2.2. [4] *Z-number can be defined as an ordered pair of fuzzy numbers as given by $Z = (T, V)$. Membership makes up the T component, and the T's dependability is represented by V.*

Definition 2.3. [23] *Considering S to be the universal set, the following formula defines the Fermatean fuzzy set as a non-empty set:*

$$B = \{(s, (\mu(s), \nu(s)))| s \in S\}$$

such that $\mu(s) : S \to [0,1]$ and $\nu(s) : S \to [0,1]$ are the membership & the non-membership grades in a set S with following constrains $0 \leq \mu^3(s) + \nu^3(s) \leq 1$.

Definition 2.4. [23] *Consider $B_1 = (\mu_1, \nu_1)$ and $B_2 = (\mu_2, \nu_2)$ be two FFNs. Then by the following relations:*
$B_1 \supseteq B_2 \Leftrightarrow \mu_1 \geq \mu_2$ and $\nu_1 \leq \nu_2$;
$B_1 = B_2 \Leftrightarrow B_1 \supseteq B_2$ and $B_2 \supseteq B_1$;
$B_1 \cup B_2 = (\mu_1 \vee \mu_2, \nu_1 \wedge \nu_2)$;
$B_1 \cap B_2 = (\mu_1 \wedge \mu_2, \nu_1 \vee \nu_2)$;
$(B_1)' = (\mu_1, \nu_1)' = (\nu_1, \mu_1)$.

3 Fermatean fuzzy Z-numbers and their operational rules

In this section we further explained FFZNs and provided an overview of basic operators and theorems.

Definition 3.1. *If S is the universal set, then the given following form defines the fermatean fuzzy Z-numbers:*

$$F_Q = \{\langle s, (\mu(T,V)(s), \nu(T,V)(s))\rangle | s \in S\}$$

such that $\mu(T,V)(s) = (\mu_T(s), \mu_V(s))$ and $\nu(T,V)(s) = (\nu_T(s), \nu_V(s))$ such that $\mu(T,V)(s) : S \to [0,1]$ and $\nu(T,V)(s) : S \to [0,1]$ are the ordered pair of membership & non-membership grades in a set S and second component V is a Fermatean measure of integrity for T under all the conditions

$$0 \leq (\mu_T(s))^3 + (\nu_T(s))^3 \leq 1 \quad \text{and} \quad 0 \leq (\mu_V(s))^3 + (\nu_V(s))^3 \leq 1.$$

The elements for the standard representation is $(\mu_T(s), \nu_T(s))$ in F_Q where

$$(\mu(T,V), \nu(T,V)) = ((\mu_T, \mu_V), (\nu_T, \nu_V))$$

which is named as FFZN.

Definition 3.2. *Consider*

$$F_{Q_1} = (\mu_1(T,V), \nu_1(T,V)) = ((\mu_{T_1}, \mu_{V_1}), (\nu_{T_1}, \nu_{V_1}))$$

and

$$F_{Q_2} = (\mu_2(T,V), \nu_2(T,V)) = ((\mu_{T_2}, \mu_{V_2}), (\nu_{T_2}, \nu_{V_2}))$$

be two FFZNs and $\lambda > 0$. *Then the operational rules are define as follows:*

$$F_{Q_1} \supseteq F_{Q_2} \Leftrightarrow \mu_{T_1} \geq \mu_{T_2}, \mu_{V_1} \geq \mu_{V_2}, \nu_{T_1} \leq \nu_{T_2}, \nu_{V_1} \leq \nu_{V_2}$$

$$F_{Q_1} = F_{Q_2} \Leftrightarrow F_{Q_1} \supseteq F_{Q_2} and F_{Q_2} \supseteq F_{Q_1}$$

$$F_{Q_1} \cup F_{Q_2} = \Big((\mu_{T_1} \vee \mu_{T_2}, \mu_{V_1} \vee \mu_{V_2}), (\nu_{T_1} \wedge \nu_{T_2}, \nu_{V_1} \wedge \nu_{V_2}) \Big)$$

$$F_{Q_1} \cap F_{Q_2} = \Big((\mu_{T_1} \wedge \mu_{T_2}, \mu_{V_1} \wedge \mu_{V_2}), (\nu_{T_1} \vee \nu_{T_2}, \nu_{V_1} \vee \nu_{V_2}) \Big)$$

$$(F_{Q_1})' = \Big((\mu_1(T,V), \nu_1(T,V))' = (\nu_1(T,V), \mu_1(T,V)) \Big)$$

$$F_{Q_1} \oplus F_{Q_2} = \Big((\sqrt[3]{\mu_{T_1}^3 + \mu_{T_2}^3 - \mu_{T_1}^3 \mu_{T_2}^3}, \sqrt[3]{\mu_{V_1}^3 + \mu_{V_2}^3 - \mu_{V_1}^3 \mu_{V_2}^3}), (\nu_{T_1}\nu_{T_2}, \nu_{V_1}\nu_{V_2}) \Big) \tag{1}$$

$$F_{Q_1} \otimes F_{Q_2} = \Big((\mu_{T_1}\mu_{T_2}, \mu_{V_1}\mu_{V_2}), (\sqrt[3]{\nu_{T_1}^3 + \nu_{T_2}^3 - \nu_{T_1}^3 \nu_{T_2}^3}, \sqrt[3]{\nu_{V_1}^3 + \nu_{V_2}^3 - \nu_{V_1}^3 \nu_{V_2}^3}) \Big) \tag{2}$$

$$\lambda F_{Q_1} = \Big((\sqrt[3]{1 - (1 - \mu_{T_1}^3)^\lambda}, \sqrt[3]{1 - (1 - \mu_{V_1}^3)^\lambda}), (\nu_{T_1}^\lambda, \nu_{V_1}^\lambda) \Big) \tag{3}$$

$$(F_{Q_1})^\lambda = \Big((\mu_{T_1}^\lambda, \mu_{V_1}^\lambda), (\sqrt[3]{1 - (1 - \nu_{T_1}^3)^\lambda}, \sqrt[3]{1 - (1 - \nu_{V_1}^3)^\lambda}) \Big) \tag{4}$$

Definition 3.3. *To compare the FFZNs* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v}))$. *We introduce the score function as follow:*

$$\Im(F_{Q_v}) = \frac{1 + (\mu_{T_v}.\mu_{V_v}) - (\nu_{T_v}.\nu_{V_v})}{2},$$

where $\Im(F_{Q_v}) \in [0,1]$.

Definition 3.4. *As for any two score values of FFZNs,* $\Im(F_{Q_v})$ *and* $\Im(F_{Q_{v'}})$. *Then*

$$\begin{aligned} &\textit{if } \Im(F_{Q_v}) \geq \Im(F_{Q_{v'}}) \textit{ then } F_{Q_v} \geq F_{Q_{v'}}, \\ &\textit{if } \Im(F_{Q_v}) \leq \Im(F_{Q_{v'}}) \textit{ then } F_{Q_v} \leq F_{Q_{v'}}, \\ &\textit{if } \Im(F_{Q_v}) = \Im(F_{Q_{v'}}) \textit{ then} \end{aligned}$$

we calculate the accuracy function: $\wp(F_{Q_v}) = (\mu_{T_v}.\mu_{V_v}) - (\nu_{T_v}.\nu_{V_v})$, *and*

$$\begin{aligned} &\textit{if } \wp(F_{Q_v}) \geq \wp(F_{Q_{v'}}) \textit{ then } F_{Q_v} \geq F_{Q_{v'}}, \\ &\textit{if } \wp(F_{Q_v}) \leq \wp(F_{Q_{v'}}) \textit{ then } F_{Q_v} \leq F_{Q_{v'}}, \\ &\textit{if } \wp(F_{Q_v}) = \wp(F_{Q_{v'}}) \textit{ then } F_{Q_v} \sim F_{Q_{v'}}. \end{aligned}$$

Example 3.5. Consider two FFZNs as $F_{Q_1} = ((0.6, 0.4)(0.7, 0.3))$ and $F_{Q_2} = ((0.2, 0.4)(0.2, 0.3))$. There ranking is then given as follows:
When using the scoring function formula we have, $\wp(F_{Q_1}) = 1 + 0.6 \times 0.4 - 0.7 \times 0.3/2 = 0.52$ and $\wp(F_{Q_2}) = 1 + 0.2 \times 0.4 - 0.2 \times 0.3/2 = 0.51$. Since $\wp(F_{Q_1}) \geq \wp(F_{Q_2})$, this implies that $F_{Q_1} \geq F_{Q_2}$.

Definition 3.6. *Consider*

$$F_{Q_1} = (\mu_1(T,V), \nu_1(T,V)) = ((\mu_{T_1}, \mu_{V_1})(\nu_{T_1}, \nu_{V_1}))$$

and

$$F_{Q_2} = (\mu_2(T,V), \nu_2(T,V)) = ((\mu_{T_2}, \mu_{V_2})(\nu_{T_2}, \nu_{V_2}))$$

be two FFZNs, then the Euclidean distance between them as follows:

$$d(F_{Q_1}, F_{Q_2}) = \sqrt{((\mu^3_{T_1}.\mu^3_{T_2}) - (\mu^3_{V_1}.\mu^3_{V_2}))^2 + ((\nu^3_{T_1}.\nu^3_{T_2}) - (\nu^3_{V_1}.\nu^3_{V_2}))^2}.$$

4 Fermatean fuzzy aggregation operators with Z-numbers

In this section, we will present some arithmetic and geometric aggregation operators derived from FFZN, namely: the FFZN weighted averaging operator (FFZNWA), the FFZN ordered weighted averaging operator (FFZNOWA), the FFZN hybrid averaging operator (FFZNHA), the FFZN weighted geometric operator (FFZNWG), the FFZN ordered weighted geometric operator (FFZNOWG), and the FFZN hybrid geometric operator (FFZNHG).

