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Higher order conservation laws of the Caudery-Dodd-Gibbon-Sawada-Kotera equation by scaling method

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Abstract. In this paper, a novel conservation law for the Caudrey-Dodd-Gibon-Sawada-Kotera equation utilizing scaling method is derived. This approach is systematic and relies on variational calculus and linear algebra. Also, the conservation law's density is developed by examining the scaling symmetry of the equation, while the corresponding flux is determined through the homotopy operator. This density-flux combination yields a conservation law for the equation. In particular, we establish a conservation law of rank 8 for the Caudrey-Dodd-Gibon-Sawada-Kotera equation.

Keywords: Caudery-Dodd-Gibbon-Sawada-Kotera, scaling symmetry, conservation law.

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1. Introduction

Conservation laws are represented as divergence expressions that equal zero when applied to the solutions of partial differential equations (in short, PDEs). These laws are essential in physics and assert that certain quantities within a system remain constant over time. Note that various methods exist for deriving the conservation laws applicable to a given system [1, 3, 5, 6]. Noether's theorem, connecting variational symmetry with conservation laws in partial differential equations (PDEs), has been employed in various established methods [9, 10]. Moreover, there exists an alternative approach that utilizes calculus of variations and linear algebra. This approach, known as the scaling method, operates in the following manner [12]. First, a primitive density characterized by arbitrary coefficients, which remains invariant under the scaling symmetry of the PDE, is

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considered. Then the time derivative of this primitive density is computed and integrated with the PDE. By applying the Euler operator, a linear system is derived which solving this system allows for the construction of the actual density. Finally, the corresponding flux is determined through the application of the inverse divergence operator, also referred to as the homotopy operator. Sawada and Kotera [15] proposed one of models in soliton theory as follows:

$$u_t + u_{5x} + 30uu_{3x} + 30u_xu_{2x} + 180u^2u_x = 0, \tag{1}$$

which is also introduced by Caudrey et al. [4]. Hence, (1) is called the Caudrey-Dodd-Gibbon-Sawada-Kotera (in short, CDGSK) equation. The CDGSK equation has been developed by many researchers. In relation to this equation, the finite dimensional reduction was investigated by Enolski et al. [7] and N-soliton solutions were discovered by Parker [11] via the dressing method. Darboux transformation [2] and Bäcklund transformation in bilinear forms [14] were also applied to study the CDGSK equation.

This research is structured in the following manner. Section 2 presents various definitions and prior findings that will be referenced in the sequel. In Section 3, we will demonstrate that the CDGSK equation maintains uniformity in rank and possesses a scaling symmetry. Section 4 focuses on the construction of the primitive density for rank 8, which the actual density is derived by eliminating divergence and divergence-equivalent terms. Finally, we compute the relevant fluxes utilizing the homotopy operator.

2. preliminaries

Consider a system of equations $P(x, u^{(n)}) = 0$, where $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ are independent space variables and dependent variables and $u^{(n)}$ is all derivatives of u up to the n-th order. A conservation law can be expressed as follows:

$$DivQ = 0 \;\text{ on }\; P = 0. \tag{2}$$

The definition is deduced from Olver [6, 1993] and Bluman et al. [8, 2010], and is generally employed in the literature of symmetries of PDEs. In dynamical problems, the time variable t and the spatial variables $x = (x^1, \ldots, x^p)$ are determined separately. Thus, one can use another definition for the conservation law by $D_t\rho + DivJ = 0$ on $P = 0$, where ρ is the conserved density and J is the corresponding flux. In the following, total time derivative operator and total divergence with respect to the space variables are explained. Total derivative operator D_t applied to function $f(x, t, u^{(n)})$ is defined as

$$D_tf = \frac{\partial f}{\partial t} + \sum_{\alpha}^{q}\sum_{J} u^{\alpha}_{J,t}\frac{\partial f}{\partial u^{\alpha}_{J}},$$

where $J = (j_1, \ldots, j_k)$ is a multi-index with $0 \leqslant k \leqslant n$

Definition 2.1 [13] The one dimensional Euler operator for dependent variable $u^j(x)$ is expressed as follows:

$$\mathcal{L}_{u^j(x)}f = \sum_{k=0}^{M_1^j}(-D_x)^k\frac{\partial f}{\partial u^j_{kx}}, \;\; j = 1, \cdots, q. \tag{3}$$

This operator plays an important role in determining the accuracy of differential functions, which is a critical aspect in the calculation of conservation laws.

