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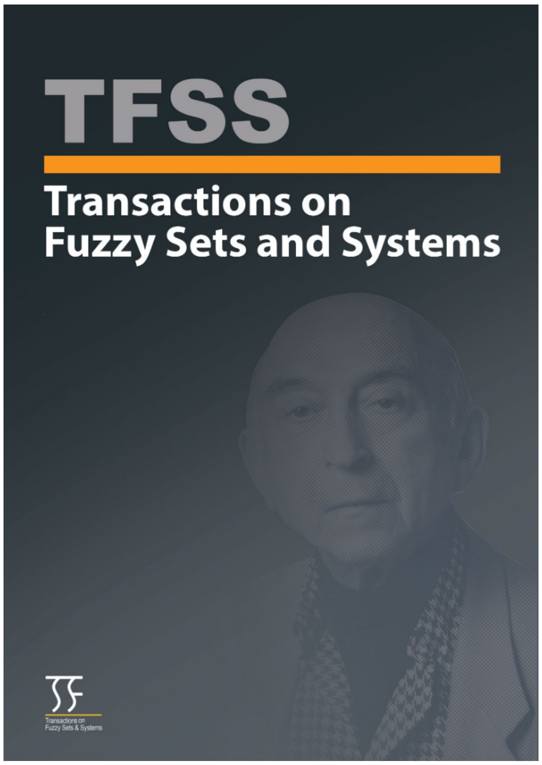

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Algebraic Perspectives of Fuzzy Sets: Structures and Properties

Satabdi Ray, **Debnarayan Khatua**, **Sayantan Mandal***

Abstract. A structure that is inherent to a specific space and involves actions within that space is highly significant. Various types of structures regarding fuzzy sets have been extensively examined in the literature. Examples of these structures encompass algebraic, topological, and analytical structures. This study examines several algebraic structures present in the domain of fuzzy sets, specifically focusing on the standard union and intersection operations of fuzzy sets. Furthermore, we provide updated depictions for specific subspaces of fuzzy sets and introduce newly defined operations. Consequently, we examine different structures on different subspaces of fuzzy sets in connection to the just-mentioned operations.

AMS Subject Classification 2020: 03E72; 08A72; 06B05, 06D72
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1 Introduction

In recent years, there have been notable breakthroughs in the field of fuzzy set theory and its related areas. Researchers have introduced novel techniques to address the complexities and challenges associated with systems that contain unclear, incomplete, or inaccurate data. Zadeh introduced fuzzy sets [1] in 1965 to provide a more nuanced and flexible approach to dealing with uncertainty, imprecision, and complex data. It has become invaluable and continues to find expanding applications across diverse fields, such as probability theory, control systems, artificial intelligence, decision-making, pattern recognition, data analysis, etc.

1.1 Importance of algebraic structure

Zadeh's theory [1] establishes that the space of all fuzzy sets forms a distributive lattice under the usual intersection and union operations on fuzzy sets. De Luca and Termini [2] have shown that for a fuzzy set $A : X \longrightarrow [0, 1]$, if the range $[0, 1]$ is a Brouwerian lattice, then the space of all fuzzy sets (denoted by $\mathcal{F}(X)$) is also a Brouwerian lattice. In addition to that, the authors have discussed various approaches to inducing a lattice to the whole or a subclass of $\mathcal{F}(X)$, e.g. if X has some lattice structure, then it is possible to have a lattice structure to a subclass of $\mathcal{F}(X)$. The authors in [3] discussed the algebraic properties of fuzzy sets and have shown that it is possible to get a unitary commutative semi-ring structure in the space of fuzzy sets under union and intersection. It has also been proven that the space of fuzzy sets forms a commutative monoid, which is neither a lattice nor a semi-ring under the algebraic sum (or probabilistic sum) t-conorm

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and the algebraic product (or product) t-norm. A lattice-ordered semigroup structure exists in the space of fuzzy sets w.r.t. union, intersection, and algebraic product. Likewise, the space has the same structure under union, intersection, and bounded product. Algebraic structures of fuzzy sets involving drastic product t-norm and drastic sum t-conorm operations have also been studied. In [4], a bounded lattice structure has been found on a collection of continuous fuzzy numbers where the operations are the usual fuzzy union and intersection. In this lattice, subsethood measures have been established showing its usefulness in fuzzy rule-based systems. The space of fuzzy sets gives rise to commutative semigroup structures with each of the drastic product norms t-norm and drastic sum t-conorm operations, whereas the space forms neither a lattice nor a semi-ring with drastic product norm t-norm and drastic sum t-conorm operations. When the drastic product (drastic sum) operation is combined with the union (intersection) operation, the space of fuzzy sets forms a commutative semi-ring. Furthermore, when the operations of union, intersection, and drastic product (union, intersection, and drastic sum) have been considered together, fuzzy sets form a lattice-ordered semigroup, exhibiting a rich interplay of mathematical structures. Since the lattice structure of fuzzy sets does not address the characteristics of repetition and negativity in uncertain information, to address these limitations, Homenda [5] introduced the concept of linear space and ring structure of fuzzy sets using sum product vector operators (a-p-v operators) with its ability to manage uncertain information in the context of fuzzy reasoning, fuzzy neural networks. Cao and tpnika [6] investigated the preservation of fundamental characteristics of residuated lattice structures in extended algebras for partial fuzzy set theory and partial fuzzy logic. The work [6] introduces a comprehensive collection of nine algebras, referred to as the "atlas," designed to handle undefined values. It provides a concise overview of how each algebra preserves or does not preserve important aspects.

In [7], algebraic properties of fuzzy sets of type-2 have also been studied. Under union and intersection operations, the space of convex fuzzy sets type-2 forms a commutative semi-ring, while the space of standard convex fuzzy sets of type-2 forms a distributive lattice. Similarly to the algebraic product and algebraic sum operations, the algebraic properties of fuzzy sets of type-2 have also been explored. Some algebraic structures such as modular quasi-lattice and distributive quasi-lattice of space of fuzzy sets of type-2 have been discussed in [8]. It has also been noticed that the space of fuzzy sets of type-2 does not form a lattice or semi-ring under these operations but forms a commutative semigroup. A semigroup structure ordered by lattice is known to exist when the space is standard convex fuzzy sets of type-2 and the corresponding operations are the union, intersection and algebraic product. Some residuated lattice-like structures have been found to exist in the space of intuitionistic fuzzy sets and fuzzy set-based systems, respectively, in [9, 10].

Chang first introduced the topological structure in fuzzy sets [11], along with specific characteristics of fuzzy topological spaces, such as continuity and compactness. In [12], Chang's concept of compactness [11] has been modified by taking fuzzy aspects into account, and several separation axioms have then been discussed. The terms fuzzy point and fuzzy Q-neighborhood were established in [11], and using these notions, fuzzy metric space and fuzzy uniform space have been studied with the condition that when a fuzzy topological space is a fuzzy metric space. Furthermore, various characterizations of the separation axioms in fuzzy subsets have been presented in continuous functions [13] and T_i, which yield various general topological results. In [14], fuzzy T_2 space, fuzzy regularity, almost regularity, and fuzzy Urysohn space have all been extensively discussed using fuzzy θ closing operators, Q neighborhood, and quasi-coincidence. Almost compactness and near compactness corresponding to fuzzy sets have been studied with their various properties in [15]. An investigation of connectedness has been carried out in fuzzy topological spaces in the literature [16]. Also, there have been discussions on some notions of functions in fuzzy topological spaces and completely distributive lattices.

Algebraic structures on space of some most significant fuzzy logic connectives viz. Fuzzy implications, t norms and t norms, which are generalizations of the classical implications, conjunctions, and disjunctions, respectively, can be found in [17, 18].

The combination of fuzzy vector spaces and the investigation of generalized roughness in LA semigroups broaden the possible uses of these mathematical notions in areas such as optimization, data analysis, and machine learning [19, 20]. The studies showcase the potential of fuzzy set theory and associated algebraic structures to improve our comprehension and management of complex systems. This can lead to more efficient decision making and system optimization.

1.2 Motivation for the study

Algebraic studies of spaces serve various purposes, encompassing theoretical exploration and practical applications. Algebra occupies a crucial component of mathematics and its applications, making it a vibrant area of research with a broad range of motivations and objectives.

Introducing different operations is essential to obtain a diverse range of structures. As these structures emerge, they can be characterized in novel ways [21], allowing the application of established results to gain deeper insight and more refined perspectives [18]. Moreover, these structures have significant potential for practical implementation, particularly in approximate reasoning [4], pattern recognition, and machine intelligence, in the broad domain of fuzzy implications, where they can be effectively applied to areas such as control theory, decision making, expert systems, and fuzzy logic.

One can study algebraic aspects by defining a closed binary operation on fuzzy sets that can produce new fuzzy sets from a given one. Suppose that one can impose a richer algebraic structure, such as a group. In that case, one can apply results to the crucial ideas that arise in group theory, viz., semigroups, monoids, normal subgroups, homomorphisms and the group of inner automorphisms, to gain a deeper and better perspective of the different families of fuzzy sets. This encourages us towards the algebraization of fuzzy sets by proposing a binary operation on the space of fuzzy sets, which would produce a sufficiently rich algebraic structure, more light on fuzzy sets, and make new links between the known families and characteristics of fuzzy sets.

Commutative monoids are produced via new and existing operations in the fuzzy set space. These monoids allow us to investigate the semi-ring structures of fuzzy sets. These semi-rings include some lattice structures used in several domains, including pattern recognition, digital image processing, and formal concept analysis. Furthermore, the monoids can be used to study homomorphism, Grothendieck group structure, left translation on a monoid, the center of a monoid, construction of monoid rings, how commutative monoids are embedded in rings, etc.

2 Preliminaries

This section contains necessary definitions and notations for various kinds of fuzzy sets, their spaces, and different fuzzy logic connectives. Additionally, we present well-known algebraic structures on a set and some operations on fuzzy sets that are required for this work.

2.1 Fuzzy sets and spaces on fuzzy sets

Let $\mathbb{R}$ be the collection of all real numbers. We follow the convention that $X \subseteq \mathbb{R}$ is a closed and bounded interval, making X totally ordered and linear.

Definition 2.1. *A fuzzy set A on a non-empty set $X \subseteq \mathbb{R}$ is a function A from X to $[0, 1]$. We denote the collection of all fuzzy sets in X by $\mathcal{F}(X)$.*

Definition 2.2. *A fuzzy set A denoted by $A = [a, b, c]; a \leq b \leq c,\ a, b, c \in X$ is called a* ***triangular fuzzy set*** *if the representation of its membership function is*

$$A(x) = \begin{cases} 0 & ;\ \textit{if } a \geq x \\ l(x) & ;\ \textit{if } a < x \leq b \\ r(x) & ;\ \textit{if } b \leq x < c \\ 0 & ;\ \textit{if } x \geq c \end{cases}$$

where $l(x)$ and $r(x)$ are respectively, linear strictly increasing and decreasing functions of x and $l(x) = r(x)$ at $x = b$.