4.1 Fermatean fuzzy Z-number weighted averaging aggregation operators

In this section, developing the Fermatean fuzzy Z-numbers weighted averaging aggregation operations is based on the Algebric t-norm and t-conorm, and further builds up the basic results and implications of these averaging aggregation operators.

Definition 4.1. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v})(\nu_{T_v}, \nu_{V_v}))$ *and* $v = (1, 2, 3, ..., h)$ *be a group of FFZNs and the FFZNWA operator in a way that* $FFZNWA : FFZN^X \to FFZN$ *is specified as:*

$$FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = \sum_{v=1}^{h} \lambda_v F_{Q_v}$$

such that $0 \leq \lambda_v \leq 1$ *with* $\sum_{v=1}^{h} \lambda_v = 1$ *and* λ_v *represents* $F_{Q_v}(v = 1, 2, 3, ..., h)$ *weight vectors.*

Theorem 4.2. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v})(\nu_{T_v}, \nu_{V_v}))$ *and* $v = (1, 2, 3, ..., h)$ *be a group of the FFZNs. Then the aggregated value of the FFZNWA is a FFZN, by using the Definition 3.2, we have:*

$$\begin{aligned} FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} \lambda_v F_{Q_v} \\ &= \left(\left(\sqrt[3]{1 - \prod_{v=1}^{h} (1 - \mu^3_{T_v})^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{h} (1 - \mu^3_{V_v})^{\lambda_v}} \right), \left(\prod_{v=1}^{h} \nu^{\lambda_v}_{T_v}, \prod_{v=1}^{h} \nu^{\lambda_v}_{V_v} \right) \right). \end{aligned}$$

such that λ_v *represents* $F_{Q_v}(v = 1, 2, 3, ..., h)$ *weight vector with* $0 \leq \lambda_v \leq 1$ *and* $\sum_{v=1}^{h} = 1$.

Proof. We will prove the above equation through the mathematical induction. If $h = 2$, by the operations Equation 1 and 3 of the Definition 3.2. We will arrive at the following outcome:

$$\lambda_1 F_{Q_1} = \Big(\big(\sqrt[3]{1-(1-\mu_{T_1}^3)^{\lambda_1}}, \sqrt[3]{1-(1-\mu_{V_1}^3)^{\lambda_1}} \big), \big(\nu_{T_1}^{\lambda_1}, \nu_{V_1}^{\lambda_1} \big) \Big)$$
$$\lambda_2 F_{Q_2} = \Big(\big(\sqrt[3]{1-(1-\mu_{T_2}^3)^{\lambda_2}}, \sqrt[3]{1-(1-\mu_{V_2}^3)^{\lambda_2}} \big), \big(\nu_{T_2}^{\lambda_2}, \nu_{V_2}^{\lambda_2} \big) \Big)$$

$$\begin{aligned}
FFZNWA(F_{Q_1}, F_{Q_2}, ..., F_{Q_h}) &= \sum_{v=1}^{2} \lambda_v F_{Q_v} \\
&= \lambda_1 F_{Q_1} \oplus \lambda_2 F_{Q_2} \\
&= \begin{pmatrix} \Big(\big(\sqrt[3]{1-(1-\mu_{T_1}^3)^{\lambda_1}}, \sqrt[3]{1-(1-\mu_{V_1}^3)^{\lambda_1}} \big), \big(\nu_{T_1}^{\lambda_1}, \nu_{V_1}^{\lambda_1} \big) \Big) \\ \oplus \Big(\big(\sqrt[3]{1-(1-\mu_{T_2}^3)^{\lambda_2}}, \sqrt[3]{1-(1-\mu_{V_2}^3)^{\lambda_2}} \big), \big(\nu_{T_2}^{\lambda_2}, \nu_{V_2}^{\lambda_2} \big) \Big) \end{pmatrix}, \\
&= \begin{pmatrix} \sqrt[3]{\Big(1-(1-\mu_{T_1}^3)^{\lambda_1}) + (1-(1-\mu_{T_2}^3)^{\lambda_2}) - (1-(1-\mu_{T_1}^3)^{\lambda_1})(1-(1-\mu_{T_2}^3)^{\lambda_2}\Big)}, \\ \sqrt[3]{\Big(1-(1-\mu_{V_1}^3)^{\lambda_1}) + (1-(1-\mu_{V_2}^3)^{\lambda_2}) - (1-(1-\mu_{V_1}^3)^{\lambda_1})(1-(1-\mu_{V_2}^3)^{\lambda_2}\Big)}, \\ \Big(\nu_{T_1}^{\lambda_1} \nu_{T_2}^{\lambda_2}, \nu_{V_1}^{\lambda_1} \nu_{V_2}^{\lambda_2} \Big) \end{pmatrix}, \\
&= \Big(\sqrt[3]{\Big(1-(1-\mu_{T_1}^3)^{\lambda_1}(1-\mu_{T_2}^3)^{\lambda_2}\Big)}, \sqrt[3]{\Big(1-(1-\mu_{V_1}^3)^{\lambda_1}(1-\mu_{V_2}^3)^{\lambda_2}\Big)}, \Big(\nu_{T_1}^{\lambda_1} \nu_{T_2}^{\lambda_2}, \nu_{V_1}^{\lambda_1} \nu_{V_2}^{\lambda_2} \Big) \Big).
\end{aligned}$$

This implies that

$$FFZNWA(F_{Q_1} \oplus F_{Q_2}) = \left(\Big(\sqrt[3]{1-\prod_{v=1}^{2}(1-\mu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{2}(1-\mu_{V_v}^3)^{\lambda_v}} \Big), \Big(\prod_{v=1}^{2} \nu_{T_v}^{\lambda_v}, \prod_{v=1}^{2} \nu_{V_v}^{\lambda_v} \Big) \right).$$

If $h = b$, then:

$$\begin{aligned}
FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_b})) &= \sum_{v=1}^{b} \lambda_v F_{Q_v} \\
&= \left(\Big(\sqrt[3]{1-\prod_{v=1}^{b}(1-\mu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b}(1-\mu_{V_v}^3)^{\lambda_v}} \Big), \Big(\prod_{v=1}^{b} \nu_{T_v}^{\lambda_v}, \prod_{v=1}^{b} \nu_{V_v}^{\lambda_v} \Big) \right).
\end{aligned}$$

If $h = b + 1$, then

$$\begin{aligned}
FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_b}, F_{Q_{b+1}}) &= \sum_{v=1}^{b} \lambda_v F_{Q_v} \oplus \lambda_{b+1} F_{Q_{b+1}} \\
&= \begin{pmatrix} \Big(\sqrt[3]{1-\prod_{v=1}^{b}(1-\mu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b}(1-\mu_{V_v}^3)^{\lambda_v}} \Big) \\ , \Big(\prod_{v=1}^{b} \nu_{T_v}^{\lambda_v}, \prod_{v=1}^{b} \nu_{V_v}^{\lambda_v} \Big) \Big) \oplus \lambda_{b+1} F_{Q_{b+1}} \end{pmatrix}, \\
&= \left(\Big(\sqrt[3]{1-\prod_{v=1}^{b+1}(1-\mu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b+1}(1-\mu_{V_v}^3)^{\lambda_v}} \Big), \Big(\prod_{v=1}^{b+1} \nu_{T_v}^{\lambda_v}, \prod_{v=1}^{b+1} \nu_{V_v}^{\lambda_v} \Big) \right).
\end{aligned}$$

This mean that $b + 1$ must hold. Thus it is true for all h, and so it give the proof of this theorem. $\square$

Property 4.3. *Inequality: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)), v \in S$ *be a group of FFZNs, if all* F_{Q_v} *are identical, then*

$$FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = F_Q$$

Proof. Consider F_{Q_v} are identical and we also know that:

$$\begin{aligned}
FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} \lambda_v F_{Q_v}, \\
&= \left(\left(\sqrt[3]{1 - \prod_{v=1}^{h} (1-\mu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{h} (1-\mu_{V_v}^3)^{\lambda_v}} \right), \left(\prod_{v=1}^{h} \nu_{T_v}^{\lambda_v}, \prod_{v=1}^{h} \nu_{V_v}^{\lambda_v} \right) \right) \\
&= \left(\left(\sqrt[3]{1 - (1-\mu_{T_v}^3)^{\sum_{v=1}^{h} \lambda_v}}, \sqrt[3]{1 - (1-\mu_{V_v}^3)^{\sum_{v=1}^{h} \lambda_v}} \right), \left(\nu_{T_v}^{\sum_{v=1}^{h} \lambda_v}, \nu_{V_v}^{\sum_{v=1}^{h} \lambda_v} \right) \right) \\
&= \left(\left(\sqrt[3]{1 - (1-\mu_{T_v}^3)}, \sqrt[3]{1 - (1-\mu_{V_v}^3)} \right), \left(\nu_{T_v}, \nu_{V_v} \right) \right) \\
&= F_Q.
\end{aligned}$$

□

Property 4.4. *A monotonous state:Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *and* $F_{Q_{v'}} = (\mu_{v'}(T,V), \nu_{v'}(T,V))$ *where* $v, v' \in S$ *be a group of FFZNs, such that* $F_{Q_v} \subseteq F_{Q_{v'}}$*. Then,*

$$FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNWA(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}}).$$