Definition 2.2 [13] Let f be a differential function of order n. In one dimension, f is termed exact if it is a total derivative, that is, there exists a differentiable function $F(x, u^{(n-1)}(x))$ such that $f = D_xF$.

Theorem 2.3 [13]A function f is considered exact if and only if $\mathcal{L}_{u(x)f}$ equals 0, 0 represents vector $(0, \ldots, 0)$, which consists of q components corresponding to the number of components in u.

Definition 2.4 [12] Let f be an exact 1 D differential function. The homotopy operator in one dimensional is expressed as

$$\mathcal{H}_{u(x)}f = \int_0^1 \left(\sum_{j=1}^{n} \mathcal{I}_{u^j(x)}f \right) [\lambda u] \frac{d\lambda}{\lambda}, \text{where } u = (u^1, \cdots, u^q).$$

$\mathcal{I}_{u^j(x)}f$ is defined by

$$\mathcal{I}_{u^j(x)}f = \sum_{k=1}^{n_1^j} \left(\sum_{i=0}^{k-1} u^j_{ix}(-D_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u^j_{kx}}, \tag{4}$$

where n_1^j represents the order of f in relation to the dependent variable u^j regarding x.

Theorem 2.5 [12] Let f be an exact differential function, that is, $D_xF = f$ for some differential function $F(x, u^{(n-1)}(x))$. Then $F = D_x^{-1}f = \mathcal{H}_{u(x)}f$.

3. Scaling symmetry of the CDGSK equation

The CDGSK equation admits the scaling symmetry $(x, t, u) \to (\lambda^{-1}x, \lambda^{-5}t, \lambda^2 u)$, where λ is an arbitrarily constant. Several algorithmic approaches exist for identifying scaling symmetries [3, 5, 10]. Here, we apply the idea of variable weights to determine it [8].

Definition 3.1 Let $x \to \lambda^p x$ be the scaling symmetry. The weight of the variable x, denoted by $W(x)$, is equal to $-p$. If $W(x)$ is equal to $-p$, then the weight of D_x is characterized as p. In this manner, one can take the rank of a monomial which is the sum of the weights of its variables. The monomials within a differential function share the same rank referred to as being uniform in rank.

An equation that exhibits scaling symmetry is consistent in rank, which allows us to derive the scaling symmetry of the CDGSK equation by considering (1) maintains uniformity in rank. Under this assumption, we can formulate a system of weight-balance equations. We can identify the scaling symmetry by solving this system. For (1), the corresponding weight-balance equation is

$$W(u) + W(D_t) = W(u) + 5W(D_x) = 2W(u) + 3W(D_x) = 3W(u) + W(D_x), \tag{5}$$

which results in $W(u) = 2, W(D_x) = 1$, and $W(D_t) = 5$. Given that (2) must be zero for all solutions of the PDE, the density and flux of the conservation law must adhere to its scaling symmetries. Hence, it is clear that the conservation law must also maintain uniformity in rank. Furthermore, based on the symmetry inherent in the CDGSK equation, we can formulate the initial density as a linear combination of monomials of the specified rank. To see details, we refer to [2].

4. Computing conservation laws of the CDGSK equation

In this section, we derive the conservation laws associated with the CDGSK equation through the scaling method. The process starts with the construction of the density, followed by the determination of the corresponding flux J. To establish the density, we start with an initial density represented as a linear combination of differential terms, each with arbitrary coefficients. It is essential that this combination is selected from a predetermined rank and it remains invariant under the scaling symmetry. Next step involves calculating the total time derivative of the initial density, where all time derivatives in the expressions are substituted with their corresponding equivalents as defined in (1). By (2), the resulting expression is required to be precise. Finally, the arbitrary coefficients are determined by solving the linear system, which is established through the application of Theorem 2.3 regarding exactness. Replacing the coefficients derived from the aforementioned system with the initial density, the actual density can be determined. Then the relevant flux is calculated using this formula.

$$J = -Div^{-1}(D_t\rho). \tag{6}$$

Candidate density as previously stated, the initial step in identifying conservation laws involves determining the density. We begin by selecting an arbitrary rank for the initial density. Next, we formulate the terms of the density by combining monomials of a designated rank that incorporate dependent variables and their partial derivatives. Then we derive the initial density of rank 8 for the CDGSK equation (1). Consider the set $\mathcal{P}$ including dependent variables up to rank 8. Using (5), we have $\mathcal{P} = u^4, u^3, u^2, u$. We then apply the total derivative operator concerning the spatial variables to $\mathcal{P}$, thereby elevating the rank of the terms in the list to 8. This generated list is referred to as the updated list.

