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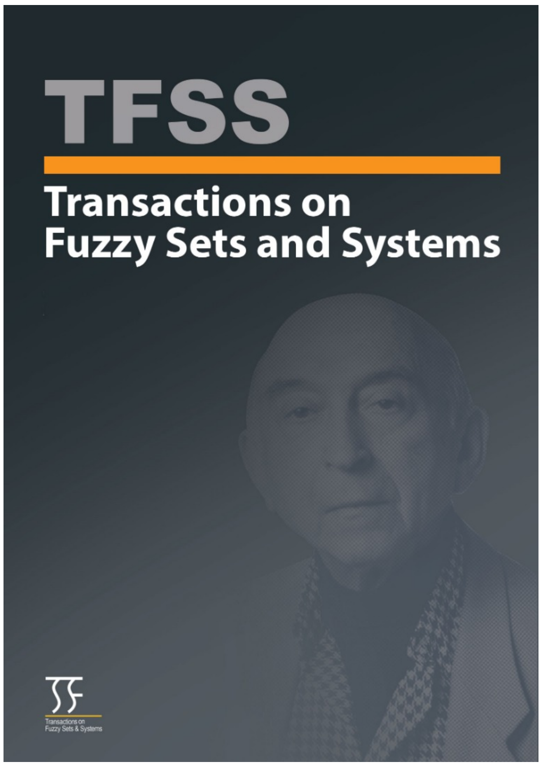

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Fuzzy Subgroup-Based Centralizer-Graph

Elahe Mohammadzadeh , **Mohammad Hamidi** * , **Somayeh Amirmofidi**

Abstract. In this paper, we introduce the notion of (non)commutative fuzzy subgroup-based centralizer-graph with respect to any given non-abelian group. Basically, we investigate on dominating set of the class of (non)commutative fuzzy subgroup-based centralizer-graphs. Also, with some additional conditions we see that the (non)commutative fuzzy subgroup-based centralizer-graphs are connected. Also, we investigate on isomorphic (non)commutative fuzzy subgroup-based centralizer-graphs and make a new isomorphic graph of (non)abelian groups derived from an isomorphic (non)commutative fuzzy subgroup-based centralizer-graph. Also, we see that if any given underlined fuzzy subset is an extra-special fuzzy subgroup, then the related fuzzy subgroup-based centralizer-graphs are isomorphic and if any given underlined fuzzy subset is good nilpotent of class 2, then it is an extra-special fuzzy subgroup.

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Keywords and Phrases: Centralizer-graph, Fuzzy subgroupiod, Good nilpotent fuzzy group, Dominating set.

1 Introduction

In the life, one of the most important applications of mathematics is modelling real problems in the form of mathematical data. Due to the lawfulness of nature, this modelling might be accurate when the modelled systems are designed based on certain rules and regulations. Algebraic groups are legal systems that play an important role in mathematics. In fact, a group is a collection of objects that forms an algebraic system under the principles and basic laws. On the other hand, each pure set alone cannot be very efficient, so labelling the elements of each pure set can fix this defect and strengthen the use of this set. The motivation to display fuzzy subsets as an distribution of pure subsets can be a way to make set theory widely used. In [1], Zadeh define a fuzzy set. This notion has been applied on several sciences such as mathematics, computer science, chemistry and so on. With the spread of the theory of fuzzy subsets, many researchers developed algebraic theories into fuzzy algebraic theories. Especially, the theory of fuzzy groups might be considered as one of the first algebraic classes developed in terms of fuzzy algebras, as structures [2, 3, 4]. Due to the importance of graph theory in mathematical modelling, the development of this important branch of mathematics to fuzzy graph was proposed and researched by many researchers, when Kaufmann displayed the view of fuzzy graph in [5]. In addition, fuzzy information might be analysed by adopting a fuzzy graph. Also, Azriel Rosenfeld displayed fuzzy graph in 1975 by the notion of fuzzy relations. Then Ameri, Davvaz, Hamidi, and Nasiri [6, 7, 8, 9] continued the investigation on fuzzy graphs and obtained several results. Recently, many researchers have developed and related the structure and algebraic hyperstructures with fuzzy graph theory,

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which has the aspect of innovation and utilization in industry and other sciences. For instance you can see [10, 11, 12, 13, 14, 15].

In this paper, we present a graph with respect to a fuzzy group A with membership function $ð$, which is known as fuzzy subgroup-based centralizer-graph and is denoted by $\Gamma^{ð(g)}(\mathcal{G})$. Basically, the dominating set of $\Gamma_{\mathcal{G}}^{ð(e)}$ is studied. In addition, we investigate some properties of $\Gamma^{ð(g)}(\mathcal{G})$ such as diameter of this graph. Also, by isomorphic graphs we obtain some results on good nilpotent fuzzy groups. Basically, extra-special analogies of fuzzy groups are defined and we obtain a relation between extra-special groups and fuzzy groups.

Motivation: We apply the view of fuzzy subgroupoid as a distribution of fuzzy subgroups as a useful tool in the construction of a new type of centralizer subset in the group structures. Since the commutator operations are helpful in the construction of nilpotent groups, we extend this view to the good nilpotent fuzzy subgroups. The main motivation o this work is based on the non- groups to generate of non-trivial good nilpotent fuzzy subgroups. On this point we display the view of the fuzzy subgroup-based centralizer-graph with underlying non-abelian groups. The dominating set is very important in networks, which is related to graph theory. In today's life, the most important issue is optimization in time, so that every user seeks to do their work with the best quality and in the shortest possible time. In doing this important issue of choice the most important principle is that the selection of the set of inclusions is the solution to the problem. In this research, first, we convert an arbitrary set into a group according to the principles of algebraic rules, and we emphasize the non-commutative of elements, and with the help of the fuzzy set theory, we convert it into a fuzzy subgroup, and considering the importance of the centralizer set we convert it to a complex network. This complex network is a modelled real example, and considering the importance of the dominating set, we discuss its dominating number. According to the properties of nilpotent groups and centralizing series, we choose the dominating set in this network in a special way.

2 Preliminaries

We remember some preliminary definitions used in this paper. A *simple graph* $\mathcal{G}$ contains two finite non-empty sets, vertices $V = V(\mathcal{G}) = \{v_i\}_{i=1}^{n}$ and edges $E = E(\mathcal{G}) = \{e_i\}_{i=1}^{m}$. For $u, v \in V$ we use $u \bowtie v$, when they are *adjacent*, i.e get an edge joining u and v. A simple graph with n vertices, in which each one $u, v \in V$ we have $u \bowtie v$, is a *complete* graph and is indicated by K_n. A *path* in $\mathcal{G}$ is a finite distinct sequence of vertices and edges of the form $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ and $v_0 = v_n$, make it a *cycle*. In a *connected* graph get a path between each $u, v \in V$. A graph is named *disconnected*, if it is not connected and $t(\mathcal{G})$ is the number of *connected components* of a graph $\mathcal{G}$. For the simple graph $\mathcal{G}$ and $x, y \in V(\mathcal{G})$, $d(x, y)$ is known as the *length* of a path with the minimum number of edges and $diam(\mathcal{G}) = sup\{d(x, y) \mid x, y \in V\}$. We write $\mathcal{G} \cong H$, for two *isomorphic* graphs. It means that get a bijective function $\varphi : V(\mathcal{G}) \longrightarrow V(H)$ that preserves edges. The function φ is then known as an isomorphism. A subgraph H of $\mathcal{G}$ with $V(H) = V(\mathcal{G})$, is named as spanning subgraph of $\mathcal{G}$ and is indicated by $H \sqsubseteq \mathcal{G}$.

