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# Fuzzy Differential Equations with Application in Electrical Circuit

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A R T I C L E  I N F O

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A B S T R A C T

Today, the production and services field faces a change in the competition pattern among independent companies and supply chains. The food supply chain is among the complex supply chains with special characteristics that can toughly be adapted to general evaluation systems. The current research aims to determine the effective indicators for evaluating the performance of the sustainable food supply chain. This research is descriptive-survey in terms of method and practical in terms of purpose. In line with the research implementation, based on the study of the theoretical foundations and the background of the research conducted concerning the effective indicators in evaluating the performance of the sustainable supply chain, the effective criteria were extracted and given to 25 research experts in the form of a questionnaire. Finally, to investigate the relationships between these 26 basic criteria, another questionnaire was prepared and given to the research experts. The final factors were structured based on the answers received and using the methods of fuzzy cognitive mapping and fuzzy DEMATEL. Regarding the centrality criterion in the fuzzy cognitive mapping method, the factors "income distribution, sustainable investment, and average annual training time of employees" have the most centrality, so they were recognized as the main factors influencing the performance evaluation of the sustainable food supply chain.

## 1. Introduction

Modeling many significant real-world problems by mathematics leads to differential equations [4,6, 23, 24]. Some information about many problems on electronics, mechanics, medicine, etc., is ambiguous or imprecise. Classical mathematics does not have the necessary tools to express these ambiguities, so it eliminates the uncertainty.

In traditional methods, a probability problem is often reduced to solving a differential or integral equation. Probability theory is related to random events and is used to predict the outcome of a random event in the future. The event is supposed to happen in the future and its result is currently unknown. To express ambiguities and

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inaccuracies of events, we need another type of mathematical relationship to simulate a different model from the models in probability theory. The fuzzy theory is not always accompanied by an event; in fact, it supports non-random uncertainty. Fuzzy theory accepts uncertainty as an important part of the real world and models it [10,22,27,28].

Briefly, differential equations are classified into two categories: certain and uncertain. According to the type of the given problems and models, the uncertain one is also divided into two parts: random and fuzzy. In studying fuzzy differential equations, we face with limitations and variations of fuzzy sets such as the type of meter, type of derivative, and types of acceptable answers for the problems.

In 1978, Kandel and Byatt used the title of fuzzy differential equations for the first time, which was completely different from its present-day concept. In the same vein, Lotfi Zadeh's theories regarding the probability of a fuzzy event were expanded, and differential equations with the membership function of a known fuzzy set were solved. Also, the concept of fuzzy numbers and the operations of addition and multiplication on fuzzy numbers were studied in 1978 by Dubois and Prade. Goetschel and Voxman in 1983 made minor changes to the previous definition of a fuzzy numbers [12].

As a result, fuzzy modeling leads to greater efficiency of the system. Since the concept of derivative is an essential part of a differential equation, the evolution of fuzzy derivatives plays a key role in the evolution of fuzzy differential equations. Chang and Lotfi Zadeh [9] introduced the concept of a fuzzy derivative for the first time. After that, other types of derivatives were defined by Dubois and Prade [11], Puri and Ralescu [25], Goetschel and Voxman [13], and Friedman and colleagues [29]. One of the most important definitions for fuzzy derivative was introduced by Seikkala in 1987, which is called Seikkala's derivative [26]. It has been proved that all different forms of fuzzy derivatives are equivalent, if they exist.

In recent years, the analysis of fuzzy differential equations has attracted the attention of many researchers. There are three perspectives in studying fuzzy differential equations. One perspective is based on the Hukuhara derivative [14]. In this perspective differential equations are defined by Kaleva in 1987 [16] for the first time. The stated perspective had problems and disadvantages that caused Hullermeier propose a different formulation of fuzzy differential equations in 1997 [15]. Another perspective was expressed by Buckley and Feuring from Lotfi Zadeh's generalization principle for generalizing classical differential equations to the fuzzy type [7,8]. By this method, the solution of classical differential equations was generalized to fuzzy differential equations. One of the disadvantages of this method was the lack of differentiability of the solution. The third perspective was expressed by Bede and Gal in 2005, in which the concept of generalized derivative was introduced and discussed [2]. This perspective was expressed along the concept of H-derivative. By this method, the mentioned disadvantages of the past methods were resolved.