$$\mathcal{Q} = \{u^4, u^2u_{2x}, {u_{2x}}^2, {u_x}^2u, u_{3x}u_x, u_{4x}u, u_{6x}\}. \tag{7}$$

To ensure that the density is nontrivial, it is necessary to eliminate the divergence terms while retaining one of the divergence-equivalent terms from the list and discarding the others. By applying (3) to (7), we obtain the following result:

$$\mathcal{L}_{u(x)}\mathcal{Q} = \{4u^3, 4uu_{2x} + 2{u_x}^2, -2u_xu_{2x} - {u_x}^2, 2u_{4x}, -2u_{4x}, 2u_{4x}, 0\}. \tag{8}$$

Theorem 2.3 indicates that the term u_{6x} is divergence and should consequently be excluded from $\mathcal{Q}$. The second and third entries in list (8) are multiples of one another, which implies that the corresponding terms in (7) are divergence-equivalent, and one of them must omit from $\mathcal{Q}$. Among the equivalent terms, the one with the lowest rank should be retained while the others are discarded. As a result, u^2u_{2x} will be eliminated from the list. Similarly, the fourth, fifth and sixth terms will follow this process. Therefore, $\mathcal{QQ} = \{u^4, u{u_x}^2, {u_{2x}}^2\}$. Next, let us check a linear combination of the elements from the aforementioned list, utilizing arbitrary coefficients to establish the initial density of rank 8 for the CDGSK equation. Consider a linear combination of the elements from the aforementioned list, utilizing arbitrary coefficients to establish the initial density of rank 8 for the CDGSK equation.

$$\rho = c_1 u^4 + c_2 u {u_x}^2 + c_3 {u_{2x}}^2. \tag{9}$$

5. Calculating the actual density

To identify the unknown coefficients in equation (9), we compute D_t of (9). Thus,

$$D_t\rho = (4C_1u^3 + C_2{u_x}^2)u_t + 2C_2uu_xu_{xt} + 2C_3u_{2x}u_{2xt}.$$

We proceed by substituting u_t and its derivatives with their corresponding values as indicated in (1). Let E be defined as D_t. Then E must be precise as stated in (2). Hence, applying Theorem 2.3, we obtain $\mathcal{L}_{u(x)}E = 0$. This establishes a system of linear equations, and by solving this system, the unknown coefficients can be identified as follows:

$$c_1 = 12c_3,\ c_2 = -18c_3,\ c_3 = c_3. \tag{10}$$

c_3 is considered arbitrary. If we set $c_3 = 1$, it follows that $c_1 = 12$ and $c_2 = -18$. Substituting (10) into (9), we can derive the actual density by $\rho = 12u^4 - 18u{u_x}^2 + {u_{xx}}^2$.

6. Determining the flux

Once the density is established, the associated flux can be computed by utilizing the relationship $J = Div^{-1}(E)$. We apply Theorem 2.5 along with the homotopy operator to derive the flux. Substituting (10) into E, we obtain

$$\begin{aligned} E = &-3240c_1u^2u_xu_{2x} - 2160c_1u{u_x}^3 - 2160c_2u^2u_xu_{2x} - 1440c_2u{u_x}^3 \\ &- 120c_1uu_xu_{4x} - 240c_1uu_{2x}u_{3x} - 240c_1{u_x}^2u_{3x} - 360c_1u_x{u_{2x}}^2 \\ &+ 120c_2uu_xu_{4x} + 240c_2uu_{2x}u_{3x} + 240c_2{u_x}^2u_{3x} + 360c_2u_x{u_{2x}}^2 \\ &+ 3600c_3uu_xu_{4x} + 7200c_3uu_{2x}u_{3x} + 7200c_3{u_x}^2u_{3x} + 10800c_3u_x{u_{2x}}^2 \\ &+ 10c_2u_xu_{6x} + 30c_2u_{2x}u_{5x} + 50c_2u_{3x}u_{4x} + 180c_3u_xu_{6x} \\ &+ 540c_3u_{2x}u_{5x} + 900c_3u_{3x}u_{4x} \end{aligned}$$