Definition 2.1. [16]*Presume $\mathfrak{X}$ be a non-void set. Then $\mathcal{A} = \{(x, ð(x)) \mid x \in \mathfrak{X}\}$ is known as a fuzzy subset of $\mathfrak{X}$(simply as a fuzzy set), in which for the membership function $ð : \mathfrak{X} \to [0, 1]$ and any $x \in \mathfrak{X}, ð(x)$ is titled as the degree of membership of x in the fuzzy set $\mathcal{A}$. Using $\mathcal{A}_ð \in \mathcal{F}(\mathfrak{X})$, means that $\mathcal{A}$ is a fuzzy set of $\mathfrak{X}$ with the membership function $ð$. Sometimes for simplify we write $\mathcal{A} \in \mathcal{F}(\mathfrak{X})$.*

For any $x, y \in \mathfrak{X}$ we use $x =_ð y$, instead of $ð(x) = ð(y)$.

Definition 2.2. [17]*Presume $(\mathcal{G}, \cdot)$ be a group and $\mathcal{A} \in \mathcal{F}(\mathcal{G})$. Then $\mathcal{A}$ is named a fuzzy subgroup if each one $x, y \in \mathcal{G}$, $ð(xy) \geq min\{ð(x), ð(y)\}$ and $ð(x^{-1}) \geq ð(x)$. Moreover, $\mathcal{A}$ with additional condition $xy =_ð yx$ is named normal. We use $\mathbb{FS}(\mathcal{G})$ and $\mathbb{FN}(\mathcal{G})$ for the class of all fuzzy subgroups and normal fuzzy subgroups of*

$\mathcal{G}$, respectively. Also, $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is called commutative, if $\mathbf{Ce}(\mathcal{A}) = \mathcal{G}$ which

$$\mathbf{Ce}(\mathcal{A}) = \{x \in \mathcal{G} \mid xy =_{\eth} yx \text{ and } xyz =_{\eth} yxz, \forall y, z \in \mathcal{G}\}$$

is named the centralizer of $\mathcal{A}$ in $\mathcal{G}$.

Using [18], each one group $(\mathcal{G}, \cdot)$ the *commutator* of $x, y \in \mathcal{G}$ is $\mathbb{C}x, y = x^{-1} \cdot y^{-1} \cdot x \cdot y$, and inductively,$\mathbb{C}^n{}_{i=1}x, y_i = \mathbb{C}x, y_1, y_2, ..., y_n = \mathbb{C}(\mathbb{C}^{n-1}{}_{i=1}x, y_i), y_n \ \forall \ y_1, y_2, ..., y_n \in \mathcal{G}$. Also, $\mathcal{G}$ is called *nilpotent* if $\mathbf{Ce}_n(\mathcal{G})$ terminates at finite steps, where

$$\mathbf{Ce}_n(\mathcal{G}) = \{x \in \mathcal{G}; \mathbb{C}^n{}_{i=1}x, y_i = e, \forall \ y_1, y_2, ..., y_n \in \mathcal{G} \ \}.$$

For any non-empty subsets $\mathfrak{X}_1$ and $\mathfrak{X}_2$ of $\mathcal{G}$ define

$$\mathbb{C}\mathfrak{X}_1, \mathfrak{X}_2 = \langle \mathbb{C}x_1, x_2 | x_1 \in \mathfrak{X}_1, x_2 \in \mathfrak{X}_2 \rangle.$$

Moreover, define $\mathbb{C}^n_{i=2}\mathfrak{X}_1, \mathfrak{X}_i = \mathbb{C}(\mathbb{C}^{n-1}_{i=2}\mathfrak{X}_1, \mathfrak{X}_i), \mathfrak{X}_n$, where $n \geq 2$ and $\mathbb{C}\mathfrak{X}_1 = \langle \mathfrak{X}_1 \rangle$. Also, derived series of $\mathcal{G}$ is defined by $... \subseteq \mathcal{G}_n \subseteq \ldots \subseteq \mathcal{G}_0 = \mathcal{G}$; where for each integer $n > 1$, $\mathcal{G}_n = \mathbf{C}\mathcal{G}_{n-1} = \mathbb{C}\mathcal{G}_{n-1}, \mathcal{G}_{n-1}$. Mohammadzadeh et al., defined the notion of nested normal series as follows. Presume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$. Put $\mathbf{Ce}_0(\mathcal{A}) = \{e\}$. Clearly $\mathbf{Ce}_0(\mathcal{A}) \trianglerighteq \mathcal{G}$. Presume $\mathbf{Ce}_1(\mathcal{A}) = \{x \in \mathcal{G} \mid \ \mathbb{C}x, y =_{\eth} e, \forall \ y \in \mathcal{G}\}$. One can see that $\mathbf{Ce}_1(\mathcal{A}) = \mathbf{Ce}(\mathcal{A})$ that is a normal subgroup of $\mathcal{G}$. Mohammadzadeh et al. defined a subgroup $\mathbf{Ce}_2(\mathcal{A})$ of $\mathcal{G}$ for to $\dfrac{\mathbf{Ce}_2(\mathcal{A})}{\mathbf{Ce}_1(\mathcal{A})} = \mathbf{Ce}\left(\dfrac{\mathcal{G}}{\mathbf{Ce}_1(\mathcal{A})}\right)$ and $\mathbf{Ce}_1(\mathcal{A}) \trianglerighteq \mathbf{Ce}_2(\mathcal{A})$, because of $\mathbf{Ce}_1(\mathcal{A}) \trianglerighteq \mathcal{G}$. Assume that $x \in \mathbf{Ce}_2(\mathcal{A})$ and $g \in \mathcal{G}$. Thus $x\mathbf{Ce}_1(\mathcal{A}) \in \dfrac{\mathbf{Ce}_2(\mathcal{A})}{\mathbf{Ce}_1(\mathcal{A})} = \mathbf{Ce}\left(\dfrac{\mathcal{G}}{\mathbf{Ce}_1(\mathcal{A})}\right)$, which implies that $\mathbb{C}x\mathbf{Ce}_1(\mathcal{A}), g\mathbf{Ce}_1(\mathcal{A}) = \mathbf{Ce}_1(\mathcal{A})$ each one $g \in \mathcal{G}$. Accordingly $\mathbb{C}x, g \in \mathbf{Ce}_1(\mathcal{A})$. Hence $\mathbb{C}\mathbf{Ce}_2(\mathcal{A}), \mathcal{G} \subseteq \mathbf{Ce}_1(\mathcal{A})$ and so $x^g = x\mathbb{C}x, g \in \mathbf{Ce}_2(\mathcal{A})$, which imply that $\mathbf{Ce}_2(\mathcal{A}) \trianglerighteq \mathcal{G}$. In a similar way each one $k \geq 2$, it is defined a normal subgroup $\mathbf{Ce}_k(\mathcal{A})$ for to $\dfrac{\mathbf{Ce}_k(\mathcal{A})}{\mathbf{Ce}_{k-1}(\mathcal{A})} = \mathbf{Ce}\left(\dfrac{\mathcal{G}}{\mathbf{Ce}_{k-1}(\mathcal{A})}\right)$. It is clear that $\ldots \supseteq \mathbf{Ce}_2(\mathcal{A}) \supseteq \mathbf{Ce}_1(\mathcal{A}) \supseteq \mathbf{Ce}_0(\mathcal{A})$.