Definition 2.3. *A fuzzy set A denoted by $A = [a, b, c]; a \leq b \leq c, a, b, c \in X$ is called a* ***normal triangular fuzzy set*** *if the representation of its membership function is*

$$A(x) = \begin{cases} 0 & ;\ \textit{if } a \geq x \\ (x-a)/(b-a) & ;\ \textit{if } a < x \leq b \\ (c-x)/(c-b) & ;\ \textit{if } b \leq x < c \\ 0 & ;\ \textit{if } x \geq c. \end{cases}$$

Definition 2.4. *A fuzzy set $A : X \longrightarrow [0, 1]$ is called*

(*i*) ***normal***, *if there exists $x_0 \in X$ such that $A(x_0) = 1$,*

(*ii*) ***bounded***, *if the Supp$(A) = \{x \in X : A(x) > 0\}$ is bounded,*

(*iii*) ***convex***, *if for any $x, y \in X$ and $0 \leq \lambda \leq 1$, $A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\}$,*

(*iv*) ***symmetric*** *around $x = c$, if $A(c + x) = A(c - x)\ \forall\ x \in X$.*

Based on the above properties of a fuzzy set, we define some subspaces of $\mathcal{F}(X)$ that will be relevant in this work, as follows:

- $\mathcal{F}_B(X)$: the collection of all bounded fuzzy sets on X,
- $\mathcal{F}_{BN}(X)$: the collection of all bounded and normal fuzzy sets on X,
- $\mathcal{F}_{BNC}(X)$: the collection of all bounded, normal, and convex fuzzy sets on X,
- $\mathcal{F}_{TBNC}(X)$: the collection of all triangular, bounded, normal, and convex fuzzy sets on X,
- $\mathcal{F}_{TBSNC}(X)$: the collection of all triangular, bounded, symmetric, normal, and convex fuzzy sets on X,
- $\mathcal{F}_{BSN}(X)$: the collection of all bounded, symmetric, and normal fuzzy sets on X,
- $\mathcal{F}_{TBSN}(X)$: the collection of all triangular, bounded, symmetric, and normal fuzzy sets on X.

We have the following relations from the above subspaces:

$$\mathcal{F}_{TBSNC}(X) \subseteq \mathcal{F}_{TBSN}(X) \subseteq \mathcal{F}_{TBNC}(X) \subseteq \mathcal{F}_{BNC}(X) \subseteq \mathcal{F}_{BN}(X) \subseteq \mathcal{F}_B(X) \subseteq \mathcal{F}(X).$$

.

2.2 Fuzzy logic connectives and operations on fuzzy sets

Fuzzy logic connectives are a generalization of classical logical operations. In the following, for completeness and well-readability, we recall some definitions of fuzzy logic connectives that will be useful in this work.

Definition 2.5. [22] *A **t-norm** is a monotonically increasing, commutative, and associative function* $T : [0,1]^2 \longrightarrow [0,1]$ *with* 1 *as the identity element.*

Definition 2.6. [22] *A **t-conorm** is a monotonically increasing, commutative and associative function* $S : [0,1]^2 \longrightarrow [0,1]$ *with* 0 *as the identity element.*

Definition 2.7. [23] *A **fuzzy implication** is a function* $I : [0,1]^2 \longrightarrow [0,1]$ *that decreases in the first variable and increases in the second variable along with the boundary conditions* $I(1,0) = 0$ *and* $I(1,1) = I(0,0) = 1$.

Remark 2.8. [24] *For a t-norm* T *and a t-conorm* S *and* $x, y, z \in [0,1]$

(i) S *is distributive over* T *i.e.* $S(x, T(y,z)) = T(S(x,y), S(x,z))$ *iff* T *is a minimum t-norm.*

(ii) T *is distributive over* S *i.e.* $T(x, S(y,x)) = S(T(x,y), T(x,z))$ *iff* S *is the maximum t-conorm.*

(iii) *The pair* (T, S) *is a distributive pair iff* T *is a minimum t-norm and* S *is the maximum t-conorm.*

Definition 2.9. [1] *Let* $A, B \in \mathcal{F}(X)$. *The **union** of* A, B *is defined by* $A \cup B = \max\{A(x), B(x)\}$, $\forall\ x \in X$.

Definition 2.10. [1] *Let* $A, B \in \mathcal{F}(X)$. *The **intersection** of* A, B *is defined by* $A \cap B = \min\{A(x), B(x)\}$, *for all* $x \in X$.

Definition 2.11. [1] *Let* $A \in \mathcal{F}(X)$. *The **complement** of* A *is* A^c *and is defined by* $A^c(x) = 1 - A(x)$, $\forall\ x \in X$.

2.3 Algebraic structures on a set

An algebraic structure is a non-empty set equipped with one or more operations that follow specific axioms. In the following, we recall some definitions of algebraic structures that are studied in the manuscript.

Definition 2.12. [25] *A **monoid** is a non-empty set* G *equipped with an associative binary operation along with an identity element. A **commutative monoid** is a monoid in which the associated binary operation is commutative.*

Definition 2.13. [25] *A **group** is a monoid* G *with an inverse element that corresponds to every element of* G.

Definition 2.14. [25] *A **quasigroup** is a nonempty set* G *with a binary operation* $*$ *such that for each* $a, b \in G$, *there exist unique elements* $x, y \in G$ *s.t.* $a * x = b$ *and* $y * a = b$ *hold. A **loop** is a quasigroup with an identity element.*

Definition 2.15. [26] *Let* G *be a non-empty set along with two binary operations* $*_1, *_2$ *such that* $*_2$ *is distributed over* $*_1$. *Then* $(G, *_1, *_2)$ *is called a **ringoid**.*

Definition 2.16. [3] *A **semi-ring** is a ringoid* $(G, *_1, *_2)$ *along with the associativity of* $*_2$ *and the associativity, the commutativity of* $*_1$. *A semi-ring* $(G, *_1, *_2)$ *is called a **commutative semi-ring** if* $*_2$ *is commutative.*

Definition 2.17. [27] *An **anti-ring** is a semi-ring* $(G, *_1, *_2)$ *if* $a *_1 b = i_{*_1}$ *implies that* $a = b = i_{*_1}$ *for* $a, b \in G$ *and* i_{*_1} *is the identity element of* G *w.r.t.* $*_1$.

Definition 2.18. [28] *A* ***near-semi-ring*** *is a non-empty set G with two binary operations $*_1$, and $*_2$ such that $(G, *_1)$ is a monoid, $*_2$ is associative, $*_2$ is distributive over $*_1$ and $i_{*_1} *_2 a = i_{*_1}$, for all $a \in G$ where i_{*_1} is the identity element of G w.r.t. $*_1$.*

Definition 2.19. [28] *A near-semi-ring $(G, *_1, *_2)$ is said to be a* ***zero-symmetric near-semi-ring*** *if $G = G_0$ where $G_0 = \{a \in G \mid a *_2 i_{*_1} = i_{*_1}\}$ for the identity element i_{*_1} of G w.r.t $*_1$.*

Definition 2.20. [29] *A non-empty set G together with two binary operations $*_1$ and $*_2$ is called a* ***near-ring*** *if $(G, *_1)$ is a group, $*_2$ is associative and distributive over $*_1$.*

Definition 2.21. [25] *A non-empty set G together with two binary operations $*_1$ and $*_2$ is called a* ***ring*** *if G is a ringoid, $(G, *_1)$ is a commutative group and $*_2$ is associative.*

Definition 2.22. [21] *A* ***semi-lattice*** *is a non-empty set G with a binary composition $*_1$ such that $*_1$ satisfies the properties of associativity, commutativity and idempotency.*

Definition 2.23. [3] *A* ***lattice*** *is an algebraic structure $(G, *_1, *_2)$ consisting of a non-empty set G and two binary operations $*_1, *_2$ that satisfy the absorption laws:*

$$a *_1 (a *_2 b) = a \text{ and } a *_2 (a *_1 b) = a$$

3 Structures on the space of bounded fuzzy sets

In this section, we study various structures in the space $\mathcal{F}_B(X)$ w.r.t. t-norm (T) and t-norm (S).

Theorem 3.1. *$(\mathcal{F}_B(X), T)$ is a commutative monoid.*

Proof. Let $A, B, C \in \mathcal{F}_B(X)$. We have the following arguments.

(i) By virtue of a t-norm being a function from $[0, 1]^2$ to $[0, 1]$ $T(A, B) \in \mathcal{F}_B(X)$.

(ii) Since T is associative, $T(A, T(B, C)) = T(T(A, B), C)$.

(iii) T being commutative, we have $T(A, B) = T(B, A)$.

(iv) $T(A, I) = T(I, A) = A$, where $I(x) = 1$; $\forall\ x \in X$. Here, $I \in \mathcal{F}_B(X)$ acts as the identity element w.r.t. T.

Hence, $(\mathcal{F}_B(X), T)$ is a commutative monoid. □

Theorem 3.2. *$(\mathcal{F}_B(X), S)$ is a commutative monoid.*

Proof. Let $A, B, C \in \mathcal{F}_B(X)$. Then the following assertions hold.

(i) By virtue of a t-conorm being a function from $[0, 1]^2$ to $[0, 1]$, $S(A, B) \in \mathcal{F}_B(X)$.

(ii) Since S is associative, $S(A, S(B, C)) = S(S(A, B), C)$

(iii) S being commutative, we have $S(A, B) = S(B, A)$

(iv) $S(A, O) = S(O, A) = A$ where $O(x) = 0\ \forall\ x \in X$. Here, $O \in \mathcal{F}_B(X)$ acts as the identity element with respect to S.

Hence, $(\mathcal{F}_B(X), S)$ is a commutative monoid. □

Remark 3.3. *In particular,* $(\mathcal{F}_B(X), \cap)$ *and* $(\mathcal{F}_B(X), \cup)$ *are commutative monoids.*

Remark 3.4. *For a t-norm* T *and an arbitrary* $x \in [0,1]$*, there may not exist a unique* y *s.t.* $T(x,y) = 1$*. Hence, for an arbitrary* $A \in \mathcal{F}_B(X)$*, there does not exist a* $A' \in \mathcal{F}_B(X)$ *such that* $T(A, A') = I$*. This leads to the conclusion that* $(\mathcal{F}_B(X), T)$ *does not become a group. Continuing on similar lines, we can again conclude that* $(\mathcal{F}_B(X), S)$ *also does not form a group.*

Theorem 3.5. $(\mathcal{F}_B(X), S, T)$ *forms a commutative semi-ring if and only if* S *is the maximum t-conorm.*

Proof. Let $A, B, C \in \mathcal{F}_B(X)$. A t-norm T and a t-conorm S being both commutative and associative, we have

$$T(A,B) = T(B,A), \quad T(A, T(B,C)) = T(T(A,B), C),$$
$$S(A,B) = S(B,A), \quad S(A, S(B,C)) = S(S(A,B), C).$$

We know from Remark 2.8, T is distributive over S, i.e., $T(A, S(B,C)) = S(T(A,B), T(A,C))$ if and only if S is the maximum t-conorm. Therefore, $(\mathcal{F}_B(X), S, T)$ forms a commutative semi-ring if and only if S is the maximum t-conorm. □

Remark 3.6. *In particular,* $(\mathcal{F}_B(X), \cup, \cap)$ *is a commutative semi-ring.*

Theorem 3.7. $(\mathcal{F}_B(X), S, T)$ *forms an anti-ring if and only if* S *is the maximum t-conorm.*

Proof. From Theorem 3.5, we know that $(\mathcal{F}_B(X), S, T)$ is a semi-ring if and only if S is the maximum t-conorm. For the maximum t-conorm S, we know that

$$S(x,y) = 0 \Longrightarrow x = y = 0, x, y \in [0,1] \ .$$

Hence for the maximum t-conorm S and $A, B, O \in \mathcal{F}_B(X)$,

$$S(A,B) = O \Longrightarrow A = B = O.$$

Therefore, $(\mathcal{F}_B(X), S, T)$ is an anti-ring if and only if S is the maximum t-conorm. □

Remark 3.8. *In particular,* $(\mathcal{F}_B(X), \cup, \cap)$ *is an anti-ring.*

Theorem 3.9. $(\mathcal{F}_B(X), S, T)$ *is a zero symmetric near semi-ring if and only if* S *is the maximum t-conorm.*

Proof. From Theorem 3.2, we find that $(\mathcal{F}_B(X), S)$ is a monoid where $O \in \mathcal{F}_B(X)$ $(O(x) = 0 \ \forall \ x \in X)$, acts as the identity element with respect to S. We also know from Remark 2.8, that T is distributive over S if and only if S is the maximum t-conorm. Again, by virtue of a t-norm, T is associative and $T(O, A) = O \ \forall$ $A, O \in \mathcal{F}_B(X)$. Hence, $(\mathcal{F}_B(X), S, T)$ is a near semi-ring.