In 2007, Bede et al. studied and solved the first-order linear fuzzy differential equations under this derivative [3]. First, they stated that one of the disadvantages of the generalized derivative versus the H-derivative is the non-uniqueness of the solution of the differential equation. In other words, by this method, the differential equation may have several solutions because the H-derivative is not unique [5]. The advantage that a differential equation has multiple solutions is that we can choose the one that provides a better description of the system in the real world.

Due to the variety of perspectives, fuzzy differential equations are an interesting subject for research. One of the important applications of the proposed fuzzy differential equations in recent years has been in the field of control theory. In this regard, the optimal control of a linear fuzzy dynamic system based on gH differentiability and SGH differentiability has been studied in [19-21]. The design of an optimal feedback control for adjusting a linear fuzzy dynamic system with a proposed application in Boeing 747 was presented in [17], in which fuzzy differential equations were considered under the concept of gr differentiability. In addition, in [18], the problem of optimal control of fuzzy time using gr differentiability was studied. A deep analysis of the stability of linear fuzzy dynamic systems under the concept of gr differentiability has been reported in [1].

Preliminary fuzzy definitions are stated in Section 2. In Section 3, at first, the definition of a fuzzy differential equation is stated. After that, a brief explanation is given for solving first-order differential equations with fuzzy coefficients of various types: separable, exact, and linear equations. Several examples are given and they are solved using the mentioned methods. In Section 4, a real example which gives the application of fuzzy differential equations in electrical circuits is examined and solved. Section 5 is devoted to conclusion and summary.

## 2. Basic Concepts

In this section, some basic definitions on fuzzy numbers and fuzzy arithmetics are reviewed.

**Definition 1.** A triangular fuzzy number which is shown by $\tilde{a} = \langle a_1, a_2, a_3 \rangle$ is a convex normalized fuzzy set $\tilde{a}$ of the real line $\mathbb{R}$ such that:

- There exists precisely one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{a}}(x_0) = 1$ ($x_0$is called the mean value of $\tilde{a}$), where $\mu_{\tilde{a}}$ is called the membership function of the fuzzy set.
- $\mu_{\tilde{a}}(x)$is piece-wise continuous.

The membership function is defined as follows:

$$\mu_{\tilde{a}} = \begin{cases} 0 & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x < a_2 \\ \dfrac{x - a_3}{a_2 - a_3} & a_2 \le x < a_3 \\ 0 & x \ge a_3 \end{cases}$$

**Definition 2.** Let $\tilde{a} = \langle a_1, a_2, a_3 \rangle$ and $\tilde{b} = \langle b_1, b_2, b_3 \rangle$ be two non-negative triangular fuzzy numbers where ($a_1 \ge 0$ and $b_1 \ge 0$). Also, let $k$ be a real number. Operations for fuzzy numbers are defined as follows [9]:

$$k\tilde{a} = \begin{cases} \langle ka_1, ka_2, ka_3 \rangle, & k \ge 0, \\ \langle ka_3, ka_2, ka_1 \rangle, & k < 0, \end{cases}$$

$$\tilde{a} + \tilde{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle,$$

$$\tilde{a} - \tilde{b} = \langle a_1 - b_3, a_2 - b_2, a_3 - b_1 \rangle$$

**Remark 1.** By Definition 2, it is obvious that if $\tilde{a} = \tilde{b}$ then $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.