Lemma 2.3. [19] *Presume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$, $x \in \mathcal{G}$ and $i \geq 1$. Then*

(i) for any $k \geq 2$, $\mathbf{Ce}_k(\mathcal{A}) = \{x \in \mathcal{G} \mid (\mathbb{C}^k{}_{i=1}x, y_i) =_{\eth} e, \ \forall y_1, y_2, \ldots, y_k \in \mathcal{G} \ \}$.

(ii) each one $y \in \mathcal{G}$, $\mathbb{C}x, y \in \mathbf{Ce}_{i-1}(\mathcal{A})$ iff $x \in \mathbf{Ce}_i(\mathcal{A})$.

Presume $\mathcal{A} \in \mathbb{FN}(\mathcal{G})$. For any $x, y \in \mathcal{G}$, define the equvalence relation $x \sim y \Leftrightarrow (xy^{-1}) =_{\eth} e$. It is a congruence relation (see [19]). Also, $x\eth = \{y \in \mathcal{G} | \ x \sim y\}$ and $\dfrac{\mathcal{G}}{\mathcal{A}} = \{x\eth \mid x \in \mathcal{G}\}$. From [19], $(\dfrac{\mathcal{G}}{\mathcal{A}}, \cdot, e\eth)$ is a group, that for any $x\eth, y\eth \in \dfrac{\mathcal{G}}{\mathcal{A}}$, $(x\eth)(y\eth) = xy\eth$, $e\eth$ is unit of $\dfrac{\mathcal{G}}{\mathcal{A}}$ and $(x\eth)^{-1} = x^{-1}\eth$.

Theorem 2.4. [19] *Presume $\mathcal{A} \in \mathbb{FN}(\mathcal{G})$ and $x, y \in \mathcal{G}$. Then*

(i) A is a good nilpotent fuzzy subgroup (we write $\mathcal{A} \in \mathcal{G}\mathbf{NF}$) iff $\dfrac{\mathcal{G}}{\mathcal{A}}$ is a nilpotent group.

(ii) Then $x =_{\eth} y$ iff $x\eth = y\eth$.

3 Graphs based on fuzzy subgroup

In this section, we define a graph with respect to a fuzzy subgroup $\mathcal{A}$, which is known as fuzzy subgroup-based centralizer-graph and is denoted by $\Gamma^{\eth(g)}(\mathcal{G})$. then $\Gamma^{\eth(g)}_{\mathcal{G}}$. Basically, the dominating set of $\Gamma^{\eth(e)}(\mathcal{G})$ is studied. In addition, we investigate some properties of $\Gamma^{\eth(g)}(\mathcal{G})$ such as diameter of this graph. Also, by isomorphic graphs we obtain some results on good nilpotent fuzzy groups.

Definition 3.1. *Presume $\mathcal{G}$ be a non-abelian group (we write $\mathcal{G} \in \mathbb{NA}$), $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ and $e \neq g \in \mathcal{G}$ be a fixed element. The fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(g)}(\mathcal{G})$ associated with group $\mathcal{G}$ and $e \neq g \in \mathcal{G}$ is the simple graph having $\mathcal{G} \setminus Z(\mathcal{A})$ as its vertex set, where $x \bowtie y$ if, $(\mathbb{C}x, y) \neq_{\eth} g$. If $g = e$ and $(\mathbb{C}x, y) \neq_{\eth} e$, then $\Gamma^{\eth(e)}(\mathcal{G})$ is non-commutative fuzzy subgroup-based centralizer-graph and if define $(\mathbb{C}x, y) =_{\eth} e$, it is called a commutative fuzzy subgroup-based centralizer-graph and will denote by $\gamma^{\eth(e)}(\mathcal{G})$.*

Example 3.2. Assume the symmetric group $(S_3, \cdot)$ with the Table 1 and define $\mathcal{A}_{\eth} \in \mathbb{FS}(S_3)$ with $\eth(x) = \begin{cases} t_0 & \text{if } x = e \\ t_1 & \text{otherwise} \end{cases}$, whence $t_1 < t_0$.

Table 1: Symmetric Group $(S_3, \cdot)$

$\cdot$	$\underline{\alpha}$	$\underline{\beta}$	$\underline{\gamma}$	$\underline{\delta}$	$\underline{\zeta}$	$\underline{\xi}$
$\underline{\alpha}$	$\underline{\beta}$	$\underline{\xi}$	$\underline{\zeta}$	$\underline{\gamma}$	$\underline{\zeta}$	$\underline{\alpha}$
$\underline{\beta}$	$\underline{\xi}$	$\underline{\alpha}$	$\underline{\delta}$	$\underline{\zeta}$	$\underline{\gamma}$	$\underline{\beta}$
$\underline{\gamma}$	$\underline{\delta}$	$\underline{\zeta}$	$\underline{\xi}$	$\underline{\alpha}$	$\underline{\beta}$	$\underline{\gamma}$
$\underline{\delta}$	$\underline{\zeta}$	$\underline{\gamma}$	$\underline{\beta}$	$\underline{\zeta}$	$\underline{\alpha}$	$\underline{\delta}$
$\underline{\zeta}$	$\underline{\gamma}$	$\underline{\delta}$	$\underline{\alpha}$	$\underline{\beta}$	$\underline{\xi}$	$\underline{\zeta}$
$\underline{\xi}$	$\underline{\alpha}$	$\underline{\beta}$	$\underline{\gamma}$	$\underline{\delta}$	$\underline{\zeta}$	$\underline{\xi}$

Clearly, $\mathbf{Ce}(S_3) = \{\underline{\xi}\}$ and each one $x \in S_3$

$$(\mathbb{C}\underline{\xi}, x) = \underline{\xi}. \text{ Also, } (\mathbb{C}\underline{\beta}, \underline{\alpha}) = (\mathbb{C}\underline{\alpha}, \underline{\beta}) = \underline{\xi}$$
$$(\mathbb{C}\underline{\zeta}, \underline{\delta}) = (\mathbb{C}\underline{\gamma}, \underline{\zeta}) = (\mathbb{C}\underline{\beta}, \underline{\zeta}) = (\mathbb{C}\underline{\delta}, \underline{\gamma}) = (\mathbb{C}\underline{\zeta}, \underline{\beta}) = (\mathbb{C}\underline{\delta}, \underline{\beta}) = (\mathbb{C}\underline{\gamma}, \underline{\beta}) = (\mathbb{C}\underline{\alpha}, \underline{\delta}) = (\mathbb{C}\underline{\alpha}, \underline{\gamma}) = \underline{\alpha}$$
$$(\mathbb{C}\underline{\delta}, \underline{\zeta}) = (\mathbb{C}\underline{\zeta}, \underline{\gamma}) = (\mathbb{C}\underline{\gamma}, \underline{\delta}) = (\mathbb{C}\underline{\beta}, \underline{\delta}) = (\mathbb{C}\underline{\beta}, \underline{\gamma}) = (\mathbb{C}\underline{\zeta}, \underline{\alpha}) = (\mathbb{C}\underline{\delta}, \underline{\alpha}) = (\mathbb{C}\underline{\gamma}, \underline{\alpha}) = (\mathbb{C}\underline{\alpha}, \underline{\zeta}) = \underline{\beta}.$$

Since $(\mathbb{C}\underline{\alpha}, \underline{\beta}) =_{\eth} (\mathbb{C}\underline{\beta}, \underline{\alpha}) = t_0 = \eth(\underline{\xi})$ and each one $x \neq y \in \{\underline{\zeta}, \underline{\delta}, \underline{\gamma}\}, \eth(\mathbb{C}x, y) = t_1 = \eth(\underline{\zeta})$, we have the fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(\underline{\zeta})}(\mathcal{G})$ as Figure 1a and the fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(\underline{\xi})}(\mathcal{G})$ as Figure 1b associated with the group $(S_3, \cdot)$.