For any t-norm we get, $\mathcal{F}_B(X)_0 = \{A \in \mathcal{F}_B(X) : T(O, A) = T(A, O) = O\}$. Thus, $\mathcal{F}_B(X) = \mathcal{F}_B(X)_0$. Hence, $(\mathcal{F}_B(X), S, T)$ is a zero-symmetic near semi-ring. □

Remark 3.10. *In particular,* $(\mathcal{F}_B(X), \cup, \cap)$ *is a zero symmetric near-semi-ring.*

Remark 3.11. *Since* $(\mathcal{F}_B(X), S)$ *does not form a group,* $(\mathcal{F}_B(X), S, T)$ *is neither a near-ring nor a ring.*

Remark 3.12. *Let* $A = [0, 2, 5], B = [1, 2, 3,], C = [2, 3, 3,] \in \mathcal{F}_B(X)$*. If we consider* T *to be a minimum t-norm, then we get*

$$T(A,B) = [0, 2, 3] = D,$$

and

$$T(C, A) = [0, 2, 3] = D.$$

Again, let $B' = [2, 2, 3], C' = [1, 3, 3,] \in \mathcal{F}_B(X)$. *Then also we get*

$$T(A, B') = [0, 2, 3] = D,$$

and

$$T(C', A) = [0, 2, 3] = D.$$

We see $B' \neq B$ *and* $C' \neq C$. *Thus* $T(A, B) = D$ *and* $T(C, A) = D$ *have no unique solution. Hence* $(\mathcal{F}_B(X), T)$ *fails to be a quasigroup.*

In the same way, we can also show that $(\mathcal{F}_B(X), S)$ *is not a quasigroup.*

Remark 3.13. *In general,* $T(A, A) \neq A$ *and* $S(A, A) \neq A$. *Hence,* $(\mathcal{F}_B(X), T)$ *and* $(\mathcal{F}_B(X), S)$ *are not semi-lattices. In particular, due to the fact that* $A \cap A = A$ *and* $A \cup A = A$, $(\mathcal{F}_B(X), \cap)$ *and* $(\mathcal{F}_B(X), \cup)$ *both form a semi-lattice.*

Remark 3.14. *If we consider* S *as the maximum t-conorm and* T *as the Lukasiewicz t-norm, then*

$$T(A, S(A, B)) \neq A.$$

Hence, $(\mathcal{F}_B(X), S, T)$ *does not form a lattice structure. However, in particular,* $(\mathcal{F}_B(X), \cup, \cap)$ *forms a lattice.*

4 Algebraic structures on triangular fuzzy sets w.r.t. newly defined operations

Let $\mathbb{R}^+$ denote the collection of all non-negative real numbers, i.e., $\mathbb{R}^+ = [0, \infty)$. $\mathcal{F}_{TBNC}(\mathbb{R}^+)$ denotes the space of all triangular, bounded, normal, and convex fuzzy sets on $\mathbb{R}^+$.

4.1 Structures on $\mathcal{F}_{TBNC}(\mathbb{R}^+)$

Let us define some new operations $+_F$ and $\cdot_F$ on the space $\mathcal{F}_{TBNC}(\mathbb{R}^+)$. Let $A = [a, b, c], B = [a', b', c'] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$ where $a, b, c, a', b', c' \in \mathbb{R}^+$ with $a \leq b \leq c$ and $a' \leq b' \leq c'$. The operations $+_F : \mathcal{F}_{TBNC}(\mathbb{R}^+) \times \mathcal{F}_{TBNC}(\mathbb{R}^+) \to \mathcal{F}_{TBNC}(\mathbb{R}^+)$ and $\cdot_F : \mathcal{F}_{TBNC}(\mathbb{R}^+) \times \mathcal{F}_{TBNC}(\mathbb{R}^+) \to \mathcal{F}_{TBNC}(\mathbb{R}^+)$ are defines as follows:

$$A +_F B = [a + a', b + b', c + c']$$
$$A \cdot_F B = [aa', bb', cc'].$$

We now study for existence of some algebraic structures on $\mathcal{F}_{TBNC}(\mathbb{R}^+)$ w.r.t. the operations defined above.

These operations are chosen for their direct correspondence with standard algebraic structures, such as monoids, semigroups, semi-rings, and lattices. This selection of operations guarantees logical coherence and compatibility with established mathematical reasoning frameworks.

Theorem 4.1. $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F)$ *forms a commutative monoid.*

Proof. Let $A = [a, b, c], B = [a', b', c'], C = [a'', b'', c''] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$ where $a, b, c, a', b', c', a'', b'', c'' \in \mathbb{R}^+$ with $a \leq b \leq c$, $a' \leq b' \leq c'$ and $a'' \leq b'' \leq c''$. The subsequent claims are true:

(*i*) Since $a, b, c, a', b', c' \in \mathbb{R}^+$ with $a \leq b \leq c$ and $a' \leq b' \leq c'$, we have $a + a', b + b', c + c' \in \mathbb{R}^+$ with $a + a' \leq b + b' \leq c + c'$. Thus,

$$A +_F B = [a + a', b + b', c + c'] \in \mathcal{F}_{TBNC}(\mathbb{R}^+).$$

(*ii*) Addition ($+_F$) is associative, since

$$A +_F (B +_F C) = [a + a' + a'', b + b' + b'', c + c' + c''] = (A +_F B) +_F C.$$

(*iii*) There exists an element $O = [0, 0, 0] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$ s.t. $A +_F O = O +_F A = A$. Here, $O = [0, 0, 0]$ acts as the identity element corresponding to the addition ($+_F$).

(*iv*) The operation, addition ($+_F$) is commutative since

$$A +_F B = [a + a', b + b', c + c'] = B +_F A \; .$$

Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F)$ is a commutative monoid. □

Theorem 4.2. *$(\mathcal{F}_{TBNC}(\mathbb{R}^+), \cdot_F)$ forms a commutative monoid.*

Proof. Let $A = [a, b, c], B = [a', b', c'], C = [a'', b'', c''] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$ where $a, b, c, a', b', c', a'', b'', c'' \in \mathbb{R}^+$ with $a \leq b \leq c$, $a' \leq b' \leq c'$ and $a'' \leq b'' \leq c''$. The following arguments hold:

(*i*) Since $a, b, c, a', b', c' \in \mathbb{R}^+$ with $a \leq b \leq c$ and $a' \leq b' \leq c'$, we have $aa', bb', cc' \in \mathbb{R}^+$ with $aa' \leq bb' \leq cc'$. Thus,

$$A \cdot_F B = [aa', bb', cc'] \in \mathcal{F}_{TBNC}(\mathbb{R}^+).$$

(*ii*) Multiplication ($\cdot_F$) is associative, because

$$A \cdot_F (B \cdot_F C) = [aa'a'', bb'b'', cc'c''] = (A \cdot_F B) \cdot_F C$$

(*iii*) There exists an element $e = [1, 1, 1] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$ s.t. $A \cdot_F e = e \cdot_F A = A$. Thus, $e = [1, 1, 1]$ is the identity element in $\cdot_F$.

(*iv*) Since $A \cdot_F B = [aa', bb', cc'] = B \cdot_F A$, multiplication ($\cdot_F$) is commutative.

Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), \cdot_F)$ forms a commutative monoid. □

Remark 4.3. *$(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F)$ fails to form a group because there does not exist any element A' $\mathcal{F}_{TBNC}(\mathbb{R}^+)$ such that $A +_F A' = [0, 0, 0] = A' +_F A$. Similarly, there is no A' such that $A \cdot_F A' = A' \cdot_F A = [1, 1, 1]$ so $(\mathcal{F}_{TBNC}(\mathbb{R}^+), \cdot_F)$ is also not a group.*

Theorem 4.4. *$(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ is a commutative semi-ring.*

Proof. From Theorem 4.1, we have the commutativity, associativity of $+_F$ and from Theorem 4.2, we have the commutativity, associativity of $\cdot_F$. Now for left distributivity,

$$\begin{aligned} A \cdot_F (B +_F C) &= [a, b, c] \cdot_F ([a', b', c'] +_F [a'', b'', c'']) \\ &= [a, b, c] \cdot_F [a' + a'', b' + b'', c' + c''] \\ &= [a(a' + a''), b(b' + b''), c(c' + c'')] \\ &= [aa' + aa'', bb' + bb'', cc' + cc''] \\ &= [aa', bb', cc'] + [aa'', bb'', cc''] \\ &= ([a, b, c] \cdot_F [a', b', c']) +_F ([a, b, c] \cdot [a'', b'', c'']) \\ &= (A \cdot_F B) +_F (A \cdot_F C) \; . \end{aligned}$$

Similarly, the right distributive law also holds in $\mathcal{F}_{TBNC}(\mathbb{R}^+)$. Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ is a commutative semi-ring. □

Theorem 4.5. *$(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ is an anti-ring.*

Proof. Let $A, B \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$ s.t. $A +_F B = [0,0,0]$. Then $A = B = [0,0,0]$. Therefore, $(\mathcal{F}_{TBNC}(X), +_F, \cdot_F)$ forms an anti-ring. □

Theorem 4.6. *$(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ is a zero-symmetric near-semi-ring.*

Proof. We find that $\mathcal{F}_{TBNC}(\mathbb{R}^+)$ is a monoid under $+_F$, associative under $\cdot_F$, and satisfies the distributive property (see Theorem 4.1, 4.2, 4.4, respectively). Now for $A = [a,b,c] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$, we have $O \cdot_F A = O$ where $O = [0,0,0]$ is the identity element under $+_F$. Hence, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ is a near-ring. Again, $\mathcal{F}_{TBNC}(\mathbb{R}^+)_0 = \{A \in \mathcal{F}_{TBNC}(\mathbb{R}^+) : O \cdot_F A = [0,0,0] \cdot_F [a,b,c] = [0,0,0] = O = A \cdot_F O\}$. Thus, $\mathcal{F}_{TBNC}(\mathbb{R}^+) = \mathcal{F}_{TBNC}(\mathbb{R}^+)_0$. Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ is a zero-symmetric near-semi-ring. □

Remark 4.7. *Since $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F)$ does not form a group, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ does not form a ring.*

Remark 4.8. *Since $A +_F A = [a+a, b+b, c+c] = [2a, 2b, 2c] \neq A$ and $A \cdot_F A = [a^2, b^2, c^2] = A^2 \neq A$, we observe that $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F)$ and $(\mathcal{F}_{TBNC}(\mathbb{R}^+), \cdot_F)$ do not form semi-lattice.*

Remark 4.9. *For arbitrary $A = [a,b,c], B = [a',b',c'] \in \mathcal{F}_{TBNC}(\mathbb{R}^+)$, we have*

$$\begin{aligned} A \cdot_F (A +_F B) &= [a,b,c] \cdot_F ([a,b,c] +_F [a',b',c']) \\ &= [a,b,c] \cdot_F [a+a', b+b', c+c'] \\ &= [aa+aa', bb+bb', cc+cc'] \\ &= (A \cdot_F A) +_F (A \cdot_F B) \neq A \end{aligned}$$

and,

$$\begin{aligned} A +_F (A \cdot_F B) &= [a,b,c] +_F ([a,b,c] \cdot_F [a',b',c']) \\ &= [a,b,c] +_F [aa', bb', cc'] \\ &= [a+aa', b+bb', c+cc'] \neq A\ . \end{aligned}$$

Hence, $(\mathcal{F}_{TBNC}(\mathbb{R}^+), +_F, \cdot_F)$ does not form a lattice.