**Definition 3.** Let $\tilde{a} = \langle a_1, a_2, a_3 \rangle$ and $\tilde{b} = \langle b_1, b_2, b_3 \rangle$. We have,

$$\tilde{a} \times \tilde{b} = \begin{cases} \min\{a_1 b_1, a_3 b_1\}, a_2 b_2, \max\{a_1 b_3, a_3 b_3\} & a_1 \ge 0 \\ \min\{a_1 b_3, a_3 b_1\}, a_2 b_2, \max\{a_1 b_1, a_3 b_3\} & a_1 < 0, a_3 \ge 0 \\ \min\{a_1 b_3, a_3 b_3\}, a_2 b_2, \max\{a_1 b_1, a_3 b_1\} & a_3 < 0 \end{cases}$$

$$\frac{\tilde{a}}{\tilde{b}} = \langle a_1, a_2, a_3 \rangle \times \langle \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1} \rangle$$

**Remark 2.** By mathematical relations, we have,

$$\max\{a_1, a_3\} = \frac{a_1 + a_3}{2} + \left|\frac{a_1 - a_3}{2}\right|,$$

$$\min\{a_1, a_3\} = \frac{a_1 + a_3}{2} - \left|\frac{a_1 - a_3}{2}\right|.$$

## 3. Fuzzy Differential Equations

The general form of the first-order fuzzy differential equation is $y'(t) = f\big(t, y(t)\big)$ where $y(t)$ is a fuzzy function of the certain variable $t$, $y'(t)$ is the fuzzy derivative of $y(t)$, and $f(t, y)$ is a fuzzy function of the variable $t$ and the fuzzy variable $y$.

### 3.1. First-order separable differential equation with fuzzy coefficients

A differential equation that can be written in the form

$$af(y)dy = bg(x)dx \quad a, b \in \mathbb{R}$$

is called a first-order separable differential equation. To solve these equations, we integrate both sides of the equality

$$\int af(y)dy = \int bg(x)dx$$

Now, if the parameters $a$ and $b$ are regarded fuzzily due to the uncertainty of the problem, then the first-order separable differential equation with fuzzy coefficients is formed as

$$\tilde{a}f(y)dy = \tilde{b}g(x)dx$$

To solve these equations, the variables are multiplied by the fuzzy coefficients $\tilde{a}$ and $\tilde{b}$ according to Definition 2. After the initial calculations, the fuzzy equation is converted into several certain equations, which should be solved.

**Example 1**. Consider the first-order separable differential equation with fuzzy coefficients:

$$y' = \frac{\langle 1,2,3\rangle x}{\langle 2,3,4\rangle y^2}$$

In the first step, the fuzzy coefficient $\langle 1,2,3\rangle$is divided by $\langle 2,3,4\rangle$

$$y' = \langle \frac{1}{4}, \frac{2}{3}, \frac{3}{2}\rangle \frac{x}{y^2}$$

After that $\frac{x}{y^2}$ is multiplied fuzzy in the resulting coefficient

$$\frac{dy}{dx} = \langle min\left\{\frac{1}{4}\frac{x}{y^2}, \frac{3}{2}\frac{x}{y^2}\right\}, \frac{2}{3}\frac{x}{y^2}, max\left\{\frac{1}{4}\frac{x}{y^2}, \frac{3}{2}\frac{x}{y^2}\right\}\rangle$$

After applying Remark 2, the following equation is obtained

$$\frac{dy}{dx} = \langle \frac{7}{8}\frac{x}{y^2} - \frac{5}{8}\left|\frac{x}{y^2}\right|, \frac{2}{3}\frac{x}{y^2}, \frac{7}{8}\frac{x}{y^2} + \frac{5}{8}\left|\frac{x}{y^2}\right|\rangle$$