(a) Fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(\underline{\zeta})}(\mathcal{G})$ **(b)** Fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(\underline{\xi})}(\mathcal{G})$

Figure 1: Fuzzy subgroup-based centralizer-graphs $\Gamma^{\eth(\underline{\zeta})}(\mathcal{G})$ and $\Gamma^{\eth(\underline{\xi})}(\mathcal{G})$

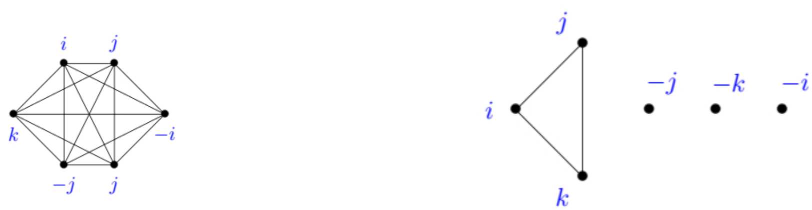
Example 3.3. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Define $\mathcal{A}_{\eth} \in \mathbb{FS}(Q_8)$ with

$$\eth(x) = \begin{cases} 1 & x = 1, \\ \frac{1}{2} & x = -1, \\ \frac{1}{3} & x \in Q_8 - \{\pm 1\}, \\ 0 & \text{otherwise.} \end{cases}.$$

Computations show that $\mathbf{Ce}_0(\mathcal{A}) = \{1\}, \mathbf{Ce}_1(\mathcal{A}) = \mathbf{Ce}(\mathcal{A}) = \{\pm 1\}, \mathbf{Ce}_2(\mathcal{A}) = Q_8$. In addition,

$$\begin{aligned}
&(\mathbb{C}i, j) =_{\eth} (\mathbb{C}i, k) =_{\eth} (\mathbb{C}j, k) =_{\eth} (-1) = \frac{1}{2} \neq \eth(i) = \frac{1}{3},\\
&(\mathbb{C}i, -i) =_{\eth} (\mathbb{C}j, -j) =_{\eth} (\mathbb{C}k, -k) =_{\eth} (\mathbb{C} - 1, -1) =_{\eth} (1) \neq \eth(i), \textit{and}\\
&(\mathbb{C} - 1, i) =_{\eth} (\mathbb{C} - 1, j) =_{\eth} (\mathbb{C} - 1, k) =_{\eth} (1) \neq \eth(i).
\end{aligned}$$

Accordingly, $\Gamma^{\eth(i)}(Q_8)$ is a complect graph with $V(\mathcal{G}) = \{\pm i, \pm j, \pm k\}$ in Figure 2a. Also, the non-commutative graph $\Gamma^{\eth(1)}(Q_8)$ is pictured as Figure 2b.



(a) Fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(i)}(\mathcal{G})$

(b) Fuzzy subgroup-based centralizer-graph $\Gamma^{\eth(1)}(\mathcal{G})$

Figure 2: Fuzzy subgroup-based centralizer-graphs $\Gamma^{\eth(i)}(\mathcal{G})$ and $\Gamma^{\eth(1)}(\mathcal{G})$

In what follows, each one $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ and $x \in \mathcal{G}$, we consider $C^{\mathcal{A}}_{\mathcal{G}}(x) = \{a \in \mathcal{G} \mid (\mathbb{C}a, x) =_{\eth} e\}$. Also, each one given non-empty subset $H \subseteq \mathcal{G}$,

$$C^{\mathcal{A}}_{\mathcal{G}}(H) = \{g \in \mathcal{G} \mid (\mathbb{C}g, h) =_{\eth} e, \forall h \in H\}.$$

Presume $\mathcal{G} = (V, E)$ be a graph and $D \subseteq V$. We recall that D is a dominating set if for every vertex $x \in V \setminus D$, get a vertex $y \in D$ for to $\{x, y\} \in E$.

Theorem 3.4. *Presume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$, $x \in \mathcal{G}$, $\mathbf{Ce}(\mathcal{A}) \neq \mathcal{G}$ and $D = \{x\}$ be a dominating set in $\Gamma^{\eth(e)}(\mathcal{G})$. Then*

(i) $\mathbf{Ce}(\mathcal{A}) = \{e\}$,

(ii) $x^2 = e$,

(iii) $C^{\mathcal{A}}_{\mathcal{G}}(x) = \{e, x\}$.

Proof. *(i)* Assume that $\mathbf{Ce}(\mathcal{A})$ is a non-trivial and $z \in \mathbf{Ce}(\mathcal{A})$. Since $D = \{x\}$ is a dominating set in $\Gamma^{\eth(e)}(\mathcal{G})$, we get that $x \bowtie w$ each one $w \in \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A})$ in the graph $\Gamma^{\eth(e)}(\mathcal{G})$ and $(\mathbb{C}x, z) =_{\eth} e$. Then we have

$$(\mathbb{C}xz, x) =_{\eth} ((\mathbb{C}x, x)^z(\mathbb{C}x, z)) \geq \eth((\mathbb{C}x, x)^z) \wedge \eth(\mathbb{C}x, z) = \eth(\mathbf{C}x^z) \wedge \eth(\mathbb{C}x, z) = \eth(e).$$

Accordingly, $xz \not\bowtie x$ that is opposite to $\{x\}$ is a dominating set . Consequently, $\mathbf{Ce}(\eth) = \{e\}$.

(ii) If $x^2 \neq e$, then $x \neq x^{-1}$, while $(\mathbb{C}x, x^{-1}) =_{\eth} e$. Accordingly, $x^{-1} \not\bowtie x$, which it is a contradiction. Hence, $x^2 = e$.

(iii) Take $x \neq y \in C^{\mathcal{A}}_{\mathcal{G}}(x)$. Then $(\mathbb{C}y, x) =_{\eth} e$ and it implies that $y \not\bowtie x$, which is a contradiction. So $y = x$ and so $C^{\mathcal{A}}_{\mathcal{G}}(x) = \{e, x\}$. $\square$

Theorem 3.5. *Assume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is non-commutative. Then $H \subseteq V(\Gamma^{\eth(e)}(\mathcal{G}))$ is a dominating set iff ${C^{\mathcal{A}}}_{\mathcal{G}(H)} \subseteq \mathbf{Ce}(\mathcal{A}) \cup H$.*

Proof. Suppose H is a dominating set. If $a \notin \mathbf{Ce}(\mathcal{A}) \cup H$, then by the definition of dominating set, we have $x \in H$ such that $(\mathbb{C}a, x) \neq_{\eth} e$. Thus, $a \notin C_{\mathcal{G}}^{\mathcal{A}}(H)$. Accordingly, $C_{\mathcal{G}}^{\mathcal{A}}(H) \subseteq \mathbf{Ce}(\mathcal{A}) \cup H$. Now, if $C_{\mathcal{G}}^{\mathcal{A}}(H) \subseteq \mathbf{Ce}(\mathcal{A}) \cup H$ and $a \notin H$, then $a \notin C_{\mathcal{G}}^{\mathcal{A}}(H)$. It implies that $(\mathbb{C}a, x) \neq_{\eth} e$ for some $x \in H$, and so $a \bowtie x$ where $x \in H$. Thus H is a dominating set in the simple graph $\Gamma^{\eth(e)}(\mathcal{G})$. $\square$