Using relation (4), $\mathcal{I}_{u(x)}E$ is determined as follows:

$$\begin{aligned} \mathcal{I}_{u(x)}E = &\ 8640u^6 + 7200u^4u_{2x} - 27000u^3{u_x}^2 + 192u^3u_{4x} \\ &- 4896u^2u_xu_{3x} + 3168u^2{u_{2x}}^2 - 1008u{u_x}^2u_{2x} + 432{u_x}^4 \\ &- 108uu_xu_{5x} + 288uu_{2x}u_{4x} - 144u{u_{3x}}^2 + 54{u_x}^2u_{4x} \\ &+ 144u_xu_{2x}u_{3x} + 192{u_{2x}}^3 + 4u_{2x}u_{6x} - 4u_{3x}u_{5x} + 2{u_{4x}}^2 \end{aligned}$$

Now, the flux is derived as follows:

$$\begin{aligned} J &= \mathcal{H}_{u(x)}E \\ &= \int_0^1 \mathcal{I}_{u(x)}E[\lambda u]\frac{d\lambda}{\lambda} \\ &= 1440u^6 + 1440u^4u_{2x} - 5400u^3{u_x}^2 + 48u^3u_{4x} \\ &\quad - 1224u^2u_xu_{3x} + 792u^2{u_{2x}}^2 - 252u{u_x}^2u_{2x} + 108{u_x}^4 \\ &\quad - 36uu_xu_{5x} + 96uu_{2x}u_{4x} - 48u{u_{3x}}^2 + 18{u_x}^2u_{4x} \\ &\quad + 48u_xu_{2x}u_{3x} + 64{u_{2x}}^3 + 2u_{2x}u_{6x} - 2u_{3x}u_{5x} + {u_{4x}}^2 \end{aligned}$$

7. Conclusion

In this article, we considered the fifth-order evolutionary integrable Caudery-Dodd-Gibbon-Sawada-Kotera equation that admits scaling symmetry and is uniform in rank. Applying the scaling method, the density of the conservation law was constructed and associated flux was computed by the homotopy operator. In fact, the conservation law of rank eight was constructed for the Caudery-Dodd-Gibbon-Sawada-Kotera equation.

References

[1] S. Anco, G. Bluman, Direct construction method for conservation laws of partial differential equations Part II: general treatment, Euro. J. Appl. Math. 13 (5) (2002), 568-585.
[2] R. N. Aiyer, B. Fuchssteiner, W. Oevel, Solitons and discrete eigenfunctions of the recursion operator of non-linear evolution equations. I. The Caudrey-Dodd-Gibbon-Sawada-Kotera equation, J. Phys.. A. 19 (18) (1986), 3755-3770.
[3] G. W. Bluman, A. F. Cheviakov, S. C. Anco, Applications of Symmetry Methods to Partial Differential Equations, Appl. Math. Sciences, Vol. 168, Springer Verlag, New York, 2010.
[4] P. J. Caudrey, R. K. Dodd, J. D. Gibbon, A new hierarchy of Kortewegde Vries equations, Proc. Royal. Soc. London. A. Math. Phys. Sci. 351 (1997), 407-422.
[5] A. F. Cheviakov, Computation of fluxes of conservation laws, J. Eng. Math. 66 (2010), 153-173.
[6] A. F. Cheviakov, GeM software package for computation of symmetries and conservation laws of differential equations, Comp. Phys. Comm. 176 (1) (2007), 18-61.
[7] V. Z. Enolski, Y. N. Fedorov, A. N. W. Hone, Generic hyperelliptic prym varieties in a generalized Henon-Heiles system, J. Geom. Phys. 87 (2014), 106-114.
[8] W. Herman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quant. Chem. 106 (2006), 278-299.
[9] E. Noether, Invariante variations probleme, Transp. Theory Stat. Phys. 1 (3) (1971) 186-207.
[10] P. J. Olver, Application of Lie Groups to Differential Equations, Springer-Verlag, New York, 1993.
[11] A. Parker, A reformulation of the dressing method for the Sawada-Kotera equation, Inverse Prob. 17 (4) (2001), 885-895.
[12] D. Poole, Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations Using Homotopy Operators, Ph.D. Dissertation, Colorado School of Mines, Golden, Colorado, 2009.
[13] D. Poole, W. Herman, Symbolic computation of conservation laws for nonlinear partial differential equations in multiple space dimensions, J. Symb. Comput. 46 (2011), 1355-1377.
[14] J. Satsuma, D. J. Kaup, A Bäcklund transformation for a higher order Korteweg-de Vries equation, J. Phys. Soc. Japan. 43 (2) (1977), 692-697.
[15] K. Sawada, T. Kotera, A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation, Prog. Theor. Phys. 51 (5) (1974), 1355-1367.