For ease of display, we remove the absolute value term; since the sign of $y^2$ is always positive, here the sign of $x$ matters, so we have:

$$\frac{dy}{dx} = \begin{cases} \langle \frac{1}{4}\frac{x}{y^2}, \frac{2}{3}\frac{x}{y^2}, \frac{3}{2}\frac{x}{y^2}\rangle & x \geq 0 \\ \\ \langle \frac{3}{2}\frac{x}{y^2}, \frac{2}{3}\frac{x}{y^2}, \frac{1}{4}\frac{x}{y^2}\rangle & x < 0 \end{cases}$$

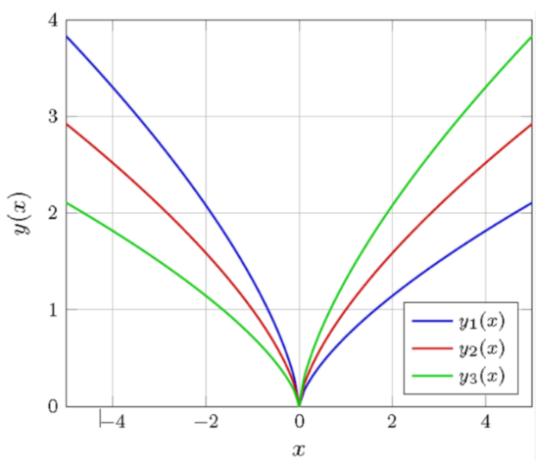
By solving the above certain first-order separable differential equations, we have:

$$\begin{cases} \langle 8y^3 = 3x^2 + c_1, y^3 = x^2 + c_2, 4y^3 = 9x^2 + c_3\rangle & x \geq 0 \\ \langle 4y^3 = 9x^2 + c_4, y^3 = x^2 + c_5, 8y^3 = 3x^2 + c_6\rangle & x < 0 \end{cases}$$

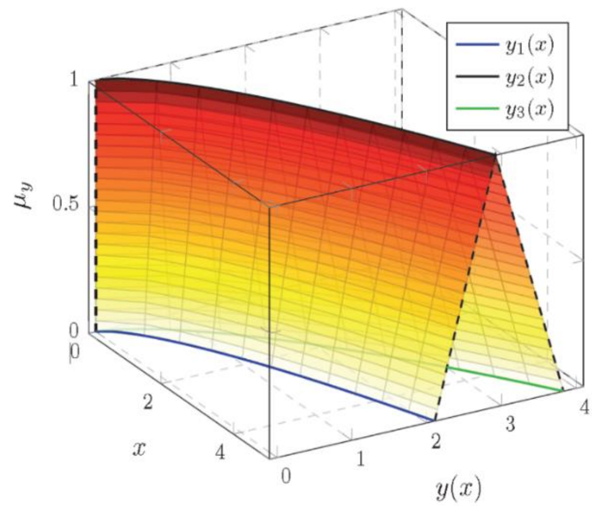
So, the final solution can be written as:

$$y(x) = \begin{cases} \langle \sqrt[3]{\frac{3}{8}x^2 + C_1}, \sqrt[3]{x^2 + C_2}, \sqrt[3]{\frac{9}{4}x^2 + C_3} \rangle & x \geq 0 \\ \langle \sqrt[3]{\frac{9}{4}x^2 + C_1}, \sqrt[3]{x^2 + C_2}, \sqrt[3]{\frac{3}{8}x^2 + C_3} \rangle & x < 0 \end{cases}$$

The graphs of the solution are shown in Figure 1. The classical graphs for $x \in [-5,5]$ are given in Figure 1a, and the fuzzy form of the solution is shown in Figure 1b for $x \in [0,5]$