Corollary 3.6. *Assume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is non-commutative and $\mathfrak{X}$ be a generating set for $\mathcal{G}$. Then $\mathfrak{X} \setminus \mathbf{Ce}(\mathcal{A})$ is a dominating set for $\Gamma^{\eth(e)}(\mathcal{G})$.*

Put $K^{\mathcal{A}}(\mathcal{G}) = \{\eth(\mathbb{C}x, y) \mid x, y \in \mathcal{G}\}$. It is obviouse that $\Gamma^{\eth(g)}(\mathcal{G})$ is a complete graph, whenever $\eth(g) \notin K^{\mathcal{A}}(\mathcal{G})$, and so every thing is known. Thus, we always assume that $\eth(e) \neq \eth(g) \in K^{\mathcal{A}}(\mathcal{G})$.

Example 3.7. Consider the cyclic group $\mathcal{G} = (\mathbb{Z}_3, +)$ and any arbitrary $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$. It is easy to see that each one $\overline{x}, \overline{y} \in \mathcal{G}$, we have $(\mathbb{C}\overline{x}, \overline{y}) =_{\eth} (\overline{0})$. Accordingly, $\gamma^{\eth(\overline{0})}(\mathcal{G})$ is a null graph.

Lemma 3.8. *Presume $e \neq g \in \mathcal{G}$. Then $\gamma^{\eth(e)}(\mathcal{G}) \sqsubseteq \Gamma^{\eth(g)}(\mathcal{G})$.*

Proof. Clearly, the vertex set is $V(\gamma^{\eth(e)}(\mathcal{G})) = V(\Gamma^{\eth(g)}(\mathcal{G})) = \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A})$. If $x \bowtie y$ in $\gamma^{\eth(e)}(\mathcal{G})$, then $(\mathbb{C}x, y) =_{\eth} e$. Accordingly, $(\mathbb{C}x, y) \neq_{\eth} g$ and so $x \bowtie y$ in $\Gamma^{\eth(g)}(\mathcal{G})$. Hence $\gamma^{\eth(e)}(\mathcal{G}) \subseteq \Gamma^{\eth(g)}(\mathcal{G})$. $\square$

Lemma 3.9. *Presume $e \neq g \in \mathcal{G}$. If $K^{\mathcal{A}}(\mathcal{G}) = \{\eth(g), \eth(e)\}$, then $\gamma^{\eth(e)}(\mathcal{G}) \cong \Gamma^{\eth(g)}(\mathcal{G})$.*

Proof. If $x \bowtie y$ in $\Gamma^{\eth(g)}(\mathcal{G})$. Then $(\mathbb{C}x, y) \neq_{\eth} g$. Since $\eth(\mathbb{C}x, y) \in K^{\mathcal{A}}(\mathcal{G})$ we conclude that $(\mathbb{C}x, y) =_{\eth} e$ and so $x \bowtie y$ in $\gamma^{\eth(e)}(\mathcal{G})$. $\square$

Lemma 3.10. *Assume $\mathcal{A} \in \mathbb{FN}(\mathcal{G})$ and $(\mathbb{C}x, y) \neq_{\eth} e$, then*

(*i*) $\operatorname{diam}(\Gamma^{\eth(g)}(\mathcal{G})) = 2$.

(*ii*) $\Gamma^{\eth(g)}(\mathcal{G})$ *is connected.*

Proof. $(i), (ii)$ Let $x \neq g$ be a vertex of $\Gamma^{\eth(g)}(\mathcal{G})$ and $g =_{\eth} (\mathbb{C}x, g)$. Then by Theorem 2.4, we have

$$\begin{aligned} g\eth = (\mathbb{C}x, g)\eth = (\mathbb{C}x\eth, g\eth) = x^{-1}\eth g^{-1}\eth x\eth g\eth &\Longleftrightarrow e\eth = x^{-1}\eth g^{-1}\eth x\eth \\ &\Longleftrightarrow g\eth = e\eth \\ &\Longleftrightarrow g =_{\eth} e, \end{aligned}$$

that is a contradiction. Accordingly, $(\mathbb{C}x, g) \neq_{\eth} g =_{\eth} g^{-1}$. Consequently, $x \bowtie g$, and so $\operatorname{diam}(\Gamma^{\eth(g)}(\mathcal{G})) \leq 2$. Since $g =_{\eth} (\mathbb{C}x_1, x_2)$ for $x_1, x_2 \in \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A})$, we have $d(x_1, x_2) \geq 2$. Accordingly, $\operatorname{diam}(\Gamma^{\eth(g)}(\mathcal{G})) = 2$. $\square$

Theorem 3.11. *Presume $\mathcal{G} \in \mathbb{NA}$ be simple, $\mathcal{A}\mathbb{FS}(\mathcal{G})$ and $|C_{\mathcal{G}}^{\mathcal{A}}(x)| = m \geq 3$. Then $\Gamma^{\eth(g)}(\mathcal{G})$ has no isolated vertex.*

Proof. Presume $x \in \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A})$. If $x^2 \neq e$, then $x \bowtie x^{-1}$. Assume x is an involution and $|C_{\mathcal{G}}^{\mathcal{A}}(x)| = m \geq 3$. Thus, get $t \in C_{\mathcal{G}}^{\mathcal{A}}(x)$ such $t \neq e, x$ and so $t \bowtie x$. Accordingly, $\Gamma^{\eth(g)}(\mathcal{G})$ has no isolated vertex. $\square$

Theorem 3.12. *Presume $\mathcal{G} \in \mathbb{NA}$ and $x \in \mathcal{G}$ be any vertex in $\Gamma^{\eth(e)}(\mathcal{G})$. Then $deg(g) =\mid \mathcal{G} \mid - \mid C_{\mathcal{G}}^{\mathcal{A}}(x) \mid$.*

Proof. Presume $x \in \mathcal{G}$. Then $deg(x)$ is equal with the number of $y \in \mathcal{G}$ for to $(\mathbb{C}x, y) \neq_{\eth} e$. So $y \notin C_{\mathcal{G}}^{\mathcal{A}}(x)$ and so $deg(g) =\mid \mathcal{G} \mid - \mid C_{\mathcal{G}}^{\mathcal{A}}(x) \mid$. $\square$

Theorem 3.13. *Presume $\mathcal{G} \in \mathbb{NA}$. Then the non-commutative fuzzy subgroup-based centralizer-graph is not isomorphic to a path graph.*

Proof. Presume the non-commutative fuzzy subgroup-based centralizer-graph is isomorphic to a path graph. Then get a $x \in \mathcal{G}$ for to $deg(x) = 1$. Now, based Theorem 3.12, we get that $1 =\mid \mathcal{G} \mid - \mid C_{\mathcal{G}}^{\mathcal{A}}(x) \mid$. So $\mid C_{\mathcal{G}}^{\mathcal{A}}(x) \mid= 1$ and it is opposite to $\{e, x\} \subseteq C_{\mathcal{G}}^{\mathcal{A}}(x)$. $\square$

Presume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$. We say that $\mathcal{A}$ is transferable, if each one $x, y, z \in \mathcal{G}, (x^{-1}y) =_{\eth} z$ implies that $y =_{\eth} (xz)$.