Proof. As we know $F_{Q_v} \subseteq F_{Q_{v'}}$.Then $\mu_v(T,V) \leq \mu_{v'}(T,V)$ and $\nu_v(T,V) \geq \nu_{v'}(T,V)$. This implies that

$$\begin{aligned}
\sqrt[3]{1 - \prod_{v=1}^{h} (1-\mu_{T_v}^3)^{\lambda_v}} &\leq \sqrt[3]{1 - \prod_{v'=1}^{h} (1-\mu_{T_v}^3)^{\lambda_{v'}}}, \\
\sqrt[3]{1 - \prod_{v=1}^{h} (1-\mu_{V_v}^3)^{\lambda_v}} &\leq \sqrt[3]{1 - \prod_{v'=1}^{h} (1-\mu_{V_v}^3)^{\lambda_{v'}}}, \\
\prod_{v=1}^{h} \nu_{T_v}^{\lambda_v} &\geq \prod_{v=1}^{h} \nu_{T_{v'}}^{\lambda_{v'}}, \\
\prod_{v=1}^{h} \nu_{V_v}^{\lambda_v} &\geq \prod_{v=1}^{h} \nu_{V_{v'}}^{\lambda_{v'}}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left(\left(\sqrt[3]{1 - \prod_{v=1}^{h} (1-\mu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{h} (1-\mu_{V_v}^3)^{\lambda_v}} \right), \left(\prod_{v=1}^{h} \nu_{T_v}^{\lambda_v}, \prod_{v=1}^{h} \nu_{V_v}^{\lambda_v} \right) \right) \\
\leq &\left(\left(\sqrt[3]{1 - \prod_{v'=1}^{h} (1-\mu_{T_v}^3)^{\lambda_{v'}}}, \sqrt[3]{1 - \prod_{v'=1}^{h} (1-\mu_{V_v}^3)^{\lambda_{v'}}} \right), \left(\prod_{v=1}^{h} \nu_{T_{v'}}^{\lambda_{v'}}, \prod_{v=1}^{h} \nu_{V_{v'}}^{\lambda_{v'}} \right) \right).
\end{aligned}$$

Hence,

$$FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNWA(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}}).$$

□

Property 4.5. *Limitations: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$, *for all* $v \in S$, *be the FFZNs, such that* $F_{Q_l} = \max_v F_{Q_v}$ *and* $F_{Q_j} = \min_v F_{Q_v}$. *Then,*

$$F_{Q_j} \leq FFZNWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq F_{Q_l}.$$

The proof of this property is rather simple to prove.

Definition 4.6. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v})(\nu_{T_v}, \nu_{V_v}))$ *and* $v = (1,2,3,...,h)$ *be a group of FFZNs and the FFZNOWA operator in a way that* $FFZNOWA : FFZN^X \to FFZN$ *is specified as:*

$$FFZNOWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = \sum_{v=1}^{h} \lambda_v F_{Q_{\delta_v}}$$

with h *measurements, such that* v*th highest weighted value is* F_{Q_v} *therefore by overall order* $F_{Q_1} \geq F_{Q_2} \geq F_{Q_3} \geq, ..., \geq F_{Q_h}$ *and the weight vector* $\lambda_v = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ *with* $\lambda_v \geq 0 (v \in S)$ *and* $\sum_{v=1}^{h} \lambda_v = 1$.

Theorem 4.7. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = \Big((\mu_{T_v}, \mu_{V_v})(\nu_{T_v}, \nu_{V_v})\Big)$ *and* $v = (1,2,3,...,h)$ *be a group of the FFZNs.Then the sum total of FFZNOWA is a FFZN, applying the Definition 3.2 as follows:*

$$\begin{aligned} FFZNOWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} \lambda_v F_{Q_{\delta_v}} \\ &= \Bigg(\Bigg(\sqrt[3]{1 - \prod_{v=1}^{h} (1 - \mu^3_{T_{\delta_v}})^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{h} (1 - \mu^3_{V_{\delta_v}})^{\lambda_v}} \Bigg), \Bigg(\prod_{v=1}^{h} \nu^{\lambda_v}_{T_{\delta_v}}, \prod_{v=1}^{h} \nu^{\lambda_v}_{V_{\delta_v}} \Bigg) \Bigg) \end{aligned}$$

with h *measurements,such that* v*th highest weighted value is* F_{Q_v} *as a result, by the consequently order* $F_{Q_1} \geq F_{Q_2} \geq F_{Q_3} \geq, ..., \geq F_{Q_h}$ *and the weight vector* $\lambda_v = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ *with* $\lambda_v \geq 0$ $(v \in S)$ *and* $\sum_{v=1}^{h} \lambda_v = 1$.

Proof. The proof of this theorem is the same as above the FFZNWA operator because it also satisfies the properties required for aggregation operator. Thus, we leave it out. □

Property 4.8. *Inequality: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)), v \in S$ *be a group of FFZNs, if all* F_{Q_v} *are identical, then*

$$FFZNOWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = F_Q.$$

Property 4.9. *A monotonous state:Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *and* $F_{Q_{v'}} = (\mu_{v'}(T,V), \nu_{v'}(T,V))$ *where* $v, v' \in S$ *be a group of FFZNs, in such a way* $F_{Q_v} \subseteq F_{Q_{v'}}$. *Then,*

$$FFZNOWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNOWA(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}}).$$

Property 4.10. *Limitations: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *for all* $v \in S$ *be the FFZNs, such that* $F_{Q_l} = \max_v F_{Q_v}$ *and* $F_{Q_j} = \min_v F_{Q_v}$*. Then,*

$$F_{Q_j} \leq FFZNOWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq F_{Q_l}.$$

All that the weighted FFZNs averaged operator does is take into consideration the relevance of the concurrent Fermatean fuzzy sets. The FFZNOWA operator only compares the position of each Fermatean fuzzy set and the ranking order related to their relevance. In our attempts to address the limitations observed in the two above mentioned FFZNs aggregation operators, the FFZNs hybrid weighed aggregation operators shall also be defined.

Definition 4.11. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = \Big((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v})\Big)$ *and* $v = 1, 2, 3, ..., h$ *be a group of FFZNs and the FFZNHA operator in such a way that* $FFZNHA : FFZN^x \rightarrow FFZN$ *is specified as:*

$$FFZNHWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = \sum_{v=1}^{h} \lambda_v F'_{Q_{\delta_v}}$$

with h measurements, such that vth highest value is $F_{Q_{\delta_v}}$ *and* $F'_{Q_v} = h\tau_v F_{Q_v}, (v \in S), \lambda = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ *is the weight vectors such that* $\lambda_v \geq 0$ *and* $\sum\limits_{v=1}^{h} = 1$*. Also* $\tau_v = \{\tau_1, \tau_2, ..., \tau_h\}$ *is the associate weight vectors such that* $\tau_v \geq 0$ $(v \in S)$ *and* $\sum\limits_{v=1}^{h} \tau_v = 1$ *and balancing cofficient is h.*

Theorem 4.12. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = \Big((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v})\Big)$ *and* $v = 1, 2, 3, ..., h$ *be the FFZNs. Then the aggregated value of FFZNHA is a FFZN, by using Definition 3.2, we have:*

$$\begin{aligned} FFZNHWA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} \lambda_v F'_{Q_{\delta_v}} \\ &= \left(\begin{array}{c} \Big((\sqrt[3]{1 - \prod\limits_{v=1}^{h} (1 - \mu'^3_{T_{\delta_v}})^{\lambda_v}}, \sqrt[3]{1 - \prod\limits_{v=1}^{h} (1 - \mu'^3_{V_{\delta_v}})^{\lambda_v}}), \\ (\prod\limits_{v=1}^{h} \nu'^3_{T_{\delta_v}}, \prod\limits_{v=1}^{h} \nu'^3_{V_{\delta_v}}) \Big) \end{array} \right) \end{aligned}$$

with h measurements, such that vth highest value is $F_{Q_{\delta_v}}$ *and* $F'_{Q_v} = h\tau_v F_{Q_v}, (v \in S), \lambda = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ *is the weight vectors such that* $\lambda_v \geq 0$ *and* $\sum\limits_{v=1}^{h} = 1$*.Also* $\tau_v = \{\tau_1, \tau_2, ..., \tau_h\}$ *is the associate weight vectors such that* $\tau_v \geq 0$ $(v \in S)$ *and* $\sum\limits_{v=1}^{h} \tau_v = 1$ *and balancing cofficient is h.*

Property 4.13. *Inequality: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)), v \in S$ *be a group of FFZNs, if all* F_{Q_v} *are identical, then*

$$FFZNHA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = F_Q.$$

Property 4.14. *A monotonous state: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *and* $F_{Q_{v'}} = (\mu_{v'}(T,V), \nu_{v'}(T,V))$ *where* $v, v' \in S$ *be a group of FFZNs, such that* $F_{Q_v} \subseteq F_{Q_{v'}}$. *Then,*

$$FFZNHA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNHA(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}}).$$

Property 4.15. *Limitations: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *for all* $v \in S$ *be the FFZNs, such that* $F_{Q_l} = \max_v F_{Q_v}$ *and* $F_{Q_j} = \min_v F_{Q_v}$. *Then,*

$$F_{Q_j} \leq FFZNHA(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq F_{Q_l}.$$

4.2 Fermatean fuzzy Z-number weighted geometric aggregation operators

In this section, the definition of the Fermatean fuzzy Z-numbers weighted geometric aggregation operations is defined based on the algebric t-norm and t-conorm also the basic properties and corollaries of these geometric aggregation operators are defined.