4.2 Structures on $\mathcal{F}_{TBNC}(\mathbb{R})$ w.r.t. $+_F^*$ and $\cdot_F^*$

Let $\mathcal{F}_{TBNC}(\mathbb{R})$ denote the space of all triangular, bounded, normal, and convex fuzzy sets in $\mathbb{R}$. Let us represent the fuzzy sets of $\mathcal{F}_{TBNC}(\mathbb{R})$ in the form $A = [\theta, x, \alpha]$ where $0 < \theta, \alpha \leq 90°$ where θ is the left end angle, $x \in \mathbb{R}$ is the point of normality and α is the right end angle. Let us assume two fuzzy sets $A = [\theta, x, \alpha]$ and $B = [\theta', x', \alpha'] \in \mathcal{F}_{TBNC}(X)$. We further define the following operations, addition $(+_F^* : \mathcal{F}_{TBNC}(\mathbb{R}) \times \mathcal{F}_{TBNC}(\mathbb{R}) \to \mathcal{F}_{TBNC}(\mathbb{R}))$ and multiplication $(\cdot_F^* : \mathcal{F}_{TBNC}(\mathbb{R}) \times \mathcal{F}_{TBNC}(\mathbb{R}) \to \mathcal{F}_{TBNC}(\mathbb{R}))$ as:

$$A +_F^* B = [\theta, x, \alpha] +_F^* [\theta', x', \alpha'] = [\theta \wedge \theta', x + x', \alpha \wedge \alpha']$$

and,

$$A \cdot_F^* B = [\theta, x, \alpha] \cdot_F^* [\theta', x', \alpha'] = [\theta \vee \theta', x + x', \alpha \vee \alpha']$$

where $\vee$ and $\wedge$ are usual maximum and minimum operation in $\mathbb{R}$. We now study the existence of some algebraic structures on $\mathcal{F}_{TBNC}(\mathbb{R})$ w.r.t. the operations $+_F^*$ and $\cdot_F^*$ as defined above.

Theorem 4.10. *$(\mathcal{F}_{TBNC}(\mathbb{R}), +_F^*)$ is a commutative monoid.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'], C = [\theta'', x'', \alpha''] \in \mathcal{F}_{TBNC}(\mathbb{R})$. The following holds:

(*i*) Since $0 < \theta \wedge \theta^{'}, \alpha \wedge \alpha^{'} \leq 90°$, $A +_F^* B \in \mathcal{F}_{TBNC}(\mathbb{R})$.

(*ii*) For associativity:

$$\begin{aligned} A +_F^* (B +_F^* C) &= [\theta, x, \alpha] +_F^* ([\theta', x', \alpha'] +_F^* [\theta'', x'', \alpha'']) \\ &= [\theta, x, \alpha] +_F^* ([\theta' \wedge \theta'', x' + x'', \alpha' \wedge \alpha'']) \\ &= [\theta \wedge \theta' \wedge \theta'', x + x' + x'', \alpha \wedge \alpha' \wedge \alpha''] \ . \end{aligned}$$

and,

$$\begin{aligned} (A +_F^* B) +_F^* C &= ([\theta, x, \alpha] +_F^* [\theta', x', \alpha']) +_F^* [\theta'', x'', \alpha''] \\ &= [\theta \wedge \theta', x + x', \alpha \wedge \alpha'] +_F^* [\theta'', x'', \alpha''] \\ &= [\theta \wedge \theta' \wedge \theta'', x + x' + x'', \alpha \wedge \alpha' \wedge \alpha''] \ . \end{aligned}$$

implies

$$A +_F^* (B +_F^* C) = (A +_F^* B) +_F^* C \ .$$

(*iii*) There exists $e^* = [90°, 0, 90°]$ such that for any $A = [\theta, x, \alpha] \in \mathcal{F}_{TBNC}(\mathbb{R})$,

$$e^* +_F^* A = [90°, 0, 90°] +_F^* [\theta, x, \alpha] = [\theta, x, \alpha] = A +_F^* e^* \ .$$

Here e^* acts as the identity element in $\mathcal{F}_{TBNC}(\mathbb{R})$ w.r.t. the operation $+_F^*$.

(*iv*) For commutativity:

$$A +_F^* B = [\theta \wedge \theta', x + x', \alpha \wedge \alpha'] = [\theta' \wedge \theta, x' + x, \alpha' \wedge \alpha] = B +_F^* A \ .$$

Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}), +_F^*)$ is a commutative monoid. □

Theorem 4.11. *$(\mathcal{F}_{TBNC}(\mathbb{R}), \cdot_F^*)$ is a semigroup.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'], C = [\theta'', x'', \alpha''] \in \mathcal{F}_{TBNC}(\mathbb{R})$. Then the following holds:

(*i*) Since $0 < \theta \vee \theta', \alpha \vee \alpha' \leq 90°$, $A \cdot_F^* B \in \mathcal{F}_{TBNC}(\mathbb{R})$.

(*ii*) We also have $A \cdot_F^* (B \cdot_F^* C) = (A \cdot_F^* B) \cdot_F^* C$, since

$$\begin{aligned} A \cdot_F^* (B \cdot_F^* C) &= [\theta, x, \alpha] \cdot_F^* ([\theta', x', \alpha'] \cdot_F^* [\theta'', x'', \alpha'']) \\ &= [\theta, x, \alpha] \cdot_F^* ([\theta^{'} \vee \theta'', x' + x'', \alpha' \vee \alpha'']) \\ &= [\theta \vee \theta' \vee \theta'', x + x' + x'', \alpha \vee \alpha' \vee \alpha''] \end{aligned}$$

and,

$$\begin{aligned} (A \cdot_F^* B) \cdot_F^* C &= ([\theta, x, \alpha] \cdot_F^* [\theta', x', \alpha']) +_F^* [\theta'', x'', \alpha''] \\ &= [\theta \vee \theta', x + x', \alpha \vee \alpha'] \cdot_F^* [\theta'', x'', \alpha''] \\ &= [\theta \vee \theta' \vee \theta'', x + x' + x'', \alpha \vee \alpha' \vee \alpha''] \ . \end{aligned}$$

Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}), \cdot_F^*)$ is a semigroup. □

Remark 4.12. *There is no element $e \in \mathcal{F}_{TBNC}(\mathbb{R})$ for which $A \cdot_F^* e = e \cdot_F^* A = A$. Hence, there exist no identity element w.r.t. $\cdot_F^*$ implying $(\mathcal{F}_{TBNC}(\mathbb{R}), \cdot_F^*)$ is neither a monoid nor a group.*

Remark 4.13. *Let $A = [40°, 1, 50°], B = [50°, -2, 30°]$ and, $C = [90°, 3, 10°] \in \mathcal{F}_{TBNC}(\mathbb{R})$. We observe that $A \cdot_F^* (B +_F^* C) = [40°, 1, 50°] \cdot_F^* [50°, 1, 10°] = [50°, 2, 50°]$ and $A \cdot_F^* B +_F^* A \cdot_F^* C = [50°, -1, 50°] +_F^* [90°, 4, 50°] = [50°, 3, 50°]$, resulting in $A \cdot_F^* (B +_F^* C) \neq A \cdot_F^* B +_F^* A \cdot_F^* C$. Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}), +_F^*, \cdot_F^*)$ is none of the ringoid, semi-ring, near-semi-ring, and near-ring.*

Remark 4.14. *Let $A = [50°, 1, 40°], B = [45°, 2, 50°], C = [45°, 2, 55°]$ and, $D = [45°, 3, 40°] \in \mathcal{F}_{TBNC}(\mathbb{R})$. Then $A +_F^* B = [45°, 3, 40°] = D$ and $C +_F^* A = [45°, 3, 40°] = D$. Choose $B' = [45°, 2, 60°] \neq B$ and $C' = [45°, 2, 65°] \neq C$. Then we also obtain $A +_F^* B' = D$ and $C' +_F^* A = D$. So, B, C are not unique solutions of $A +_F^* B = D$ and $C +_F^* A = D$ respectively. Therefore, $(\mathcal{F}_{TBNC}(\mathbb{R}), +_F^*)$ is neither a quasigroup nor a loop.*

Along similar lines, one might draw the conclusion that $(\mathcal{F}_{TBNC}(\mathbb{R}), \cdot_F^*)$ is also not a loop as it is not a quasi-group.

Remark 4.15. *Since $A +_F^* A = [\theta \wedge \theta, x + x, \alpha \wedge \alpha] = [\theta, 2x, \alpha] \neq A$ and $A \cdot_F^* A = [\theta \vee \theta, x + x, \alpha \vee \alpha] = [\theta, 2x, \alpha] \neq A$, neither $(\mathcal{F}_{TBNC}(\mathbb{R}), +_F^*)$ nor $(\mathcal{F}_{TBNC}(\mathbb{R}), \cdot_F^*)$ form a semi-lattice.*

Remark 4.16. *For $A = [40°, 1, 50°], B = [50°, -2, 30°] \in \mathcal{F}_{TBNC}(\mathbb{R})$, we see $A \cdot_F^* (A +_F^* B) = [40°, 1, 50°] \cdot_F^* [40°, -1, 30°] = [40°, 0, 50°] \neq A$ implying to the fact that $(\mathcal{F}_{TBNC}(\mathbb{R}), +_F^*, \cdot_F^*)$ is not a lattice.*

4.3 Structures on $\mathcal{F}_{TBNC}(X)$ w.r.t. $\vee_F^1$ and $\wedge_F^1$

Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'] \in \mathcal{F}_{TBNC}(\mathbb{R})$ as in Section 4.2. We define another pair of operations union $(\vee_F^1)$ and the intersection $(\wedge_F^1)$ on $\mathcal{F}_{TBNC}(X)$ as follows:

$$A \vee_F^1 B = \max\left\{[\theta, x, \alpha], [\theta', x', \alpha']\right\} = \left[\max\{\theta, \theta'\}, \max\{x, x'\}, \max\{\alpha, \alpha'\}\right],$$
$$A \wedge_F^1 B = \min\{[\theta, x, \alpha], [\theta', x', \alpha']\} = [\min\{\theta, \theta'\}, \min\{x, x'\}, \min\{\alpha, \alpha'\}] .$$

Let us define a relation $\preceq_F^1$ in $\mathcal{F}_{TBNC}(X)$ as follows: for $A = [\theta, x, \alpha], B = [\theta', x', \alpha'] \in F_{TBNC}(X)$,

$$A \preceq_F^1 B \text{ iff } \theta \leq \theta', x \leq x' \text{ and, } \alpha \leq \alpha'.$$

Consequently, we discuss the structures existing on $\mathcal{F}_{TBNC}(X)$ in relation to the operations mentioned above.