(a) classical graphs




(b) fuzzy form

**Figure 1.** Obtained results for Example 1

**Example 2**. Consider the differential equation

$$y' = \langle 0.9,1,1.1 \rangle e^{x+y}.$$

In the first step, multiply the fuzzy number $\langle 0.9,1,1.1 \rangle$ by $e^{x+y}$:

$$y' = \langle e^{x+y} - 0.1|e^{x+y}|, e^{x+y}, e^{x+y} + 0.1|e^{x+y}| \rangle$$

Since $e^{x+y}$ is always positive, it is equivalent to its absolute value:

$$\frac{dy}{dx} = \langle 0.9e^{x+y}, e^{x+y}, 1.1e^{x+y} \rangle$$

For ease of calculation, the last equation is converted into the following three certain first-order separable differential equations, which are solved as follows:

$$\begin{cases} \frac{dy}{dx} = 0.9e^{x+y} \to \frac{dy}{e^y} = 0.9e^x dx \to e^{-y} + 0.9e^x = c_1 \\ \frac{dy}{dx} = e^{x+y} \to \frac{dy}{e^y} = e^x dx \to e^{-y} + e^x = c_2 \\ \frac{dy}{dx} = 1.1e^{x+y} \to \frac{dy}{e^y} = 1.1e^x dx \to e^{-y} + 1.1e^x = c_3 \end{cases}$$

Finally, the fuzzy answer is:

$$y(x) = \langle -\ln(-0.9e^x + c_1), -\ln(-e^x + c_2), -\ln(-1.1e^x + c_3) \rangle$$

### 3.2. Solving first-order linear differential equations with fuzzy coefficients

A differential equation in the form

$$y' + ap(x)y = bq(x), \quad a, b \in \mathbb{R}$$

is called a first-order linear equation. The general solution of the equation is as follows:

$$y = \frac{1}{\mu(x)}\left[\int bq(x)\mu(x)dx + c\right],$$

where $\mu(x) = e^{\int ap(x)dx}$. Now, if the parameters $a$ and $b$ are presented fuzzy due to the uncertainty of the problem, then we have a first-order linear differential equation with fuzzy coefficients:

$$y' + \tilde{a}p(x)y = \tilde{b}q(x).$$

To solve this equation, at first, the fuzzy parameters $\tilde{a}$ and $\tilde{b}$ are multiplied by the variables according to Definition 2. After the initial calculations, the equation with fuzzy coefficients is converted into several certain first-order linear equations, which should be solved.

**Example 3.** Consider the following first-order linear differential equation with fuzzy coefficients:

$$\frac{dy}{dx} + \langle 1,1,2\rangle y = \langle 0.5,1,2\rangle e^x$$

Initially, by multiplying the fuzzy coefficients in the variables, we will have:

$$\frac{dy}{dx} + \langle 1.5y - 0.5|y|, y, 1.5y + 0.5|y|\rangle = \langle 1.25e^x - 0.75e^x, e^x, 1.25e^x + 0.75e^x\rangle$$

After performing the fuzzy addition, the equation is converted as follows:

$$\left\langle \frac{dy}{dx} + 1.5y - 0.5|y|, \frac{dy}{dx} + y, \frac{dy}{dx} + 1.5y + 0.5|y| \right\rangle = \langle 0.5e^x, e^x, 2e^x. \rangle$$

So, the final equations are as follows:

$$\begin{cases} \dfrac{dy}{dx} + 1.5y - 0.5|y| = 0.5e^x & (1) \\ \dfrac{dy}{dx} + y = e^x & (2) \\ \dfrac{dy}{dx} + 1.5y + 0.5|y| = 2e^x & (3) \end{cases}$$

To obtain the solution, each of the equations should be solved separately. Here, we give the solution for Eq. (1) for instance. The other ones will be solved in the same manner.