Definition 4.16. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v}))$ *and* $v = 1, 2, 3, ..., h$ *be the FFZNs,and the FFZNWG operators such that* $FFZNWG : FFZN^x \to FFZN$ *is specified as:*

$$FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = \sum_{v=1}^{h} F_{Q_v}^{\lambda_v}$$

such that λ_v *and* $F_{Q_v}(v = 1, 2, 3, ..., h)$ *weight vector with* $0 \leq \lambda_v \leq 1$ *and* $\sum_{v=1}^{h} \lambda_v = 1$.

Theorem 4.17. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v})(\nu_{T_v}, \nu_{V_v}))$ *and* $v = (1, 2, 3, ..., h)$ *is a group of the FFZNs.Then we can aggregate the value of the FFZNWG, by using the Definition 3.2, we have:*

$$\begin{aligned} FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} F_{Q_v}^{\lambda_v} \\ &= \left(\Big(\prod_{v=1}^{h} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{h} \mu_{V_v}^{\lambda_v}\Big), \Big(\sqrt[3]{1 - \prod_{v=1}^{h}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{h}(1-\nu_{V_v}^3)^{\lambda_v}}\Big) \right) \end{aligned}$$

such that λ_v *represents* $F_{Q_v}(v = 1, 2, 3, ..., h)$ *weight vector with* $0 \leq \lambda_v \leq 1$ *and* $\sum_{v=1}^{h} = 1$.

Proof. We shall prove the above equation by mathematics induction. If $h = 2$, from operation Equation 2 and 4 in the Definition 3.2. We will arrive at the following conclusion:

$$F_{Q_1}^{\lambda_1} = \Big((\mu_{T_1}^{\lambda_1}, \mu_{V_1}^{\lambda_1}), \big(\sqrt[3]{1-(1-\nu_{T_1}^3)^{\lambda_1}}, \sqrt[3]{1-(1-\nu_{V_1}^3)^{\lambda_1}}\big)\Big)$$

$$F_{Q_2}^{\lambda_2} = \Big((\mu_{T_2}^{\lambda_2}, \mu_{V_2}^{\lambda_2}), \big(\sqrt[3]{1-(1-\nu_{T_2}^3)^{\lambda_2}}, \sqrt[3]{1-(1-\nu_{V_2}^3)^{\lambda_2}}\big)\Big)$$

$$
\begin{aligned}
FFZNWG(F_{Q_1}^{\lambda_1} \otimes F_{Q_2}^{\lambda_2}) &= \sum_{v=1}^{h} F_{Q_v}^{\lambda_v} \\
&= F_{Q_1}^{\lambda_1} \otimes F_{Q_2}^{\lambda_2} \\
&= \left(\begin{array}{c} \Big(\big(\mu_{T_1}^{\lambda_1}, \mu_{V_1}^{\lambda_1} \big), \big(\sqrt[3]{1-(1-\nu_{T_1}^3)^{\lambda_1}}, \sqrt[3]{1-(1-\nu_{V_1}^3)^{\lambda_1}} \big) \Big) \\ \otimes \Big(\big(\mu_{T_2}^{\lambda_2}, \mu_{V_2}^{\lambda_2} \big), \big(\sqrt[3]{1-(1-\nu_{T_2}^3)^{\lambda_2}}, \sqrt[3]{1-(1-\nu_{V_2}^3)^{\lambda_2}} \big) \Big) \end{array} \right) \\
&= \left(\begin{array}{c} \big(\mu_{T_1}^{\lambda_1} \mu_{T_2}^{\lambda_2}, \mu_{V_1}^{\lambda_1} \mu_{V_2}^{\lambda_2} \big), \\ \Big(\sqrt[3]{(1-(1-\nu_{T_1}^3)^{\lambda_1}) + (1-(1-\nu_{T_2}^3)^{\lambda_2}) - (1-(1-\nu_{T_1}^3)^{\lambda_1})(1-(1-\nu_{T_2}^3)^{\lambda_2}}\Big), \\ \Big(\sqrt[3]{(1-(1-\nu_{V_1}^3)^{\lambda_1}) + (1-(1-\nu_{V_2}^3)^{\lambda_2}) - (1-(1-\nu_{V_1}^3)^{\lambda_1})(1-(1-\nu_{V_2}^3)^{\lambda_2}}\Big) \end{array} \right) \\
&= \left(\ \Big(\mu_{T_1}^{\lambda_1} \mu_{T_2}^{\lambda_2}, \mu_{V_1}^{\lambda_1} \mu_{V_2}^{\lambda_2} \Big), \Big(\sqrt[3]{(1-(1-\nu_{T_1}^3)^{\lambda_1}(1-\nu_{T_2}^3)^{\lambda_2})}, \sqrt[3]{(1-(1-\nu_{T_1}^3)^{\lambda_1}(1-\nu_{T_2}^3)^{\lambda_2}\Big)} \ \right).
\end{aligned}
$$

This implies that

$$
FFZNWG(F_{Q_1}^{\lambda_1} \otimes F_{Q_2}^{\lambda_2}) = \left(\ \Big(\prod_{v=1}^{2} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{2} \mu_{V_v}^{\lambda_v} \Big), \Big(\sqrt[3]{1-\prod_{v=1}^{2}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{2}(1-\nu_{V_v}^3)^{\lambda_v}} \Big) \ \right).
$$

If $h=b$, then

$$
\begin{aligned}
FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \prod_{v=1}^{b} F_{Q_v}^{\lambda_v} \\
&= \left(\begin{array}{c} \Big(\prod_{v=1}^{b} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{b} \mu_{V_v}^{\lambda_v} \Big), \\ \Big(\sqrt[3]{1-\prod_{v=1}^{b}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b}(1-\nu_{V_v}^3)^{\lambda_v}} \Big) \end{array} \right)
\end{aligned}
$$

If $h=b+1$, then

$$
\begin{aligned}
FFZWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_b}, F_{Q_{b+1}}) &= \prod_{v=1}^{b} F_{Q_v}^{\lambda_v} \otimes F_{Q_{b+1}}^{\lambda_{b+1}} \\
&= \left(\begin{array}{c} \Big(\prod_{v=1}^{b} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{b} \mu_{V_v}^{\lambda_v} \Big), \\ \Big(\sqrt[3]{1-\prod_{v=1}^{b}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b}(1-\nu_{V_v}^3)^{\lambda_v}} \Big)) \\ \otimes F_{Q_{b+1}}^{\lambda_{b+1}} \end{array} \right) \\
&= \left(\begin{array}{c} \Big(\prod_{v=1}^{b+1} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{b+1} \mu_{V_v}^{\lambda_v} \Big), \\ \Big(\sqrt[3]{1-\prod_{v=1}^{b+1}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b+1}(1-\nu_{V_v}^3)^{\lambda_v}} \Big) \end{array} \right).
\end{aligned}
$$

This it means that $b+1$ holds. Thus we are true for all h and in this way it also completes the proof. $\square$

Property 4.18. *Inequality: Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)), v \in S$ be a group of FFZNs, if all F_{Q_v} are identical, then*

$$FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = F_Q.$$

Proof. Consider F_{Q_v} are identical and we also know that:

$$\begin{aligned}
FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} F_{Q_v}^{\lambda_v}, \\
&= \left(\left(\prod_{v=1}^{b} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{b} \mu_{V_v}^{\lambda_v}\right), \left(\sqrt[3]{1-\prod_{v=1}^{b}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{b}(1-\nu_{V_v}^3)^{\lambda_v}}\right) \right) \\
&= \left(\left(\mu_{T_v}^{\sum_{v=1}^{b}\lambda_v}, \mu_{V_v}^{\sum_{v=1}^{b}\lambda_v}\right), \left(\sqrt[3]{1-(1-\nu_{T_v}^3)^{\sum_{v=1}^{b}\lambda_v}}, \sqrt[3]{1-(1-\nu_{V_v}^3)^{\sum_{v=1}^{b}\lambda_v}}\right) \right) \\
&= \left(\left(\mu_{T_v}, \mu_{V_v}\right), \left(\sqrt[3]{1-(1-\nu_{T_v}^3)}, \sqrt[3]{1-(1-\nu_{V_v}^3)}\right) \right) \\
&= F_Q.
\end{aligned}$$

□

Property 4.19. *A monotonous state:Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ and $F_{Q_{v'}} = (\mu_{v'}(T,V), \nu_{v'}(T,V))$ where $v, v' \in S$ be a group of FFZNs, such that $F_{Q_v} \subseteq F_{Q_{v'}}$. Then,*

$$FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNWG(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}})$$