Theorem 4.17. $\left(\mathcal{F}_{TBNC}(X), \preceq_F^1\right)$ *is a partially ordered set.*

Proof. It is easy to check that the relation $\preceq_F^1$ is reflexive, antisymmetric, and transitive. Hence $\left(\mathcal{F}_{TBNC}(X), \preceq_F^1\right)$ is a partially ordered set. □

Theorem 4.18. $(\mathcal{F}_{TBNC}(X), \wedge_F^1)$ *is a semigroup.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'], \text{and} C = [\theta'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. Clearly, $A \wedge_F^1 (B \wedge_F^1 C) = (A \wedge_F^1 B) \wedge_F^1 C$, since

$$\begin{aligned} A \wedge_F^1 (B \wedge_F^1 C) &= [\theta, x, \alpha] \wedge_F^1 [\min\{\theta', \theta''\}, \min\{x', x''\}, \min\{\alpha', \alpha''\}] \\ &= [\min\{\theta, \theta', \theta''\}, \min\{x, x', x''\}, \min\{\alpha, \alpha', \alpha''\}] \end{aligned}$$

and

$$\begin{aligned} (A \wedge_F^1 B) \wedge_F^1 C &= [\min\{\theta, \theta'\}, \min\{x, x'\}, \min\{\alpha, \alpha'\}] \wedge_F^1 [\theta'', x'', \alpha''] \\ &= [\min\{\theta, \theta', \theta''\}, \min\{x, x', x''\}, \min\{\alpha, \alpha', \alpha''\}] . \end{aligned}$$

Therefore, $(\mathcal{F}_{TBNC}(X), \wedge_F^1)$ forms a semigroup. □

Theorem 4.19. *$(\mathcal{F}_{TBNC}(X), \vee_F^1)$ forms a semigroup.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'], C = [\theta'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. Clearly $A \vee_F^1 (B \vee_F^1 C) = (A \vee_F^1 B) \vee_F^1 C$, since

$$\begin{aligned} A \vee_F^1 (B \vee_F^1 C) &= [\theta, x, \alpha] \vee_F^1 [\max\{\theta', \theta''\}, \max\{x', x''\}, \max\{\alpha', \alpha''\}] \\ &= [\max\{\theta, \theta', \theta''\}, \max\{x, x', x''\}, \max\{\alpha, \alpha', \alpha''\}] \end{aligned}$$

and

$$\begin{aligned} (A \vee_F^1 B) \vee_F^1 C &= [\max\{\theta, \theta'\}, \max\{x, x'\}, \max\{\alpha, \alpha'\}] \vee_F^1 [\theta'', x'', \alpha''] \\ &= [\max\{\theta, \theta', \theta''\}, \max\{x, x', x''\}, \max\{\alpha, \alpha', \alpha''\}] \ . \end{aligned}$$

Therefore, $(\mathcal{F}_{TBNC}(X), \vee_F^1)$ forms a semigroup. □

Remark 4.20. *There does not exist $e \in \mathcal{F}_{TBNC}(X)$ s.t. $e \wedge_F^1 A = A \wedge_F^1 e = A$ for arbitrary $A \in \mathcal{F}_{TBNC}(X)$. Hence $(\mathcal{F}_{TBNC}(X), \wedge_F^1)$ is neither a monoid nor a group. For a similar reason, we can also conclude that $(\mathcal{F}_{TBNC}(X), \vee_F^1)$ is neither a monoid nor a group.*

Theorem 4.21. *$(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ is a ringoid.*

Proof. To validate the theorem, we must demonstrate that

$$A \wedge_F^1 (B \vee_F^1 C) = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C)$$

where $A = [\theta, x, \alpha], B = [\theta', x', \alpha'], C = [\theta'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. To prove the above, the following cases arise:
(i) $A = B \neq C$ *(ii)* $A \neq B = C$, *(iii)* $A = C \neq B$, *(iv)* $A = B = C$, *(v)* $A \neq B \neq C$.

Now we discuss cases (i) and (v). Due to the similar nature of the proofs, we discard the proofs of the other cases.
Case-(*i*)**: ($A = B \neq C$):** In this case, the following sub-cases arise:

(a) If $C \preceq_F^1 B$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 B = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(b) If $B \preceq_F^1 C$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 C = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(c) If $\theta'' \leq \theta'$, $x'' \leq x'$ and $\alpha' \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x', \alpha''] = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(d) If $\theta'' \leq \theta'$, $x' \leq x''$ and $\alpha' \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha''] = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(e) If $\theta'' \leq \theta'$, $x' \leq x''$ and $\alpha'' \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha'] = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(f) If $\theta' \leq \theta''$, $x' \leq x''$ and $\alpha'' \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x'', \alpha'] = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(g) If $\theta' \leq \theta''$, $x'' \leq x'$ and $\alpha'' \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha'] = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(h) If $\theta' \leq \theta''$, $x' \leq x''$ and $\alpha' \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha''] = A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

Case-(v): ($A \neq B \neq C$): In this case, the following sub-cases arise:

(a) If $C \preceq_F^1 B \preceq_F^1 A$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 B = B = B \vee_F^1 C = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(b) If $A \preceq_F^1 B \preceq_F^1 C$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 C = A = A \vee_F^1 A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(c) If $B \preceq_F^1 C \preceq_F^1 A$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 C = C = B \vee_F^1 C = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(d) If $A \preceq_F^1 C \preceq_F^1 B$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 B = A = A \vee_F^1 A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(e) If $C \preceq_F^1 A \preceq_F^1 B$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 B = A = A \vee_F^1 C = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(f) If $B \preceq_F^1 A \preceq_F^1 C$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 C = A = B \vee_F^1 A = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C).$$

(g) If $\theta'' \leq \theta' \leq \theta$, $x'' \leq x' \leq x$, $\alpha \leq \alpha' \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x', \alpha''] = [\theta', x', \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x', \alpha] \vee_F^1 [\theta'', x'', \alpha] = [\theta', x', \alpha].$$

(h) If $\theta'' \leq \theta' \leq \theta$, $x \leq x' \leq x''$, $\alpha \leq \alpha' \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha''] = [\theta', x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x, \alpha] \vee_F^1 [\theta'', x, \alpha] = [\theta', x, \alpha].$$

(i) If $\theta'' \leq \theta' \leq \theta$, $x \leq x' \leq x''$, $\alpha'' \leq \alpha' \leq \alpha$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha'] = [\theta', x, \alpha'],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x, \alpha'] \vee_F^1 [\theta'', x, \alpha''] = [\theta', x, \alpha'].$$

(j) If $\theta \leq \theta' \leq \theta''$, $x'' \leq x' \leq x$, $\alpha \leq \alpha' \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha''] = [\theta, x', \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x', \alpha] \vee_F^1 [\theta, x'', \alpha] = [\theta, x', \alpha].$$

(k) If $\theta \leq \theta' \leq \theta''$, $x \leq x' \leq x''$, $\alpha'' \leq \alpha' \leq \alpha$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x'', \alpha'] = [\theta, x, \alpha'],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x, \alpha'] \vee_F^1 [\theta, x, \alpha''] = [\theta, x, \alpha'].$$

(l) If $\theta \leq \theta' \leq \theta''$, $x'' \leq x' \leq x$, $\alpha'' \leq \alpha' \leq \alpha$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha'] = [\theta, x', \alpha'],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x', \alpha'] \vee_F^1 [\theta, x'', \alpha''] = [\theta, x', \alpha'].$$

(m) If $\theta' \leq \theta'' \leq \theta$, $x' \leq x'' \leq x$, $\alpha \leq \alpha'' \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x'', \alpha'] = [\theta'', x'', \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x', \alpha] \vee_F^1 [\theta'', x'', \alpha] = [\theta'', x'', \alpha].$$

(n) If $\theta' \leq \theta'' \leq \theta$, $x \leq x'' \leq x'$, $\alpha \leq \alpha'' \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha'] = [\theta'', x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x, \alpha] \vee_F^1 [\theta'', x, \alpha] = [\theta'', x, \alpha].$$

(o) If $\theta' \leq \theta'' \leq \theta$, $x \leq x'' \leq x'$, $\alpha' \leq \alpha'' \leq \alpha$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha''] = [\theta'', x, \alpha''],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x, \alpha'] \vee_F^1 [\theta'', x, \alpha''] = [\theta'', x, \alpha''].$$

(p) If $\theta \leq \theta'' \leq \theta'$, $x' \leq x'' \leq x$, $\alpha \leq \alpha'' \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha'] = [\theta, x'', \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x', \alpha] \vee_F^1 [\theta, x'', \alpha] = [\theta, x'', \alpha].$$

(q) If $\theta \leq \theta'' \leq \theta'$, $x \leq x'' \leq x'$, $\alpha' \leq \alpha'' \leq \alpha$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x', \alpha''] = [\theta, x, \alpha''],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x, \alpha'] \vee_F^1 [\theta, x, \alpha''] = [\theta, x, \alpha''].$$

(r) If $\theta \leq \theta'' \leq \theta'$, $x' \leq x'' \leq x$, $\alpha' \leq \alpha'' \leq \alpha$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha''] = [\theta, x'', \alpha''],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x', \alpha'] \vee_F^1 [\theta, x'', \alpha''] = [\theta, x'', \alpha''].$$

(s) If $\theta'' \leq \theta \leq \theta'$, $x'' \leq x \leq x'$, $\alpha' \leq \alpha \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x', \alpha''] = [\theta, x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x, \alpha'] \vee_F^1 [\theta'', x'', \alpha] = [\theta, x, \alpha].$$

(t) If $\theta'' \leq \theta \leq \theta'$, $x' \leq x \leq x''$, $\alpha' \leq \alpha \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha''] = [\theta, x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x', \alpha'] \vee_F^1 [\theta'', x, \alpha] = [\theta, x, \alpha].$$

(u) If $\theta'' \leq \theta \leq \theta'$, $x' \leq x \leq x''$, $\alpha'' \leq \alpha \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta', x'', \alpha''] = [\theta, x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta, x', \alpha'] \vee_F^1 [\theta'', x, \alpha''] = [\theta, x, \alpha].$$

(v) If $\theta' \leq \theta \leq \theta''$, $x'' \leq x \leq x'$, $\alpha' \leq \alpha \leq \alpha''$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha''] = [\theta, x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x, \alpha'] \vee_F^1 [\theta, x'', \alpha] = [\theta, x, \alpha].$$

(w) If $\theta' \leq \theta \leq \theta''$, $x' \leq x \leq x''$, $\alpha'' \leq \alpha \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x'', \alpha'] = [\theta, x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x', \alpha] \vee_F^1 [\theta, x, \alpha''] = [\theta, x, \alpha].$$

(x) If $\theta' \leq \theta \leq \theta''$, $x'' \leq x \leq x'$, $\alpha'' \leq \alpha \leq \alpha'$ then

$$A \wedge_F^1 (B \vee_F^1 C) = A \wedge_F^1 [\theta'', x', \alpha'] = [\theta, x, \alpha],$$
$$(A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C) = [\theta', x, \alpha] \vee_F^1 [\theta, x'', \alpha''] = [\theta, x, \alpha].$$

Thus, $A \wedge_F^1 (B \vee_F^1 C) = (A \wedge_F^1 B) \vee_F^1 (A \wedge_F^1 C)$ for $A \neq B \neq C$.