Theorem 3.14. *Presume $\mathcal{G} \in \mathbb{NA}$, $e \neq g \in \mathcal{G}$, $g^2 = e$ and $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is transferable. If each one $x \in \mathcal{G}$, get $y \in \mathcal{G}$ such that $(x^y) =_{\eth} (xg)$, then $deg(x) \geq\mid \mathcal{G} \mid - \mid C_{\mathcal{G}}^{\mathcal{A}}(x) \mid$.*

Proof. Presume $\mathcal{G} \in \mathbb{NA}$ and $x \in \mathcal{G}$. Since $g \neq e$ and $g^2 = e$, we have $g = g^{-1}$. If for some $y \in \mathcal{G}$, $(x^y) =_{\eth} (xg)$, then $T^{\mathcal{A}} = \{y \in \mathcal{G} \mid (x^y) =_{\eth} (xg)\} \neq \emptyset$, because by $(y^{-1}xy) =_{\eth} ((x^{-1})^{-1}g)$ and by hypotheses, we get that $(x^{-1}y^{-1}xy) =_{\eth} g$, which it means that $(\mathbb{C}x, y) =_{\eth} g$.

Also, each one $y \in T^{\mathcal{A}}$, we have $(\mathbb{C}x, y) =_{\eth} g$, thus $y \not\bowtie x$. Now, for $y \in \mathcal{G}$, we prove that

$$y \not\bowtie x \Leftrightarrow y \in T^{\mathcal{A}}.$$

Since $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is transferable, we obtain that

$$y \not\bowtie x \Leftrightarrow (\mathbb{C}x, y) =_{\eth} g \Leftrightarrow (x^{-1}x^y) =_{\eth} g \Leftrightarrow (x^y) =_{\eth} (xg) \Leftrightarrow y \in T^{\mathcal{A}}.$$

Hence, each one $x \in T^{\mathcal{A}}$, we obtain that $(x^x) =_{\eth} (xg)$ and since $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is transferable, we get that $e =_{\eth} g$, that is opposit to assumption. Accordingly, $1+ \mid T^{\mathcal{A}} \mid=\mid \{y \in \mathcal{G} \mid y \text{ is not adjacent to } x\} \mid$. But, $T^{\mathcal{A}} \subseteq C_{\mathcal{G}}^{\mathcal{A}}(x)y_1$, whence $y_1 \in T^{\mathcal{A}}$. Because each one given $y_2 \in T^{\mathcal{A}}$, we have $({y_2}^{-1}xy_2) =_{\eth} (xg) =_{\eth} ({y_1}^{-1}xy_1)$. Because $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is transferable, we obtain:

$$\begin{aligned}
({y_1}^{-1}xy_1) =_{\eth} ({y_2}^{-1}xy_2) &\Rightarrow (y_2{y_1}^{-1}xy_1) =_{\eth} (x({y_2}^{-1})^{-1}) \\
&\Rightarrow (y_2{y_1}^{-1}xy_1{y_2}^{-1}) =_{\eth} x =_{\eth} ((x^{-1})^{-1}e) \\
&\Rightarrow (x^{-1}y_2{y_1}^{-1}xy_1{y_2}^{-1}) =_{\eth} e \\
&\Rightarrow (\mathbb{C}y_2{y_1}^{-1}, x) =_{\eth} e \\
&\Rightarrow y_2{y_1}^{-1} \in C_{\mathcal{G}}{}^{\mathcal{A}}(x) \\
&\Rightarrow y_2 \in C_{\mathcal{G}}^{\mathcal{A}}(x)y_1.
\end{aligned}$$

Accordingly, $|T^{\mathcal{A}}| \leq |C_{\mathcal{G}}^{\mathcal{A}}(x)|$. Consequently,

$$|\{y \in \mathcal{G} \mid y \not\bowtie x\}| = 1 + |T^{\mathcal{A}}| \leq |C_{\mathcal{G}}^{\mathcal{A}}(x)| + 1$$

and so $deg(x) = |\mathcal{G}| - |\{y \in \mathcal{G} \mid y \not\bowtie x\}| \geq |\mathcal{G}| - C_{\mathcal{G}}^{\mathcal{A}}(x)| - 1$. $\square$

Definition 3.15. *Assume $\mathcal{A}_{\eth} \in \mathbb{FS}(\mathcal{G})$ and $B_{\nu} \in \mathbb{FS}(H)$ and $g \in \mathcal{G}, h \in H$. Then $\Gamma^{\eth(g)}(\mathcal{G})$ and $\Gamma^{\nu(h)}(H)$ are said to be isomorphic, and is denoted by $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$, if get a bijection map $\varphi : \mathcal{G} \backslash \mathbf{Ce}(\mathcal{A}) \to H \backslash \mathbf{Ce}(B)$ preserving edges, i.e, for two distinct vertices $x, y \in \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A})$, we have $x \bowtie y$ iff $\varphi(x) \bowtie \varphi(y)$ in H.*

Theorem 3.16. *Presume $\mathcal{A} \in \mathbb{FN}(\mathcal{G})$, $B \in \mathbb{FN}(H)$ and $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$. Then*

(i) $\Gamma^{g\eth}_{\frac{\mathcal{G}}{\mathcal{A}}} \simeq \Gamma^{h\nu}_{\frac{H}{B}}$.

(ii) If $\mathcal{A}$ is non-commutative, then B is non-commutative.

Proof. Since $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$, get an isomorphism $f : \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A}) \to H \setminus \mathbf{Ce}(B)$ preserving edges. It means that each one $x, y \in \mathcal{G}$, $(\mathbb{C}x, y) \neq_{\eth} g$ iff $(\mathbb{C}f(x), f(y)) \neq_{\nu} h$. Then each one $x\eth, y\eth \in \dfrac{\mathcal{G}}{\mathcal{A}}$, we define a map $\varphi : \dfrac{\mathcal{G}}{\mathcal{A}} \setminus \mathbf{Ce}(\dfrac{\mathcal{G}}{\mathcal{A}}) \to \dfrac{H}{B} \setminus \mathbf{Ce}(\dfrac{H}{B})$ by $\varphi(x\eth) = f(x)\nu$. Clearly, φ is an isomorphism. Also, if $x\eth \sim y\eth$, then

$$\begin{aligned}
\mathbb{C}x\eth, y\eth \neq g\eth &\Leftrightarrow (\mathbb{C}x, y) \neq_{\eth} y \\
&\Leftrightarrow (\mathbb{C}f(x), f(y)) \neq_{\nu} h \\
&\Leftrightarrow (\mathbb{C}f(x), f(y))\nu \neq h\nu \\
&\Leftrightarrow \mathbb{C}f(x)\nu, f(y)\nu \neq h\nu \\
&\Leftrightarrow \mathbb{C}\varphi(x\eth), \varphi(y\eth) \neq h\nu.
\end{aligned}$$

Consequently, $\varphi(x\eth) \sim \varphi(y\eth)$ and so $\Gamma^{g\eth}_{\frac{\mathcal{G}}{\mathcal{A}}} \simeq \Gamma^{h\nu}_{\frac{H}{B}}$.
(ii) Assume $\mathcal{A}$ is non-commutative and B is commutative, then $\mathcal{G} \setminus \mathbf{Ce}(\mathcal{A}) \neq \phi$ and $H = \mathbf{Ce}(B)$. Accordingly $\Gamma^{\nu(h)}(H)$ has no vertices. On the other hand since $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$ get a bijection map $\varphi : \mathcal{G} \setminus \mathbf{Ce}(\mathcal{A}) \to H \setminus \mathbf{Ce}(B)$, that is a contradiction. $\square$

In [9], Nasiri et al. proved the following theorem in regard to cardinal of groups. In what follows, we get this result on isomorphic non-commutative fuzzy subgroup-based centralizer-graphs.