Proof. As we know $F_{Q_v} \subseteq F_{Q_{v'}}$. Then $\mu_v(T,V) \geq \mu_{v'}(T,V)$ and $\nu_v(T,V) \leq \nu_{v'}(T,V)$. This implies that

$$\begin{aligned}
\prod_{v=1}^{h} \mu_{T_v}^{\lambda_v} &\geq \prod_{v=1}^{h} \mu_{T_{v'}}^{\lambda_{v'}}, \\
\prod_{v=1}^{h} \mu_{V_v}^{\lambda_v} &\geq \prod_{v=1}^{h} \mu_{V_{v'}}^{\lambda_{v'}}, \\
\sqrt[3]{1-\prod_{v=1}^{h}(1-\nu_{T_v}^3)^{\lambda_v}} &\leq \sqrt[3]{1-\prod_{v'=1}^{h}(1-\nu_{T_v}^3)^{\lambda_{v'}}}, \\
\sqrt[3]{1-\prod_{v=1}^{h}(1-\mu_{V_v}^3)^{\lambda_v}} &\leq \sqrt[3]{1-\prod_{v'=1}^{h}(1-\nu_{V_v}^3)^{\lambda_{v'}}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left(\left(\prod_{v=1}^{h} \mu_{T_v}^{\lambda_v}, \prod_{v=1}^{h} \mu_{V_v}^{\lambda_v}\right), \left(\sqrt[3]{1-\prod_{v=1}^{h}(1-\nu_{T_v}^3)^{\lambda_v}}, \sqrt[3]{1-\prod_{v=1}^{h}(1-\nu_{V_v}^3)^{\lambda_v}}\right) \right) \\
\leq &\left(\left(\prod_{v=1}^{h} \mu_{T_{v'}}^{\lambda_{v'}}, \prod_{v=1}^{h} \mu_{V_{v'}}^{\lambda_{v'}}\right), \left(\sqrt[3]{1-\prod_{v'=1}^{h}(1-\nu_{T_v}^3)^{\lambda_{v'}}}, \sqrt[3]{1-\prod_{v'=1}^{h}(1-\nu_{V_v}^3)^{\lambda_{v'}}}\right) \right)
\end{aligned}$$

Hence, $FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNWG(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}})$. □

Property 4.20. *Limitation: Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ for all $v \in S$ be the FFZNs, such that $F_{Q_l} = \max_v F_{Q_v}$ and $F_{Q_j} = \min_v F_{Q_v}$. Then,*

$$F_{Q_j} \leq FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq F_{Q_l}$$

Proof. The proof of this property isnt very complicated at all. □

Definition 4.21. *Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = \Big((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v}) \Big)$ and $v = 1, 2, 3, ..., h$ be the FFZNs, and the FFZNOWG operators such that $FFZNOWG : FFZN^x \to FFZN$ is specified as:*

$$FFZNOWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = \sum_{v=1}^{h} F_{Q_{\delta_v}}^{\lambda_v}$$

with h measurements, such that vth highest weighted value is F_{Q_v} as a result, by the consequently order $F_{Q_1} \geq F_{Q_2} \geq F_{Q_3} \geq, ..., \geq F_{Q_h}$ and the weight vector $\lambda_v = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ with $\lambda_v \geq 0$ $(v \in S)$ and $\sum_{v=1}^{h} \lambda_v = 1$.

Theorem 4.22. *Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = \Big((\mu_{T_v}, \mu_{V_v}) (\nu_{T_v}, \nu_{V_v}) \Big)$ and $v = (1, 2, 3, ..., h)$ be a group of the FFZNs. Then the aggregated value of the FFZNOWG is a FFZN, based on the Definition 3.2. Hence, we have*

$$\begin{aligned} FFZNOWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \sum_{v=1}^{h} F_{Q_{\delta_v}}^{\lambda_v} \\ &= \left(\Big(\prod_{v=1}^{h} \mu_{T_{\delta_v}}^{\lambda_v}, \prod_{v=1}^{h} \mu_{V_{\delta_v}}^{\lambda_v} \Big), \Big(\sqrt[3]{1 - \prod_{v=1}^{h} (1 - \nu_{T_{\delta_v}}^3)^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{h} (1 - \nu_{V_{\delta_v}}^3)^{\lambda_v}} \Big) \right), \end{aligned}$$

with h measurements, such that the th largest weighted value is F_{Q_v}. Therefore, by the consequently order $F_{Q_1} \geq F_{Q_2} \geq F_{Q_3} \geq, ..., \geq F_{Q_h}$ and the weight vector $\lambda_v = \{\lambda_1, \lambda_2, ..., \lambda_h\}$ with $\lambda_v \geq 0$ $(v \in S)$ and $\sum_{v=1}^{h} \lambda_v = 1$.

Proof. The proof of this theorem is exactly the same as the one in the above described FFZNWG operator. Thus, it is left out. □

Property 4.23. *Inequality: Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)), v \in S$ be a group of FFZNs, if all F_{Q_v} are identical, then*

$$FFZNOWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = F_Q$$

Property 4.24. *A monotonous state: Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ and $F_{Q_{v'}} = (\mu_{v'}(T,V), \nu_{v'}(T,V))$ where $v, v' \in S$ be a group of FFZNs, such that $F_{Q_v} \subseteq F_{Q_{v'}}$. Then*

$$FFZNOWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNOWG(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}})$$

Property 4.25. *Limitations: Consider $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ for all $v \in S$ be the FFZNs, such that $F_{Q_l} = \max_v F_{Q_v}$ and $F_{Q_j} = \min_v F_{Q_v}$. Then*

$$F_{Q_j} \leq FFZNOWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq F_{Q_l}$$

All that the average operator, tied with weights, of the FFZNs does is incorporate the relevance of the combination of the Fermatean fuzzy sets. The position and ranking order of each Fermatean fuzzy set is the only circulating parameter considered by the output FFZNOWG operator. As would be seen next, we shall propose the FFZNs hybrid weighed aggregation operators in order to overcome the shortfalls of the two FFZNs aggregation operators discussed above.

Definition 4.26. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = ((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v}))$ *and* $v = 1, 2, 3, ..., h$ *be the FFZNs. Then the aggregated value of the amount of FFZNHG is a FFZN, based on Definition 3.2, we have*

$$FFZNWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = \sum_{v=1}^{h} F'^{\lambda_v}_{Q_{\delta_v}}$$

with h *measurements,such that the biggest weighted value is* $F_{Q_{\delta_v}}$ *and*

$$F'_{Q_v} = F^{h\tau_v}_{Q_v}, (v \in S), \lambda = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_h\}$$

is the weight vectors in such way that $\lambda_v \geq 0$ $(v \in S)$ *and* $\sum_{h}^{v=1} \lambda_v = 1$*. Also* $\tau_v = \{\tau_1, \tau_2, \tau_3, ..., \tau_h\}$ *is the associated weight vector such that* $\tau_v \geq 0$ $(v \in S)$ *and* $\sum_{h}^{v=1} \tau_v = 1$ *and balancing cofficient is* h*.*

Theorem 4.27. *Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)) = \Big((\mu_{T_v}, \mu_{V_v}), (\nu_{T_v}, \nu_{V_v})\Big)$ *and* $v = \{1, 2, 3, ..., h\}$ *be the FFZNs. Then the aggregated value of FFZHG is a FFZN, by using Definition 3.2, we have*

$$\begin{aligned} FFZNHWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) &= \prod_{h}^{v=1} (F'_{Q_{\delta_v}})^{\lambda_v} \\ &= \left(\frac{\Big(\prod_{v=1}^{b} \mu'^{\lambda_v}_{T_{\delta_v}}, \prod_{v=1}^{b} \mu'^{\lambda_v}_{V_{\delta_v}} \Big),}{\Big(\sqrt[3]{1 - \prod_{v=1}^{b} (1 - \nu'^{3}_{T_{\delta_v}})^{\lambda_v}}, \sqrt[3]{1 - \prod_{v=1}^{b} (1 - \nu'^{3}_{V_{\delta_v}})^{\lambda_v}} \Big)} \right), \end{aligned}$$

with h *measurements,such that the biggest weighted value is* $F_{Q_{\delta_v}}$ *and*

$$F'_{Q_v} = F^{h\tau_v}_{Q_v}, (v \in S), \lambda = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_h\}$$

is the weight vectors in such a way that $\lambda_v \geq 0$ $(v \in S)$ *and* $\sum_{h}^{v=1} \lambda_v = 1$*. Also* $\tau_v = \{\tau_1, \tau_2, \tau_3, ..., \tau_h\}$ *is the associated weight vector such that* $\tau_v \geq 0$ $(v \in S)$ *and* $\sum_{h}^{v=1} \tau_v = 1$ *and balancing cofficient is* h*.*

Property 4.28. *Inequality: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V)), v \in S$ *be a group of FFZNs, if all* F_{Q_v} *are identical, then*

$$FFZNHWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) = F_Q$$

Property 4.29. *A monotonous state: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *and* $F_{Q_{v'}} = (\mu_{v'}(T,V), \nu_{v'}(T,V))$ *where* $v, v' \in S$ *be a group of FFZNs, such that* $F_{Q_v} \subseteq F_{Q_{v'}}$*. Then*

$$FFZNHWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq FFZNHWG(F_{Q_{1'}}, F_{Q_{2'}}, F_{Q_{3'}}, ..., F_{Q_{h'}})$$

Property 4.30. *Limitations: Consider* $F_{Q_v} = (\mu_v(T,V), \nu_v(T,V))$ *for all* $v \in S$ *be the FFZNs, such that* $F_{Q_l} = \max_v F_{Q_v}$ *and* $F_{Q_j} = \min_v F_{Q_v}$*. Then*

$$F_{Q_j} \leq FFZNHWG(F_{Q_1}, F_{Q_2}, F_{Q_3}, ..., F_{Q_h}) \leq F_{Q_l}$$

5 Multi-criteria decision making

This section provides an illustration of an MCDM and an algorithm to deal with the recommended MCDM according to the averaging and geometric aggregation operators.