Consequently, $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ is a ringoid. □

Theorem 4.22. *$(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ is a commutative semi-ring.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'] \in \mathcal{F}_{TBNC}(X)$. Then we have

$$\begin{aligned} A \wedge_F^1 B &= \min\{[\theta, x, \alpha], [\theta', x', \alpha']\} \\ &= [\min\{\theta, \theta'\}, \min\{x, x'\}, \min\{\alpha, \alpha'\}] = B \wedge_F^1 A \end{aligned}$$

Hence $\wedge_F^1$ is commutative. From Theorem 4.18 and Theorem 4.19 we have seen that $(\mathcal{F}_{TBNC}(X)$ is associative under $\wedge_F^1$ and $\vee_F^1$ and from Theorem 4.21 we know that $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ is a ringoid. Therefore, $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ is a commutative semi-ring. □

Remark 4.23. *Since* $(\mathcal{F}_{TBNC}(X), \vee_F^1)$ *fails to be a monoid,* $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ *is neither a near-semi-ring nor a near ring.*

Remark 4.24. *Let* $A = [80°, 1, 20°], B = [50°, 0.5, 40°], C = [60°, 0.9, 40°] \in \mathcal{F}_{TBNC}(X)$. *Then* $A \vee_F^1 B = [80°, 1, 40°] = F$ *and* $C \vee_F^1 A = [80°, 1, 40°] = F$. *Again, we consider* $B' = [60°, 0, 40°]$ *and* $C' = [50°, 0, 40°]$. *Then* $A \vee_F^1 B = A \vee_F^1 B' = [80°, 1, 40°] = F$ *and* $C \vee_F^1 A = C' \vee_F^1 A = [80°, 1, 40°] = F$. *We see that* $B' \neq B$ *and* $C' \neq C$. *Hence* B, C *are not unique solutions of* $A \vee_F^1 B = F$ *and* $C \vee_F^1 A = F$, *respectively. Thus,* $(\mathcal{F}_{TBNC}(X), \vee_F^1)$ *is not a quasigroup.*

Again, if we assume $A = [80°, 1, 20°], B = [50°, 0.5, 40°], D = [55°, 2, 20°] \in \mathcal{F}_{TBNC}(X)$. *Then* $B \wedge_F^1 A = [50°, 0.5, 20°] = E$ *and* $D \wedge_F^1 B = [50°, 0.5, 20°] = E$. *Now, if we take* $A' = [70°, 2, 20°]$ *and* $D' = [60°, 3, 20°]$ *then we have* $B \wedge_F^1 A = [50°, 0.5, 20°] = E = B \wedge_F^1 A'$ *and* $D \wedge_F^1 B = [50°, 0.5, 20°] = E = D' \wedge_F^1 B$. *we again see* $A' \neq A$ *and* $D' \neq D$. *Hence* A *and* D *are not unique solutions of* $B \wedge_F^1 A = E$ *and* $D \wedge_F^1 B = E$, *respectively. Thus,* $(\mathcal{F}_{TBNC}(X), \wedge_F^1)$ *fails to be a quasigroup.*

Theorem 4.25. $(\mathcal{F}_{TBNC}(X), \wedge_F^1)$ *is a semi-lattice.*

Proof. Let $A, B, C \in \mathcal{F}_{TBNC}(X)$. We possess the following:

(*i*) $A \wedge_F^1 B = B \wedge_F^1 A$ from Theorem 4.22,

(*ii*) $A \wedge_F^1 A = A \ \forall \ A \in \mathcal{F}_{TBNC}(X)$, and

(*iii*) $A \wedge_F^1 (B \wedge_F^1 C) = (A \wedge_F^1 B) \wedge_F^1 C$ from Theorem 4.18.

From the above facts, we conclude that $(\mathcal{F}_{TBNC}(X), \wedge_F^1)$ is a semi-lattice. □

Theorem 4.26. $(\mathcal{F}_{TBNC}(X), \vee_F^1)$ *is a semi-lattice.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'], C = [\theta'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. We have the following.

(*i*) $A \vee_F^1 B = [\max\{\theta, \theta'\}, \max\{x, x'\}, \max\{\alpha, \alpha'\}] = B \vee_F^1 A$,

(*ii*) $A \vee_F^1 A = A$ for all $A \in \mathcal{F}_{TBNC}(X)$, and

(*iii*) $A \vee_F^1 (B \vee_F^1 C) = (A \vee_F^1 B) \vee_F^1 C$ from Theorem 4.19.

From the above facts, we conclude that $(\mathcal{F}_{TBNC}(X), \vee_F^1)$ is a semi-lattice. □

Theorem 4.27. $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ *forms a lattice.*

Proof. Let $A = [\theta, x, \alpha], B = [\theta', x', \alpha'] \in \mathcal{F}_{TBNC}(X)$. For $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ to be a lattice, we need to prove the following: (I) $A \vee_F^1 (A \wedge_F^1 B) = A$, and (II) $A \wedge_F^1 (A \vee_F^1 B) = A$.

We need to discuss two cases to prove this.
Case-(*i*) **(** $A = B$**):** $A \vee_F^1 (A \wedge_F^1 B) = A$ and $A \wedge_F^1 (A \vee_F^1 B) = A$.
Case-(*ii*) **(**$A \neq B$**) :**

(a) If $B \preceq_F^1 A$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 B = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 A.$$

(b) If $A \preceq_F^1 B$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 A = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 B = A.$$

(c) If $\theta' \leq \theta$, $x' \leq x$ and $\alpha \leq \alpha'$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 [\theta', x', \alpha] = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 [\theta, x, \alpha'] = A.$$

(d) If $\theta' \leq \theta$, $x \leq x'$ and $\alpha \leq \alpha'$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 [\theta', x, \alpha] = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 [\theta, x', \alpha'] = A.$$

(e) If $\theta' \leq \theta$, $x \leq x'$ and $\alpha' \leq \alpha$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 [\theta', x, \alpha'] = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 [\theta, x', \alpha] = A.$$

(f) If $\theta \leq \theta'$, $x' \leq x$ and $\alpha \leq \alpha'$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 [\theta, x', \alpha] = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 [\theta', x, \alpha'] = A.$$

(g) If $\theta \leq \theta'$, $x \leq x'$ and $\alpha' \leq \alpha$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 [\theta, x, \alpha'] = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 [\theta', x', \alpha] = A.$$

(h) If $\theta \leq \theta'$, $x' \leq x$ and $\alpha' \leq \alpha$ then

$$A \vee_F^1 (A \wedge_F^1 B) = A \vee_F^1 [\theta, x', \alpha'] = A, \text{ and}$$
$$A \wedge_F^1 (A \vee_F^1 B) = A \wedge_F^1 [\theta', x, \alpha] = A.$$

Therefore, $(\mathcal{F}_{TBNC}(X), \vee_F^1, \wedge_F^1)$ forms a lattice. □

4.4 Structures on $\mathcal{F}_{TBNC}(X)$ w.r.t. $\vee_F^2$ and $\wedge_F^2$

Let $A = [a, x, \alpha], B = [a', x', \alpha'] \in \mathcal{F}_{TBNC}(X)$ where a, a' are the left end points; x, x' are the normal points; $\alpha, \alpha'(0 \leq \alpha, \alpha' < 180°)$ are the top angles, respectively. We define new type of operations union ($\vee_F^2$) and intersection ($\wedge_F^2$) $\mathcal{F}_{TBNC}(X)$ as follows:

$$A \vee_F^2 B = \max\{[a, x, \alpha], [a', x', \alpha']\} = [\max(a, a'), \max(x, x'), \max(\alpha, \alpha')]$$
$$A \wedge_F^2 B = \min\{[a, x, \alpha], [a', x', \alpha']\} = [\min(a, a'), \min(x, x'), \min(\alpha, \alpha')] \ .$$

Let us define a relation $\preceq_F^2$ in $\mathcal{F}_{TBNC}(X)$ as follows: for $A = [a, x, \alpha], B = [a', x', \alpha'] \in F_{TBNC}(X)$,

$$A \preceq_F^2 B \text{ iff } a \leq a', x \leq x' \text{ and, } \alpha \leq \alpha.$$

Consequently, we discuss the structures found on $\mathcal{F}_{TBNC}(X)$ w.r.t. the operations mentioned above.

Theorem 4.28. *$\left(\mathcal{F}_{TBNC}(X), \preceq_F^2\right)$ is a partially ordered set.*

Proof. It is easy to check that the relation $\preceq_F^2$ is reflexive, antisymmetric, and transitive. Hence $\left(\mathcal{F}_{TBNC}(X), \preceq_F^2\right)$ is a partially ordered set. □

Theorem 4.29. *$(\mathcal{F}_{TBNC}(X), \vee_F^2)$ is a semigroup.*

Proof. Let $A = [a, x, \alpha], B = [a', x', \alpha'], C = [a'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. Then

$$\begin{aligned} A \vee_F^2 (B \vee_F^2 C) &= A \vee_F^2 [\max(a', a''), \max(x', x''), \max(\alpha', \alpha'')] \\ &[\max(a, a', a''), \max(x, x', x''), \max(\alpha, \alpha', \alpha'')] \\ &= (A \vee_F^2 B) \vee_F^2 C. \end{aligned}$$

Therefore, $(\mathcal{F}_{TBNC}(X), \vee_F^2)$ is a semigroup. □

Theorem 4.30. *$(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ is a semigroup.*

Proof. Let $A = [a, x, \alpha], B = [a', x', \alpha'], C = [a'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. Then

$$\begin{aligned} A \wedge_F^2 (B \wedge_F^2 C) &= A \wedge_F^2 [\min(a', a''), \min(x', x''), \min(\alpha', \alpha'')] \\ &= [\min(a, a', a''), \min(x, x', x''), \min(\alpha, \alpha', \alpha'')] \\ &= (A \wedge_F^2 B) \wedge_F^2 C. \end{aligned}$$

Therefore, $(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ is a semigroup. □

Remark 4.31. *Since there does not exist any $e \in \mathcal{F}_{TBNC}(X)$ such that $e \vee_F^2 A = A \vee_F^2 e = A$, $(\mathcal{F}_{TBNC}(X), \vee_F^2)$ is neither a monoid nor a group. Similarly, $(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ is neither a monoid nor a group.*

Theorem 4.32. *$(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is a ringoid.*

Proof. To prove the theorem, we need to show that

$$A \wedge_F^2 (B \vee_F^2 C) = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C)$$

where $A = [a, x, \alpha], B = [a', x', \alpha'], C = [a'', x'', \alpha''] \in \mathcal{F}_{TBNC}(X)$. To prove the above, the following cases arise:

(i) $A = B \neq C$ $\quad (ii)$ $A = C \neq B$, $\quad (iii)$ $A \neq B = C$, $\quad (iv)$ $A = B = C$, $\quad (v)$ $A \neq B \neq C$.