Theorem 3.17. *Presume $\mathcal{G}, H \in \mathbb{NA}$ be finite groups for to $\Gamma^g_{\mathcal{G}} \simeq \Gamma^h_H$, for some non-identity element $h \in H$. Then $\mid \mathcal{G} \mid = \mid H \mid$.*

Theorem 3.18. *Presume $\mathcal{G}, H \in \mathbb{NA}$ be finite groups with $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$, for some non-identity element $h \in H$. Then*

(i) $|\mathcal{G}| = |H|$.

(ii) $|\mathbf{Ce}(\mathcal{A})| = |\mathbf{Ce}(B)|$.

Proof. (i) Since $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$, by Theorem 3.16, we have $\Gamma^{g\eth}_{\frac{\mathcal{G}}{\mathcal{A}}} \simeq \Gamma^{h\nu}_{\frac{H}{B}}$. Then by Theorem 3.17, $|\dfrac{\mathcal{G}}{\mathcal{A}}| = |\dfrac{H}{B}|$, accordingly, $|\mathcal{G}| = |H|$.
(ii) Since $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$, get that $|\mathcal{G}| - |\mathbf{Ce}(\mathcal{A})| = |H| - |\mathbf{Ce}(B)|$. Then $|\mathcal{G}| = |H|$ implies that $|\mathbf{Ce}(\mathcal{A})| = |\mathbf{Ce}(B)|$. $\square$

Nasiri et al. in [9], proved the following theorem, which we apply in the next results. In addition, we assert that in isomorphic graphs, nilpotent property is inherited, i.e., nil-potency of one side implies that the other side is nilpotent too.

Theorem 3.19. *Consider $\mathcal{G} \in \mathbb{NA}$ is a finite group of odd order, and assume that $\Gamma^g(\mathcal{G})$ has no vertex adjacent to all other vertices. If $\Gamma^g(\mathcal{G}) \cong \Gamma^h(H)$, and if $\mathcal{G}$ is nilpotent, then H is too.*

Theorem 3.20. *Presume $\mathcal{G} \in \mathbb{NA}$ be a finite group of odd order, $\Gamma^{\eth(g)}(\mathcal{G})$ has no vertex adjacent to all other vertices and $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$.*
(i) If $x\nu = e\nu$ implies that $x = e$ and $\mathcal{G}$ be a nilpotent group, then H is nilpotent.
(ii) if $\eth \in \mathcal{G}\mathbf{NF}$, then $\nu \in \mathcal{G}\mathbf{NF}$.

Proof. (i) Since $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$ and $\mathcal{G}$ is a nilpotent group, we conclude by Lemma 3.16, $\Gamma^{g\eth}_{\frac{\mathcal{G}}{\mathcal{A}}} \simeq \Gamma^{h\nu}_{\frac{H}{B}}$ and $\dfrac{\mathcal{G}}{\mathcal{A}}$ is nilpotent. So by Theorem 4.4 [9], that $\dfrac{H}{B}$ is nilpotent and so $\mathbf{Ce}_n(\dfrac{H}{B}) = \dfrac{H}{B}$. Take $x \in H$,

then $x\nu \in \dfrac{H}{B} = \mathbf{Ce}_n(\dfrac{H}{B})$ implies that each one $y_1, ..., y_n \in H$, $\mathbb{C}^n{}_{i=1}x\nu, y_i\nu = e\nu$. Then by Theorem 2.4, $(\mathbb{C}^n{}_{i=1}x, y_i)\nu = e\nu$. It imply that $B \in \mathcal{G}\mathbb{NF}$. Consequently, by assumption H is nilpotent.
(ii) Since $\Gamma^{\eth(g)}(\mathcal{G}) \simeq \Gamma^{\nu(h)}(H)$ and $\eth \in \mathcal{G}\mathbf{NF}$, we conclude by Lemma 3.16 and Theorem 2.4, $\Gamma^{g\eth}_{\frac{\mathcal{G}}{\mathcal{A}}} \simeq \Gamma^{h\nu}_{\frac{H}{B}}$ and $\dfrac{\mathcal{G}}{\mathcal{A}}$ is nilpotent. So by Theorem 3.19, $\dfrac{H}{B}$ is nilpotent. Consequently, using Theorem 2.4, we obtain $\nu \in \mathcal{G}\mathbf{NF}$. $\square$

Definition 3.21. *Presume $\mathcal{G}\mathbb{NA}$ be a finite group. It is titled as extra-special p-group (we write $\mathcal{G} \in \mathbf{ES}$) if its center is exactly equal to its commutator subgroup and $\mid \mathcal{G}' \mid = \mid \mathbf{Ce}(\mathcal{G}) \mid = p$.*

Definition 3.22. *Presume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$. Then $\mathcal{A}$ is named extra-spacial (we write $\mathcal{A} \in \mathbf{ESF}$) if $\mathcal{G}' \subseteq \mathbf{Ce}(\mathcal{A})$.*

Example 3.23. Assume $\mathcal{A} \in \mathbb{FS}(\mathcal{G})$ is as Example 3.3. Since $(\mathbb{C}i, j) = (\mathbb{C}i, k) = (\mathbb{C}j, k) = -1$, $(\mathbb{C}i, -i) = (\mathbb{C}j, -j) = (\mathbb{C}k, -k) = (\mathbb{C} - 1, -1) = 1$, and $(\mathbb{C} - 1, i) = (\mathbb{C} - 1, j) = (\mathbb{C} - 1, k) = 1$, we get that $\{\pm 1\} = \mathcal{G}' = \mathbf{Ce}(\mathcal{A})$. Accordingly, $\mathcal{A} \in \mathbf{ESF}$.

Theorem 3.24. *Assume $\mathcal{A} \in \mathbb{FN}(\mathcal{G})$. If $\dfrac{\mathcal{G}}{\mathcal{A}} \in \mathbf{ES}$, then $\mathcal{A} \in \mathbf{ESF}$.*

Proof. Since $\dfrac{\mathcal{G}}{\mathcal{A}} \in \mathbf{ES}$, we have $(\dfrac{\mathcal{G}}{\mathcal{A}})' = \mathbf{Ce}(\dfrac{\mathcal{G}}{\mathcal{A}})$. Then each one $(\mathbb{C}x, y) \in \mathcal{G}'$, we have $x\eth, y\eth \in \dfrac{\mathcal{G}}{\mathcal{A}}$ and $(\mathbb{C}x\eth, y\eth) \in (\dfrac{\mathcal{G}}{\mathcal{A}})' = \mathbf{Ce}(\dfrac{\mathcal{G}}{\mathcal{A}})$. Accordingly, for any $t \in \mathcal{G}$, $(\mathbb{C}(\mathbb{C}x\eth, y\eth), t\eth) = e\eth$. So by Theorem 2.4, that $(\mathbb{C}(\mathbb{C}x, y), t)) =_{\eth} e$. Hence $(\mathbb{C}x, y) \in \mathbf{Ce}(\mathcal{A})$ and so $\mathcal{G}' \subseteq \mathbf{Ce}(\mathcal{A})$. Consequently, $\mathcal{A} \in \mathbf{ESF}$. $\square$