Assume that the attributes $\mathfrak{W} = \{\mathfrak{w}_1, \mathfrak{w}_1, ..., \mathfrak{w}_h\}$ and a set of alternatives $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_\mathfrak{m}\}$ have weight vectors $\lambda = \{\lambda_1, \lambda_1, ..., \lambda_h\}$. The weight vector defines that the total of the weights has to be equal to one and all of the weights has to be a member of a closed interval of the unit interval. Similarly, the next steps to take were synthesized to arrive at the most implementable solution following recommended aggregation operators.

Algorithm

Step 1 Regard the universal set and the weight vectors and attribute sets as inputs and establish the FFZNs decision matrix in accordance with the following procedures. Following the gathering of expert evaluation data regarding the attributes of each choice, $F_Q = [F_{Qjv}]m \times h$.

Step 2 Two main types of criteria that are most commonly employed are known collectively as the positive criterion, the negative criterion. Therefore by adhering to the complement of the negative criterion, it becomes necessary to translate negative requirements into positive requirements.

Step 3 Employ the arithmetic and geometric aggregation operators of FFZNs which it has described above to integrate the attributes for each choice.

Step 4 Applying Definitions 3.3 and 3.4 find out the score values for each possibility.

Step 5 Arrange all the given choices in descending order to select the optimal one.

The flowchart of the proposed algorithm is given in Figure 1.

6 Numerical illustration

The above-mentioned methodology has been explained with the help of an example, the essence of which is given below. The algorithms mentioned here can be used to assess the eye lenses and select the one that produces the maximum effect.

"Evaluating Eye Lenses: A Comparative Assessment of Performance, Comfort, and Cost" means that a high follow-up of lenses necessitates the need for a detailed evaluation of lenses. Three key characteristicsperformance, comfort, and costare used to analyze four different kinds of eye lenses: Categories of sun lenses include high-index plastic, polycarbonate, photochromic, and bifocal. This is because performance is made of certain factors, including scratch resistance and optical clarity, while comfort is best described as factors such as weight and fit. Cost focuses on the manufacturer's recommended price together with additional charges for the coatings and for the special features. An evaluation such as this one is beneficial to the customers in providing a solution to a specific business need and preference.

Since Fermatean fuzzy Z-numbers are an enhanced approach in comparing the features of eye lenses, this assessment can be enhanced using this technique. It becomes possible to combine subjectivity and ambiguity in the resulting ratings, which gives a much more profound vision of each kind of lens. The quantitative indicators of performance, comfort, and cost can be depicted at different levels by the fuzzy Z numbers,

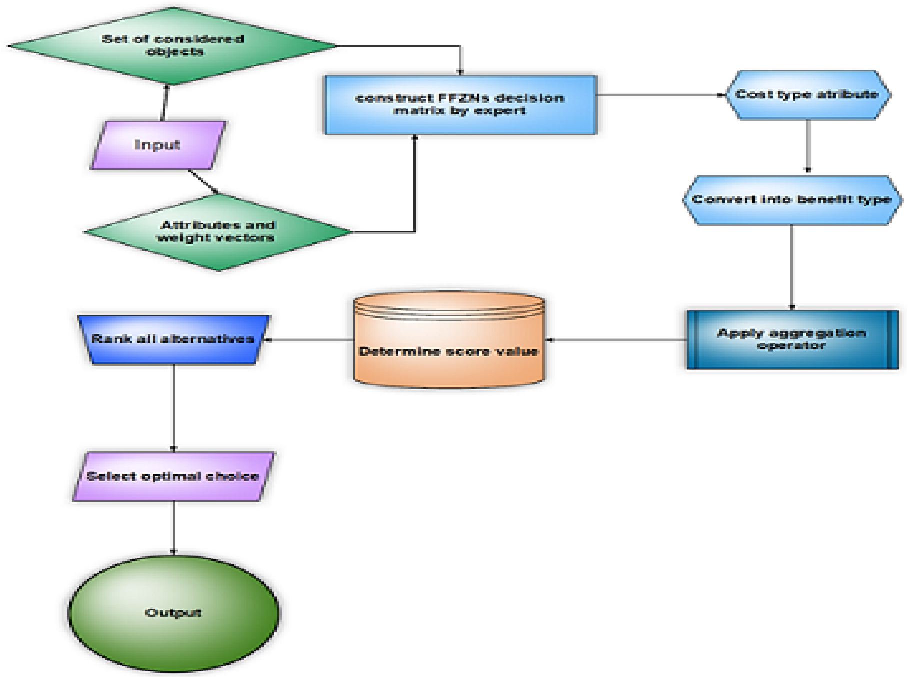

Figure 1: Flowchart of the Proposed Algorithm

which illustrate the customer experience complexity of the qualities. This method benefits the analysis by adding structure to it and also aids the customers in their decision-making by guiding them through the choices available and ensuring that what they choose is what they want or need. The four options under consideration for the evaluation of eyeglasses are as follows:

High-Index Plastic Lenses

Actually, as a plus, high-index plastic lenses are lighter and less thick than conventional lenses particularly for people who have a strong prescription. It is well preferred by designers and users because of its high light transmission and polish retention features. It is possible to achieve a wider field of view and elimination of distortions using these lenses. Despite the fact that the added layers might be cheaper in this regard, increasing the number of coatings can also increase the cost. They are popular particularly since they may enhance the look of usage of eyewear accessories.

Polycarbonate Lenses

Polycarbonate lenses are ideal in safety glasses and for anyone who is physically active, mainly because these lenses have excellent impact strength. But that design is better appreciated by children and sport-oriented people because it is lightweight and comfortable to wear for a long time. They may have a slightly lower scratch resistance than the high-index polycarbonate lenses but have wonderful optical density. As is understood, customers on a limited budget prefer polycarbonate lenses since they are normally cheaper. For use outside, their built-in UV ray protection adds at least another layer of assurance to provide.

Photochromic Lenses

The progressive lenses popularly referred to as photochromic lenses socially alter their tone in response to UV light, making it convenient for people who transition between indoor and outdoor clothes. They can reduce glare and provide better UV protection and therefore increase comfort while driving. While they are as slim and as light as high-index lenses, their speed of transition will depend on the temperature. Regardless of the fact that the cost of these lenses is relatively higher initially, it can effectively eliminate the necessity of having to own prescription sunglasses. Some of the wearers appreciate sandals for these shoes because they are easily accessible to them.

Bifocal Lenses

Different areas are provided for each distance, and thus bifocal lenses assist the people with near-sightedness and far-sightedness. It gives a good chance to use the product, which is very handy to correct presbyopia and to be able to see near and far without having to change a pair of glasses. However, they can sometimes limit the peripheral vision field, and new users might face some kind of difficulty with them in the beginning. Overall, bifocal lenses are cheaper than progressive lenses, and although not as eye-pleasing, single-vision options exist. The latter is true for numerous people, as it provides them with an uncomplicated design they could rely on.

Assume that the attributes $\mathfrak{W} = \{\mathfrak{w}_1, \mathfrak{w}_1, ..., \mathfrak{w}_h\}$ and a set of alternatives $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_\mathfrak{m}\}$ have weight vectors $\lambda = \{\lambda_1, \lambda_1, ..., \lambda_h\}$. The weight vector means that the sum of the components shall be equal to one and each of the components must be a closed unit interval. Next, we compiled the next steps to identify the best practicable solution based on the suggested aggregation operators.

6.1 By using FFZNWA and FFZNWG operator

Algorithm

Step 1 First, the information provided in the field by the expert in the FFZNs form is listed in Table 1.

Step 2 Since the introduced criterion is benefit-type, there is no necessity to normalize it.

Step 3 Employ the FFZNWA and FFZNWG operators in order to combine each of the alternatives properties as indicated in Tables 2 and 3.

Step 4 Determine the values of the scores as indicated in Tables 4 and 5.

Step 5 Rank all feasible options in descending order and choose the most suitable one.

By using FFZNWA operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4)$$

By using FFZNWG operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_4)$$

Therefore we concluded that $\mathfrak{p}_3$ is the maximum decision.

Table 1: Expert 1's Decision Matrix by the expert

F_{Q_j}	$\mathfrak{w}_1$	$\mathfrak{w}_2$	$\mathfrak{w}_3$
$\mathfrak{p}_1$	((0.6,0.2),(0.3,0.4))	((0.7,0.3),(0.4,0.4))	((0.4,0.4),(0.2,0.6))
$\mathfrak{p}_2$	((0.4,0.3),(0.3,0.5))	((0.2,0.7),(0.1,0.3))	((0.4,0.5),(0.4,0.6))
$\mathfrak{p}_3$	((0.6,0.3),(0.2,0.6))	((0.7,0.3),(0.2,0.5))	((0.2,0.5),(0.3,0.4))
$\mathfrak{p}_4$	((0.7,0.3),(0.4,0.6))	((0.6,0.4),(0.3,0.4))	((0.4,0.3),(0.2,0.7))

Table 3: Using FFZNWG operator

F_{Q_j}	$\mathfrak{w}_v$
$\mathfrak{p}_1$	((0.5131,0.3194),(0.3069,0.5234))
$\mathfrak{p}_2$	((0.3249,0.4994),(0.3365,0.5253))
$\mathfrak{p}_3$	((0.3628,0.3873),(0.2599,0.4858))
$\mathfrak{p}_4$	((0.5052,0.3270),(0.2929,0.6241))

Table 2: Using FFZNWA operator

F_{Q_j}	$\mathfrak{w}_v$
$\mathfrak{p}_1$	((0.5725,0.3476),(0.2670,0.4899))
$\mathfrak{p}_2$	((0.3623,0.5645),(0.2491,0.4699))
$\mathfrak{p}_3$	((0.5470,0.4260),(0.2449,0.4638))
$\mathfrak{p}_4$	((0.5573,0.3369),(0.2595,0.5738))

Table 4: By using FFZNWA operator

F_{Q_j}	scoring
$\mathfrak{p}_1$	0.5341
$\mathfrak{p}_2$	0.5437
$\mathfrak{p}_3$	0.5597
$\mathfrak{p}_4$	0.5194

Table 5: By using FFZNWG operator

F_{Q_j}	scoring
$\mathfrak{p}_1$	0.5016
$\mathfrak{p}_2$	0.4927
$\mathfrak{p}_3$	0.5071
$\mathfrak{p}_4$	0.4912

6.2 By using FFZNOWA & FFZNOWG operator

Algorithm

Step 1 First, the information provided in the field by the expert in the FFZNs form is listed in Table 6.