Now we discuss cases (i) and (v). Due to the similar nature of the proofs, we discard the proofs of the other cases.

Case-(i): $(A = B \neq C)$: In this case, the following sub-cases arise:

(a) If $C \preceq_F^2 B$ then
$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 C = A = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^2 C).$$

(b) If $B \preceq_F^2 C$ then
$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 B = A = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^2 C).$$

(c) If $a'' \leq a'$, $x'' \leq x'$ and $\alpha' \leq \alpha''$ then
$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 [a'', x'', \alpha'] = A = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^1 C).$$

(d) If $a'' \leq a'$, $x' \leq x''$ and $\alpha' \leq \alpha''$ then

$$A \vee_F^2 (B \wedge_F^1 C) = A \vee_F^2 [a'', x', \alpha'] = A = (A \vee_F^2 B) \wedge_F^1 (A \vee_F^2 C).$$

(e) If $a'' \leq a'$, $x' \leq x''$ and $\alpha'' \leq \alpha'$ then

$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 [a^{''}, x^{'}, \alpha^{''}] = A = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^2 C).$$

(f) If $a' \leq a''$, $x' \leq x''$ and $\alpha'' \leq \alpha'$ then

$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 [a', x', \alpha''] = A = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^2 C).$$

(g) If $a' \leq a''$, $x'' \leq x'$ and $\alpha'' \leq \alpha'$ then

$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 [a', x'', \alpha''] = A = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^2 C).$$

(h) If $a' \leq a''$, $x'' \leq x'$ and $\alpha' \leq \alpha''$ then

$$A \vee_F^2 (B \wedge_F^2 C) = A \vee_F^2 [a', x'', \alpha'] = [a', x', \alpha''] = (A \vee_F^2 B) \wedge_F^2 (A \vee_F^2 C).$$

Case-(V)**: (**$A \neq B \neq C$**):** In this case, the following sub-cases arise:

(a) If $C \preceq_F^2 B \preceq_F^2 A$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 B = B = B \vee_F^2 C = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C).$$

(b) If $A \preceq_F^2 B \preceq_F^2 C$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 C = A = A \vee_F^2 A = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C).$$

(c) If $B \preceq_F^2 C \preceq_F^2 A$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 C = C = B \vee_F^2 C = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C).$$

(d) If $A \preceq_F^2 C \preceq_F^2 B$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 B = A = A \vee_F^2 A = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C).$$

(e) If $C \preceq_F^2 A \preceq_F^2 B$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 B = A = A \vee_F^2 C = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C).$$

(f) If $B \preceq_F^2 A \preceq_F^2 C$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 C = A = B \vee_F^2 A = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C).$$

(g) If $a'' \leq a' \leq a$, $x'' \leq x' \leq x$, $\alpha \leq \alpha' \leq \alpha''$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x', \alpha''] = [a', x', \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x', \alpha] \vee_F^2 [a'', x'', \alpha] = [a', x', \alpha].$$

(h) If $a'' \leq a' \leq a$, $x \leq x' \leq x''$, $\alpha \leq \alpha' \leq \alpha''$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x'', \alpha''] = [a', x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x, \alpha] \vee_F^2 [a'', x, \alpha] = [a', x, \alpha].$$

(i) If $a'' \leq a' \leq a$, $x \leq x' \leq x''$, $\alpha'' \leq \alpha' \leq \alpha$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x'', \alpha'] = [a', x, \alpha']$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^1 C) = [a', x, \alpha'] \vee_F^2 [a'', x, \alpha''] = [', x, \alpha'].$$

(j) If $a \leq a' \leq a''$, $x'' \leq x' \leq x$, $\alpha \leq \alpha' \leq \alpha''$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x', \alpha''] = [a, x', \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x', \alpha] \vee_F^2 [a, x'', \alpha] = [a, x', \alpha].$$

(k) If $a \leq a' \leq a''$, $x \leq x' \leq x''$, $\alpha'' \leq \alpha' \leq \alpha$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x'', \alpha'] = [a, x, \alpha']$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x, \alpha'] \vee_F^2 [a, x, \alpha''] = [a, x, \alpha'].$$

(l) If $a \leq a' \leq a''$, $x'' \leq x' \leq x$, $\alpha'' \leq \alpha' \leq \alpha$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x', \alpha'] = [a, x', \alpha']$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x', \alpha'] \vee_F^2 [a, x'', \alpha''] = [a, x', \alpha'].$$

(m) If $a' \leq a'' \leq a$, $x' \leq x'' \leq x$, $\alpha \leq \alpha'' \leq \alpha'$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x'', \alpha'] = [a'', x'', \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x', \alpha] \vee_F^2 [a'', x'', \alpha] = [a'', x'', \alpha].$$

(n) If $a' \leq a'' \leq a$, $x \leq x'' \leq x'$, $\alpha \leq \alpha'' \leq \alpha'$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x', \alpha'] = [a'', x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x, \alpha] \vee_F^2 [a'', x, \alpha] = [a'', x, \alpha].$$

(o) If $a' \leq a'' \leq a$, $x \leq x'' \leq x'$, $\alpha' \leq \alpha'' \leq \alpha$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x', \alpha''] = [a'', x, \alpha'']$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x, \alpha'] \vee_F^2 [a'', x, \alpha''] = [a'', x, \alpha''].$$

(p) If $a \leq a'' \leq a'$, $x' \leq x'' \leq x$, $\alpha \leq \alpha'' \leq \alpha'$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x'', \alpha'] = [a, x'', \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x', \alpha] \vee_F^2 [a, x'', \alpha] = [a, x'', \alpha].$$

(q) If $a \leq a'' \leq a'$, $x \leq x'' \leq x'$, $\alpha' \leq \alpha'' \leq \alpha$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x', \alpha''] = [a, x, \alpha'']$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x, \alpha'] \vee_F^2 [a, x, \alpha''] = [a, x, \alpha''].$$

(r) If $a \leq a'' \leq a'$, $x' \leq x'' \leq x$, $\alpha' \leq \alpha'' \leq \alpha$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x'', \alpha''] = [a, x'', \alpha'']$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x', \alpha'] \vee_F^2 [a, x'', \alpha''] = [a, x'', \alpha''].$$

(s) If $a'' \leq a \leq a'$, $x'' \leq x \leq x'$, $\alpha' \leq \alpha \leq \alpha''$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x', \alpha''] = [a, x, \alpha]$$
$$(A \wedge_F^1 B) \vee_F^2 (A \wedge_F^2 C) = [a, x, \alpha'] \vee_F^2 [a'', x'', \alpha] = [a, x, \alpha].$$

(t) If $a'' \leq a \leq a'$, $x' \leq x \leq x''$, $\alpha' \leq \alpha \leq \alpha''$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x'', \alpha''] = [a, x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x', \alpha'] \vee_F^2 [a'', x, \alpha] = [a, x, \alpha].$$

(u) If $a'' \leq a \leq a'$, $x' \leq x \leq x''$, $\alpha'' \leq \alpha \leq \alpha'$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a', x'', \alpha''] = [a, x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a, x', \alpha'] \vee_F^2 [a'', x, \alpha''] = [a, x, \alpha].$$

(v) If $a' \leq a \leq a''$, $x'' \leq x \leq x'$, $\alpha' \leq \alpha \leq \alpha''$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x', \alpha''] = [a, x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x, \alpha'] \vee_F^2 [a, x'', \alpha] = [a, x, \alpha].$$

(w) If $a' \leq a \leq a''$, $x' \leq x \leq x''$, $\alpha'' \leq \alpha \leq \alpha'$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x'', \alpha'] = [a, x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x', \alpha] \vee_F^2 [a, x, \alpha''] = [a, x, \alpha].$$

(x) If $a' \leq a \leq a''$, $x'' \leq x \leq x'$, $\alpha'' \leq \alpha \leq \alpha'$ then

$$A \wedge_F^2 (B \vee_F^2 C) = A \wedge_F^2 [a'', x', \alpha'] = [a, x, \alpha]$$
$$(A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C) = [a', x, \alpha] \vee_F^2 [a, x'', \alpha''] = [a, x, \alpha].$$

Hence, $A \wedge_F^2 (B \vee_F^2 C) = (A \wedge_F^2 B) \vee_F^2 (A \wedge_F^2 C)$ for $A \neq B \neq C$. Consequently, $(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is a ringoid. □

Theorem 4.33. *$(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is a commutative semi-ring.*

Proof. Let $A = [a, x, \alpha], B = [a', x', \alpha'] \in \mathcal{F}_{TBNC}(X)$. Then we have (*i*) $\vee_F^2$ is commutative since

$$A \vee_F^2 B = \max\{[a, x, \alpha], [a', x', \alpha']\}$$
$$= [\max\{a, a'\}, \max\{x, x'\}, \max\{\alpha, \alpha'\}] = B \vee_F^2 A.$$

(*ii*) $\wedge_F^2$ is commutative since

$$A \wedge_F^2 B = \min\{[a, x, \alpha], [a', x', \alpha']\}$$
$$= [\min\{a, a'\}, \min\{x, x'\}, \min\{\alpha, \alpha'\}] = B \wedge_F^2 A.$$

(*iii*) Both $\vee_F^2$ and $\wedge_F^2$ are associative from Theorem 4.29 and Theorem 4.30, respectively.

Hence $(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is a commutative semi-ring. □

Remark 4.34. *Since $(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ and $(\mathcal{F}_{TBNC}(X), \vee_F^2)$ are not monoids, $(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is neither near-semi-ring nor a near ring.*

Remark 4.35. *Let $A = [-2, 1, 50°], B = [1, 0.5, 60°], C = [1, 0.9, 60°] \in \mathcal{F}_{TBNC}(X)$. Then $A \vee_F^2 B = [1, 1, 60°] = E$ and $C \vee_F^2 A = [1, 1, 60°] = E$. If we choose $B' = [1, 0.3, 60°]$ and $C' = [1, 0.7, 60°]$ then $A \vee_F^2 B' = [1, 1, 60°] = E = A \vee_F^2 B$ and $C' \vee_F^2 A = [1, 1, 60°] = E = C \vee_F^2 A$. We observe $B' \neq B$ and $C' \neq C$ implying B, C are not unique solutions of $A \vee_F^2 B = E$ and $C \vee_F^2 A = E$, respectively. Hence, $(\mathcal{F}_{TBNC}(X), \vee_F^2)$ is not a quasi-group.*

Again, let $A = [-2, 1, 50°], B = [1, 0.5, 60°], D = [-2, 2, 50°] \in \mathcal{F}_{TBNC}(X)$. Then $B \wedge_F^2 A = [-2, 0.5, 50°] = F$ and $D \wedge_F^2 B = [-2, 0.5, 50°] = F$. Let us take $A' = [-2, 3, 50°]$, $D' = [-2, 4, 50°]$, then $B \wedge_F^2 A' = [-2, 0.5, 50°] = F = B \wedge_F^2 A$ and $D' \wedge_F^2 B = [-2, 0.5, 50°] = F = D \wedge_F^2 B$. We see $A \neq A'$ and $D \neq D'$ implying $B \wedge_F^2 A = F$ and $D \wedge_F^2 B = F$ have no unique solutions. Hence, $(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ is not a quasi-group.