Theorem 3.25. *Presume $\frac{\mathcal{G}}{\mathcal{A}} \in \mathbf{ES}$ and $\Gamma^{\eth(g)}_{\mathcal{G}} \simeq \Gamma^{\nu(h)}_{H}$. If $B \in \mathcal{G}\mathbf{NF}$ is of class 2, then $B \in \mathbf{ESF}$.*

Proof. By Lemma 3.16, $\Gamma^{g\eth}_{\frac{\mathcal{G}}{\mathcal{A}}} \simeq \Gamma^{h\eth}_{\frac{H}{B}}$ and so Theorem 3.17 implies that $\mid \mathbf{Ce}(\dfrac{\mathcal{G}}{\mathcal{A}}) \mid = \mid \mathbf{Ce}(\dfrac{H}{B}) \mid = p$. Also, by Theorem 2.4 and $\nu \in \mathcal{G}\mathbf{NF}$ is of class 2, we have $\dfrac{H}{B}$ is a nilpotent group of class 2 and so $\dfrac{(\frac{H}{B})'}{\mathbf{Ce}(\frac{H}{B})} \in \mathbb{NA}$. Then $(\dfrac{H}{B})' < \mathbf{Ce}(\dfrac{H}{B})$, so $|\dfrac{H}{B}|' \leq |\mathbf{Ce}(\dfrac{H}{B})| = p$. Hence, $\dfrac{H}{B} \in \mathbf{ES}$. It implies by Theorem 3.24, $B \in \mathbf{ESF}$. $\square$

4 Conclusion

In this paper, non-commutative fuzzy subgroup-based centralizer graphs and commutative fuzzy subgroup-based centralizer graphs are displaied. Also, some properties of this fuzzy graph, such as connectivity is stated. Also, some results on dominating sets and spanning subgraphs are stated. Finally, extra-special fuzzy groups are defined to get a relation between extra-special fuzzy groups and extra-special p-groups. In addition, we see that with some additional conditions a good nilpotent fuzzy group is an extra-special group. This paper would be beneficial for studying nilpotent fuzzy groups and extra-special p-groups. This research has some limitations such as the computations on fuzzy subgroups of non-abelian groups are very hard and need to some algorithm and programming. Also obtain some advantages such as the find of a connection between of complex networks and algebraic systems. Indeed, we have extracted the following results separately:

(i) The simple graphs are constructed based on the non-abelian groups and fuzzy subgroups.

(ii) The domination numbers are computed based on the commutative fuzzy subgroup-based centralizer-graph and non-commutative fuzzy subgroup-based centralizer-graph.

(*iii*) Is showed that the commutative fuzzy subgroup-based centralizer-graph is a spanning subgraph of non-commutative fuzzy subgroup-based centralizer-graph.

(*iv*) The relation of path graphs are analysed to the commutative fuzzy subgroup-based centralizer-graph and non-commutative fuzzy subgroup-based centralizer-graph.

(*v*) The notion of transferable fuzzy subgroup of any group is displaied and presented its rule in the non-commutative fuzzy subgroup-based centralizer-graph.

(*vi*) The isomorphic non-commutative fuzzy subgroup-based centralizer-graphs are displaied and analysed their properties.

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References

[1] Zadeh LA. Fuzzy sets. *Information and Control*. 1965; 8(3): 338-353. DOI: https://doi.org/0.1016/S0019-9958(65)90241-X

[2] Rosenfeld A. Fuzzy groups. *Journal of Mathematical Analysis and Applications*. 1971; 35(3): 512-517. DOI: https://doi.org/10.1016/0022-247X(71)90199-5

[3] Rasoulzadeh M, Edalatpanah SA, Fallah M, Najafi SE. A hybrid model for choosing the optimal stock portfolio under intuitionistic fuzzy sets. *Iranian Journal of Fuzzy Systems*. 2024; 21(2): 161-179. DOI: https://doi.org/10.22111/IJFS.2024.45118.7968

[4] Smarandache F. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic Sets and Systems*. 2018; 22: 168-170.

[5] Kaufmann A. *Introduction à la théorie des sous-ensembles flous à l'usage des ingénieurs: Éléments théoriques de base.* Paris: Masson et Cie; 1973.

[6] Ameri R, Borzooei RA, Mohammadzadeh E. Engel fuzzy subgroups. *Italian Journal of Pure and Applied Mathematics*. 2015; 34: 251-262.

[7] Davvaz B. *Polygroup theory and related system.* Washington: World scientific; 2012. DOI: https://doi.org/10.1142/8593

[8] Mohammadzadeh E, Hamidi M, Aminmofidi S. Commutators-based graph in polygroup. *Journal of Mahani Mathematical Research*. 2024; 13(4): 39-52. DOI: https://doi.org/10.22103/JMMR.2024.22105.1501

[9] Nasiri M, Erfanian A, Ganjali M, Jafarzadeh A. Isomorphic g-noncommuting graph finite groups. *Publicationes Mathematicae Debrecen*. 2017; 91/1-2: 33-42. DOI: https://doi.org/10.5486/PMD.2017.7577

[10] Dogra S, Pal M. Picture fuzzy subgroup. *Kragujevac Journal of Mathematics*. 2023; 47(6): 911-933. DOI: https://doi.org/10.46793/KgJMat2306.911D

[11] Paryzad B, Eshghi K. A new approach in fuzzy optimization of time-cost-quality trade-off problem. *Journal of Optics* . 2024; 1(1): 34-45. DOI: https://doi.org/10.22105/opt.v1i1.42

[12] Hamidi M, Jahanpanah S, Radfar A. Extended graphs based on KMfuzzy metric spaces. *Iranian Journal of Fuzzy Systems*. 2020; 17(5): 81-95. DOI: https://doi.org/10.22111/IJFS.2020.5517

[13] Hamidi M, Smarandache F, Taghinezhad M. Decision making based on valued fuzzy super-hypergraphs. *Computer Modeling in Engineering and Sciences*. 2024; 138(2): 1907-1923. DOI: https://doi.org/10.32604/cmes.2023.030284

[14] Hamidi M, Taghinezhad M. Application of superhypergraphs-based domination number in real world. *Journal of Mahani Mathematical Research*. 2023; 13(1): 210-228. DOI: https://doi.org/10.22103/jmmr.2023.21203.1415

[15] Sezgin A, Onur B. Soft intersection almost bi-ideals of semigroups. *Systemic Analytic*. 2024; 2(1): 95-105. DOI: https://doi.org/10.31181/sa21202415

[16] Nguyen HT, Walker C, Walker EA. *A first course in fuzzy logic*. 4th edition. Boca Raton: Taylor & Francis, CRC Press; 2018. DOI: https://doi.org/10.1201/9780429505546

[17] Mondeson JN, Bhutani KR, Rosenfeld A. *Fuzzy group theory*. Berlin: Springer; 2005. DOI: https://doi.org/10.1007/b12359

[18] Suzuki M. *Group theory I*. Berlin: Springer; 1982. DOI: https://doi.org/10.1007/978-3-642-61804-8

[19] Mohammadzadeh H, Borzooei RA. Nilpotent fuzzy subgroups. *Mathematics*. 2018; 6(2): 1-12. DOI: https://doi.org/10.3390/math6020027

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