Step 2 Since the criterion introduced is benefit-type in nature, there is no need for normalization.

Step 3 Reorder the FFZN in relation to each attribute as shown in Table 7 after evaluating the score value of each FFZN.

Step 4 Integrating the attributes for each possibility used the FFZNOWA and FFZNOWG operators demonstrated in the Tables 8 and 9 respectively.

Step 5 Determine the values of the scores as indicated in Tables 6 and 10.

Step 6 Rank all feasible options in descending order and choose the most suitable one.
By using FFZNOWA operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4)$$

By using FFZNOWG operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_4)$$

Table 6: By using FFZNOWA operator

F_{Q_j}	scoring
$\mathfrak{p}_1$	0.5265
$\mathfrak{p}_2$	0.5304
$\mathfrak{p}_3$	0.5550
$\mathfrak{p}_4$	0.5127

Table 7: Re-ordered Decision Matrix by the expert

$F_{Q_{j_v}}$	$\mathfrak{w}_{\delta(1)}$	$\mathfrak{w}_{\delta(2)}$	$\mathfrak{w}_{\delta(3)}$
$\mathfrak{p}_1$	((0.7,0.3),(0.4,0.4))	((0.4,0.4),(0.2,0.6))	((0.6,0.2),(0.3,0.4))
$\mathfrak{p}_2$	((0.2,0.7),(0.1,0.3))	((0.4,0.3),(0.3,0.5))	((0.4,0.5),(0.4,0.6))
$\mathfrak{p}_3$	((0.7,0.3),(0.2,0.5))	((0.6,0.3),(0.2,0.6))	((0.2,0.5),(0.3,0.4))
$\mathfrak{p}_4$	((0.6,0.4),(0.3,0.4))	((0.4,0.3),(0.2,0.7))	((0.7,0.3),(0.4,0.6))

Table 8: Using FFZNOWA operator

F_{Q_j}	$\mathfrak{w}_{\delta(v)}$
$\mathfrak{p}_1$	((0.5867,0.3069),(0.2814,0.4517))
$\mathfrak{p}_2$	((0.3758,0.5278),(0.2781,0.4945))
$\mathfrak{p}_3$	((0.5298,0.4260),(0.2449,0.4724))
$\mathfrak{p}_4$	((0.6241,0.3256),(0.3067,0.5795))

Table 9: Using FFZNOWG operator

F_{Q_j}	$\mathfrak{w}_{\delta(v)}$
$\mathfrak{p}_1$	((0.5479,0.2670),(0.3069,0.4827))
$\mathfrak{p}_2$	((0.3482,0.4588),(0.3438,0.5360))
$\mathfrak{p}_3$	((0.3573,0.3873),(0.2599,0.4991))
$\mathfrak{p}_4$	((0.5738,0.3178),(0.3424,0.6124))

Table 10: By using FFZNOWG operator

F_{Q_j}	scoring
$\mathfrak{p}_1$	0.4991
$\mathfrak{p}_2$	0.4878
$\mathfrak{p}_3$	0.5043
$\mathfrak{p}_4$	0.4863

6.3 By using FFZNHWA and FFZNHWG operator

Algorithm

Step 1 First, the information provided in the field by the expert in the FFZNs form is listed in Table 1.

Step 2 Since the criterion introduced is benefit-type in nature, there is no need for normalization.

Step 3 Use τ_v weighted vectors, which are shown in Tables 11 and 12, to evaluate the weighted values matrix.

Step 4 Reorder the FFZN in Table 13, and Table 14 shows the relation to each characteristic after evaluating the score value of each FFZN weighted value.

Step 5 Use the FFZNHA and FFZNHG operators to integrate the properties for each possibility in Table 15, and Table 16 respectively.

Step 6 Determine the values of the scores.

Step 7 Rank all feasible options in descending order and choose the most suitable one.
By using FFZNHWA operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4)$$

By using FFZNHWG operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4) > \Im(\mathfrak{p}_2)$$

Therefore we concluded that $\mathfrak{p}_3$ is the maximum decision.

Table 11: By using FFZNHWA operator

F'_{Q_j}	scoring
$\mathfrak{p}_1$	0.5095
$\mathfrak{p}_2$	0.5293
$\mathfrak{p}_3$	0.5376
$\mathfrak{p}_4$	0.5000

Table 12: By using FFZNHWG operator

F'_{Q_j}	scoring
$\mathfrak{p}_1$	0.5220
$\mathfrak{p}_2$	0.5169
$\mathfrak{p}_3$	0.5352
$\mathfrak{p}_4$	0.5177

Table 13: Weighted value Decision Matrix using HA operator

$F'_{Q_{jv}}$	$\mathfrak{w}_{\delta(1)}$	$\mathfrak{w}_{\delta(2)}$	$\mathfrak{w}_{\delta(3)}$
$\mathfrak{p}_1$	((0.6136,0.3299),(0.2611,0.4301))	((0.6582,0.3574),(0.1798,0.3403))	((0.4974,0.2626),(0.5110,0.6559))
$\mathfrak{p}_2$	((0.3666,0.5736),(0.2035,0.4203))	((0.3969,0.6167),(0.1307,0.3303))	((0.2922,0.4631),(0.4511,0.6483))
$\mathfrak{p}_3$	((0.5988,0.4032),(0.2035,0.4661))	((0.6429,0.4361),(0.1307,0.3771))	((0.4847,0.3218),(0.4511,0.6827))
$\mathfrak{p}_4$	((0.6136,0.3492),(0.2611,0.5262))	((0.6582,0.3782),(0.1798,0.4402))	((0.4974,0.2782),(0.5110,0.7254))

Table 14: Weighted value Decision Matrix using HG operator

$F'_{Q_{jv}}$	$\mathfrak{w}_{\delta(1)}$	$\mathfrak{w}_{\delta(2)}$	$\mathfrak{w}_{\delta(3)}$
$\mathfrak{p}_1$	((0.5262,0.2611),(0.3299,0.5020))	((0.4402,0.1798),(0.3574,0.5414))	((0.7254,0.5110),(0.2626,0.4028))
$\mathfrak{p}_2$	((0.2896,0.4443),(0.3222,0.5135))	((0.2053,0.3546),(0.3491,0.5536))	((0.5382,0.6665),(0.2565,0.4125))
$\mathfrak{p}_3$	((0.4100,0.3275),(0.2494,0.5283))	((0.3200,0.2401),(0.2705,0.5691))	((0.6403,0.5722),(0.1982,0.4248))
$\mathfrak{p}_4$	((0.5262,0.3022),(0.3299,0.6136))	((0.4402,0.2167),(0.3574,0.6582))	((0.7254,0.5497),(0.2626,0.4974))

Table 15: Using FFZNHWA operator

F_{Q_j}	$\mathfrak{w}_{\delta(v)}$
$\mathfrak{p}_1$	((0.5814,0.3107),(0.3266,0.4951))
$\mathfrak{p}_2$	((0.3454,0.5428),(0.2653,0.4856))
$\mathfrak{p}_3$	((0.5672,0.3801),(0.2653,0.5294))
$\mathfrak{p}_4$	((0.5814,0.3290),(0.3266,0.5856))

Table 16: Using FFZNHWG operator

F_{Q_j}	$\mathfrak{w}_{\delta(v)}$
$\mathfrak{p}_1$	((0.5856,0.3266),(0.3107,0.4741))
$\mathfrak{p}_2$	((0.3561,0.5086),(0.3035,0.4852))
$\mathfrak{p}_3$	((0.4756,0.3944)(0.2348,0.4993))
$\mathfrak{p}_4$	((0.5856,0.3689),(0.3107,0.5814))

7 COCOSO technique for FFZNs

The CoCoSo approach is obtained based on the formulation of Fermatean fuzzy Z-numbers, and it can be implemented to solve decision-making problems in the case of the information provided in the form of Fermatean fuzzy Z-numbers. Fuzzy Z-numbers are an enrichment of the classical fuzzy numbers because they use both, reliability and feasibility aspects of uncertainty. The decision-making problem in a situation of uncertainty can be solved using the proposal of CoCoSo in the framework of Fermatean fuzzy Z-numbers. It combines reliability and feasibility issues, which makes decision making more objective and comprehensive since the prioritized aspects are considered for evaluation of choices.

Assume that the attributes $\mathfrak{W} = \{\mathfrak{w}_1, \mathfrak{w}_1, ..., \mathfrak{w}_h\}$ and a set of alternatives $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_\mathfrak{m}\}$ have weight vectors $\lambda = \{\lambda_1, \lambda_1, ..., \lambda_h\}$. The weight vector means that the sum of the components shall be equal to one and each of the components must be a closed unit interval. Next, we compiled the next steps to identify the best practicable solution based on the suggested aggregation operators.