Theorem 4.36. *$(\mathcal{F}_{TBNC}(X), \vee_F^2)$ is a semi-lattice.*

Proof. We know that $\vee_F^2$ is associative and commutative from Theorem 4.29 and Theorem 4.33, respectively. Again, for any $A = [a, x, \alpha] \in \mathcal{F}_{TBNC}(X)$ we have

$$A \vee^2 A = [\max(a, a), \max(x, x), \max(\alpha, \alpha)] = A.$$

Therefore, $(\mathcal{F}_{TBNC}(X), \vee_F^2)$ is a semi-lattice. □

Theorem 4.37. *$(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ is a semi-lattice.*

Proof. We know that $\wedge_F^2$ is associative and commutative from Theorem 4.30 and Theorem 4.33, respectively. Again, $A \vee^2 A = A$ for any $A = [a, x, \alpha] \in \mathcal{F}_{TBNC}(X)$. Therefore, $(\mathcal{F}_{TBNC}(X), \wedge_F^2)$ is a semi-lattice. □

Theorem 4.38. *$(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is a lattice.*

Proof. Let $A = [a, x, \alpha], B = [a', x', \alpha'] \in \mathcal{F}_{TBNC}(X)$. For $\left(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2\right)$ to be a lattice, we need to prove the following:

$$A \vee_F^2 (A \wedge_F^2 B) = A \quad \text{and} \quad A \wedge_F^2 (A \vee_F^2 B) = A.$$

We need to discuss two cases to prove this.
Case-(*i*) **(** $A = B$**):** $A \vee_F^2 (A \wedge_F^2 B) = A$ and $A \wedge_F^2 (A \vee_F^2 B) = A$.
Case-(*ii*) **(**$A \neq B$**) :**

(a) If $A \preceq_F^2 B$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 A = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 B = A.$$

(b) If $B \preceq_F^2 A$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 B = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 A = A.$$

(c) If $a' \leq a, x' \leq x, \alpha \leq \alpha'$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 [a', x', \alpha] = [a, x, \alpha] = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 [a, x, \alpha'] = [a, x, \alpha] = A.$$

(d) If $a' \leq a, x \leq x', \alpha \leq \alpha'$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 [a', x, \alpha] = [a, x, \alpha] = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 [a, x', \alpha'] = [a, x, \alpha] = A.$$

(e) If $a' \leq a, x \leq x', \alpha' \leq \alpha$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 [a', x, \alpha'] = [a, x, \alpha] = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 [a, x', \alpha] = [a, x, \alpha] = A.$$

(f) If $a \leq a', x \leq x', \alpha' \leq \alpha$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 [a, x, \alpha'] = [a, x, \alpha] = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 [a', x', \alpha] = [a, x, \alpha] = A.$$

(g) If $a \leq a', x' \leq x, \alpha' \leq \alpha$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 [a, x', \alpha'] = [a, x, \alpha] = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 [a', x, \alpha] = [a, x, \alpha] = A.$$

(h) If $a \leq a', x' \leq x, \alpha \leq \alpha'$ then

$$A \vee_F^2 (A \wedge_F^2 B) = A \vee_F^2 [a, x', \alpha] = [a, x, \alpha] = A, \text{ and}$$
$$A \wedge_F^2 (A \vee_F^2 B) = A \wedge_F^2 [a', x, \alpha'] = [a, x, \alpha] = A.$$

Therefore, $(\mathcal{F}_{TBNC}(X), \vee_F^2, \wedge_F^2)$ is a lattice. $\square$

5 Concluding Remarks

In this work, we have seen that the space of fuzzy sets, along with different operations on them, gives rise to nice algebraic structures. The space of all bounded fuzzy sets w.r.t. arbitrary t-norms and t-conorms (in particular, the usual fuzzy set union and intersection operation) gives rise to structures as rich as commutative monoid, anti-ring, and near semi-ring. Further, in different subspaces of fuzzy sets, new operations have been defined, and different algebraic structures have been studied w.r.t. the newly defined operations. In our work, we have obtained nice structures as rich as commutative semi-rings, lattices, etc. Future work will include discussing richer algebraic and topological structures on space or different subspaces of fuzzy sets.

Mainly, we have studied algebraization on the space and different subspaces of fuzzy sets by using existing operations and defining new operations on it and we obtained structures like commutative monoid, anti-ring, near-semi-ring, commutative semi-rings, and lattices, etc. Later on, the ideal of semi-rings, the semi-module of a commutative semi-ring, topological semi-rings, the morphisms (homomorphism, endomorphism) of those structures etc. may be studied. Also, we can study richer structures on the space or different subspaces of fuzzy sets by using existing operations or defining new operations on them.

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References

[1] Zadeh LA. Fuzzy sets. *Information and control*. 1965; 8(3): 338-353. DOI: http://doi.org/10.1016/S0019-9958(65)90241-X

[2] De Luca A, Termini S. Algebraic properties of fuzzy sets. *Journal of mathematical analysis and applications*. 1972; 40(2): 373-386. DOI: http://doi.org/10.1016/0022-247X(72)90057-1

[3] Mizumoto M, Tanaka K. Fuzzy sets and their operations. *Information and Control*. 1981; 48(1): 30-48. DOI: https://doi.org/10.1016/S0019-9958(81)90578-7

[4] Sussner P, Torres RP. Subsethood Measures on a Bounded Lattice of Continuous Fuzzy Numbers with an Application in Approximate Reasoning. *In: North American Fuzzy Information Processing Society Annual Conference*. Cham: Springer; 2022. p. 267-278. DOI: https://doi.org/10.1007/978-3-031-16038-7_26

[5] Homenda W. Algebraic operators: an alternative approach to fuzzy sets. *International Journal of Applied Mathematics and Computer Science*. 1996; 6(3): 505527. https://www.amcs.uz.zgora.pl/?action=paper&paper=1125

[6] Cao N, tpnika M. Preservation of properties of residuated algebraic structure by structures for the partial fuzzy set theory. *International Journal of Approximate Reasoning*. 2023; 154: 1-26. DOI: https://doi.org/10.1016/j.ijar.2022.12.001

[7] Mizumoto M, Tanaka K. Some properties of fuzzy sets of type 2. *Information and control*. 1976; 31(4): 312- 340. DOI: https://doi.org/10.1016/S0019-9958(76)80011-3

[8] Nieminen J. On the algebraic structure of fuzzy sets of type 2. *Kybernetika*. 1977; 13(4): 261273. https://eudml.org/doc/28219

[9] Eslami E. An algebraic structure for intuitionistic fuzzy logic. *Iranian Journal of Fuzzy Systems.* 2012; 9(6): 31-41. DOI: https://doi.org/10.22111/ijfs.2012.111

[10] Turunen E. Algebraic structures in fuzzy logic. *Fuzzy Sets and Systems.* 1992; 52(2): 181-188. DOI: https://doi.org/10.1016/0165-0114(92)90048-9

[11] Chang CL. Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications.* 1968; 24(1): 182-190. DOI: https://doi.org/10.1016/0022-247X(68)90057-7

[12] Sarkar M. On fuzzy topological spaces. *Journal of mathematical analysis and applications.* 1981; 79(2): 384-394. DOI: https://doi.org/10.1016/0022-247X(81)90033-0

[13] Ganguly S, Sara S. On separation axioms and T_i-fuzzy continuity. *Fuzzy sets and systems.* 1985; 16(3): 265-275. DOI: https://doi.org/10.1016/0165-0114(85)90030-2

[14] Sinha SP. Separation axioms in fuzzy topological spaces. *Fuzzy sets and systems.* 1992; 45(2): 261270. DOI: https://doi.org/10.1016/0165-0114(92)90127-P

[15] Mukherjee MN, Sinha SP. Almost compact fuzzy sets in fuzzy topological spaces. *Fuzzy Sets and Systems.* 1990; 38(3): 389-396. DOI: https://doi.org/10.1016/0165-0114(90)90211-N

[16] Lowen R. Connectedness in fuzzy topological spaces. *The Rocky Mountain Journal of Mathematics.* 1981; 11(3): 427-433. DOI: https://doi.org/10.1216/RMJ-1981-11-3-427

[17] Baczynski M, Drewniak J, Sobera J. Semigroups of fuzzy implications. *atra Mountains Mathematical Publications.* 2001; 21(1): 61- 71.

[18] Vemuri NR, Jayaram B. Homomorphisms on the Monoid of Fuzzy Implications -A Complete Characterization. *International Conference on Pattern Recognition and Machine Intelligence, 2013, 10-14 Dec 2013, Kolkata, India.* Berlin: Springer; 2013. p. 563568. DOI: https://doi.org/10.1007/978-3-642-45062-4_78

[19] Bejines C. Aggregation of fuzzy vector spaces. *Kybernetika.* 2023; 59(5): 752-767. DOI: https://doi.org/10.14736/kyb-2023-5-0752

[20] Assiry A, Baklouti A. Exploring roughness in left almost semigroups and its Cconnections tofuzzy lie algebras. *Symmetry.* 2023; 15(9): 1717. DOI: https://doi.org/10.3390/sym15091717

[21] Gupta VK, Jayaram B. *Importation Algebras. In: Halas R, Gagolewski M, Mesiar R, editors. New Trends in Aggregation Theory.* Cham: Springer International Publishing; 2019. DOI: https://doi.org/10.1007/978-3-030-19494-9_8

[22] Zimmermann H-J. *Fuzzy Set Theory and Its Applications.* 4th ed. Berlin: Springer Dordrecht; 2011. DOI: https://doi.org/10.1007/978-94-010-0646-0

[23] Baczynski M, Jayaram B, Massanet S, Torrens J. *Fuzzy implications: past, present, and future.* Berlin: Springer Handbook of Computational Intelligence; 2015. DOI: https://doi.org/10.1007/978-3-662-43505-2_ 12

[24] Klement EP, Mesiar R, Pap E. *Triangular Norms.* vol. 8 of Trends in Logic. Dordrecht: Kluwer Academic Publishers; 2000. DOI: https://doi.org/10.1007/978-94-015-9540-7

[25] Gallian JA. *Contemporary Abstract Algebra.* 10th ed. New York: Chapman and Hall/CRC; 2021. DOI: https://doi.org/10.1201/9781003142331

[26] Rezaei A, Kim HS, Borzooei RA, Borumand SA. Matrix theory over ringoids. *Filomat*. 2023; 37(30): 10275-10288. DOI: https://doi.org/10.2298/FIL2330275R

[27] Dolzan D, Oblak P. Idempotent matrices over antirings. *Linear Algebra and its Applications*. 2009; 431(5-7): 823-832. DOI: https://doi.org/10.1016/j.laa.2009.03.035

[28] Khan WA, Rehman A, Taouti A. Soft near-semi-rings. *AIMS Mathematics*. 2020; 5(6): 6464-6478. DOI: https://doi.org/10.3934/math.2020417

[29] Pilz G. *Near-rings, the Theory and Its Applications*. vol. 23 of North-Holland Mathematics Studies. Amsterdam: North-Holland; 1983. https://www.sciencedirect.com/bookseries/north-holland-mathematics-studies/vol/23/suppl/C

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