Algorithm

Step 1 Regard the universal set and the weight vectors and attribute sets as inputs and establish the FFZNs decision matrix in accordance with the following procedures. Following the gathering of expert evaluation data regarding the attributes of each choice, $F_Q = [F_{Qjv}]m \times h$.

Step 2 Two main types of criteria that are most commonly employed are known collectively as a positive criterion, the negative criterion. Therefore by adhering to the complement of the negative criterion, it becomes necessary to translate negative requirements into positive requirements.

Step 3 Apply the arithmetic and geometric aggregation operators of FFZNs, which were covered previously, to integrate the attributes for each choice.

Step 4 Integrating the attributes for each possibility used the FFZNOWA and FFZNOWG operators demonstrated in the Tables 8 and 9 respectively.

Step 5 Finally, we compute the scores for the weighted sum and weighted product measures using a proposed scoring function.

$$\Im(F_{Q_v}) = \frac{1 + (\mu_{T_v}.\mu_{V_v}) - (\nu_{T_v}.\nu_{V_v})}{2}$$

Step 6 There are three methods for rating the evaluation that can be used to evaluate the relative importance measurement of the plan:

$$\mathfrak{W}_v^1 = \frac{F_{Z_A}+F_{Z_G}}{\sum\limits_{v=1}^{h}(F_{Z_A}+F_{Z_G})}$$

$$\mathfrak{W}_v^2 = \frac{F_{Z_A}}{minF_{Z_A}} + \frac{F_{Z_G}}{minF_{Z_G}}$$

$$\mathfrak{W}_v^3 = \frac{\phi F_{Z_A}+(1-\phi)F_{Z_G}}{\phi maxF_{Z_A}+(1-\phi)maxF_{Z_G}}, \phi \in [0,1]$$

Step 7 The final appraisal index $\mathfrak{W}_v$ calculated by summing outcomes of the three score methods that were previously covered.

$$\mathfrak{W}_v = \sqrt[3]{\mathfrak{W}_v^1\mathfrak{W}_v^2\mathfrak{W}_v^3} + \frac{\mathfrak{W}_v^1\mathfrak{W}_v^2\mathfrak{W}_v^3}{3}$$

Step 8 Designs are arranged in descending order of $\mathfrak{W}_v$ value.

7.1 Numerical illustration by using COCOSO method

In this section, the applicability and efficiency of CoCoSO algorithm are presented in the context of MCDM, and for this purpose, the same case of the previous section is considered again.

Algorithm

Step 1 First, the information provided in the field by the expert in the FFZNs form is listed in Table 1.

Step 2 Since the criterion introduced is benefit-type in nature, there is no need for normalization.

Step 3 Apply arithmetic & geometric aggregation operators of FFZNs, which were covered previously, to integrate the attributes for each choice.

Step 4 Using the above proposed scoring function, we find the score of the summation of the weighted sum and weighted product measures.

$$\Im(F_{Q_v}) = \frac{1 + (\mu_{T_v}.\mu_{V_v}) - (\nu_{T_v}.\nu_{V_v})}{2}$$

Step 5 Apply the three methods for rating the evaluation that can be used to evaluate the relative importance measurement of the plan.

Step 6 The final appraisal index $\mathfrak{W}_v$ calculated in Table 17 by summing outcomes of the three score methods that were previously covered. Calculated the score values by using CoCoCo technique in Table 18.

Step 8 Designs are arranged in descending order of $\mathfrak{W}_v$ value.
By using CoCoSo operator

$$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_4)$$

Therefore we concluded that $\mathfrak{p}_3$ is the maximum decision.

Table 17: Using CoCoSo operators

$F_{Q_{jv}}$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$
$\mathfrak{p}_1$	0.2496	2.0494	0.9779
$\mathfrak{p}_2$	0.2498	2.0498	0.9716
$\mathfrak{p}_3$	0.2571	2.1099	1
$\mathfrak{p}_4$	0.2436	2	0.9556

Table 18: By using CoCoSo operator

F_{Q_j}	scoring
$\mathfrak{p}_1$	1.8861
$\mathfrak{p}_2$	1.8827
$\mathfrak{p}_3$	1.9379
$\mathfrak{p}_4$	1.8414

8 Comparison Analysis

An analysis has been made according to the results of both the recommended algorithms and some of the Fermatean fuzzy Z-number measures. The best value is the similar in all three methods of ranking, but the order of ranking is a little different.CoCoSo technique further explains the ranking and graphical representation of all the operators consisting the following options, FFZNWA, FFZNOWA, FFZNHA, FFZNWG, FFZNOWG AND FFZNHG. The weighted FFZNs averaged operator only considers the importance of the sets of aggregated Fermatean fuzzy Z-numbers. The relevance of each Fermatean fuzzy set's position and ranking order are the only factors taken into account by the FFZNOWA operator. The hybrid weighted aggregating operator of FFZNs takes into account both the order and weighted operator abilities at the same time.The recommended strategy works better since the decision-makers can select the operators and features according to the situations and their own demands.
The ranking of proposed operators with CoCoSo approach is given in Table 19.

Table 19: Ranking using various operator and CoCoSo approach

Operators	Final Ranking
$FFZNWA$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4)$
$FFZNWG$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_4)$
$FFZNOWA$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4)$
$FFZNOWG$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_4)$
$FFZNHWA$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4)$
$FFZNHWG$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_4) > \Im(\mathfrak{p}_2)$
$CoCoSo$	$\Im(\mathfrak{p}_3) > \Im(\mathfrak{p}_1) > \Im(\mathfrak{p}_2) > \Im(\mathfrak{p}_4)$

The graphical representation of the FFZNs' rank is presented in Figure 2.

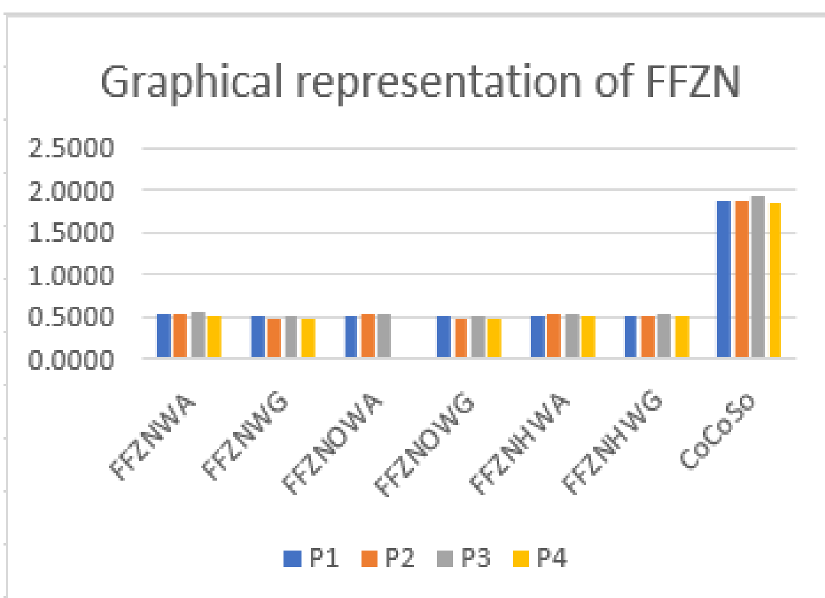

Figure 2: Graphical Representation of FFZNs Ranking

9 Conclusion

The purpose of this work is to design initial working rules for FFZNs that implement aggregation operators. The created operating laws are then used to create new operators, such as FFZNWA, FFZNOWA, FFZNHA, FFZNWG, FFZNOWG, and FFZNHG.

For the suggested aggregation operators, a number of basic characteristics, theorems, and traits are also provided. The problems related to MADM have been solved using a DM method based on recommended operators and the CoCoSo strategy. It includes the parameter in the calculation process so that judgments are made by taking into consideration both the positive and negative factors.But the main drawback of traditional CoCoSo is that it can't handle ambiguity or imperfect information when making decisions. However, it still has certain shortcomings, such as inclusive and illogical problems. To address this issue, standard CoCoSo will be used together with Fermatean fuzzy sets and fuzzy Z- numbers. We also give a thorough mathematical example. Ultimately, it is concluded from the data that the method this study proposes is the most advantageous and successful way to address the MADM problem.Future work will concentrate on creating new approaches to decision-making for the FFZNs scenario that utilize a variety of operators, including EDAS and the more generalized geometric and average operators proposed by Einstein and Frank, to improve the effectiveness of DM.

Future Works

The following are some directions for further study of Fermatean fuzzy Z-numbers:

1. FFZ's mathematical characteristics It is imperative to conduct additional research on the mathematical features of FFZ, including distance measures, aggregation operators, and algebraic operations.

2. Using FFZ in Decision-Making: FFZ can be used to make decisions in a variety of situations, such as risk assessment, fuzzy decision-making, and multi-criteria decision-making. Future research may explore decisions made using FFZ in such and other circumstances such as those mentioned above.

3. Hybrid models: To increase the precision of decision-making models, hybrid models that incorporate FFZ with additional methods such artificial neural networks, evolutionary algorithms, and approximate sets may be created.

4. Optimization algorithms: Another possible direction for future research is the creation of optimization techniques for FFZ-based decision-making issues.

5. Real-world applications: Lastly, empirical research that applies FFZ to actual decision-making situations is required.

Conflict of Interest: The authors declare no conflict of interest.

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