**Solving equation (1):**

$$\frac{dy}{dx} + 1.5y - 0.5|y| = 0.5e^x \rightarrow \begin{cases} \dfrac{dy}{dx} + y = 0.5e^x, & y \geq 0 \\ \dfrac{dy}{dx} + 2y = 0.5e^x & y < 0 \end{cases}$$

First, we compute the integral $\mu(x) = e^{\int ap(x)dx}$:

$$\begin{cases} \dfrac{dy}{dx} + y = 0.5e^{x} \rightarrow \mu(x) = e^{\int 1dx} = e^{x} & y \geq 0 \\ \dfrac{dy}{dx} + 2y = 0.5e^{x} \rightarrow \mu(x) = e^{\int 2dx} = e^{2x} & y < 0 \end{cases}$$

Finally, the general solution is obtained:

For $y \geq 0$:

$$y = \frac{1}{e^{x}}\left[\int 0.5e^{x} \times e^{x}dx + c_1\right] = e^{-x}\left[\int 0.5e^{2x}dx + c_1\right] = e^{-x}\left(0.5 \times \frac{1}{2}e^{2x} + c_1\right) = \frac{1}{4}e^{x} + c_1e^{-x}$$

For $y < 0$:

$$y = \frac{1}{e^{2x}}\left[\int 0.5e^{x} \times e^{2x}dx + c_2\right] = e^{-x}\left[\int 0.5e^{3x}dx + c_2\right] = e^{-2x}\left(0.5 \times \frac{1}{3}e^{3x} + c_2\right) = \frac{1}{6}e^{x} + c_2e^{-2x}$$

**Solving equation (2):**

$$y = \frac{1}{e^{x}}\left[\int e^{x} \times e^{x}dx + c_3\right] = e^{-x}\left[\int e^{2x}dx + c_3\right] = e^{-x}\left(\times\frac{1}{2}e^{2x} + c_3\right) = \frac{1}{2}e^{x} + c_3e^{-x}$$

**Solving equation (3):**

For $y \geq 0$:

$$y = \frac{1}{e^{2x}}\left[\int 2e^{x} \times e^{2x}dx + c_4\right] = e^{-x}\left[\int 2e^{3x}dx + c_4\right] = e^{-2x}\left(2 \times \frac{1}{3}e^{3x} + c_4\right) = \frac{2}{3}e^{x} + c_4e^{-2x}$$

For $y < 0$:

$$y = \frac{1}{e^{x}}\left[\int 2e^{x} \times e^{x}dx + c_5\right] = e^{-x}\left[\int 2e^{2x}dx + c_5\right] = e^{-x}\left(2 \times \frac{1}{2}e^{2x} + c_5\right) = e^{x} + c_5e^{-x}.$$

**Example 4.** Consider the following first-order linear differential equation with fuzzy coefficients:

$$y' + \frac{\langle 1,2,4\rangle y}{\langle 1,1,2\rangle x} = \langle 1,2,2\rangle x$$

At first, the fuzzy coefficient $\langle 1,2,4\rangle$ is divided to $\langle 1,1,2\rangle$, we will have:

$$y' + \left\langle \frac{1}{2}, 2, 4\right\rangle \frac{y}{x} = \langle 1,2,2\rangle x$$

By multiplying $\frac{y}{x}$ and $x$ by fuzzy coefficients, we have:

$$y' + \left\langle \min\left\{\frac{1}{2}\frac{y}{x}, 4\frac{y}{x}\right\}, 2\frac{y}{x}, \max\left\{\frac{1}{2}\frac{y}{x}, 4\frac{y}{x}\right\}\right\rangle = \langle 1.5x - 0.5|x|, 2x, 1.5x + 0.5|x|\rangle.$$

After performing the operations, the equation is converted as follows:

$$y' + \left\langle 2.25\frac{y}{x} - 1.75\left|\frac{y}{x}\right|, 2\frac{y}{x}, 2.25\frac{y}{x} + 1.75\left|\frac{y}{x}\right|\right\rangle = \langle 1.5x - 0.5|x|, 2x, 1.5x + 0.5|x|\rangle.$$

At last, the following equations are obtained:

$$\begin{cases} y' + 2.25\frac{y}{x} - 1.75\left|\frac{y}{x}\right| = 1.5x - 0.5|x| & (4) \\ y' + 2\frac{y}{x} = 2x & (5) \\ y' + 2.25\frac{y}{x} + 1.75\left|\frac{y}{x}\right| = 1.5x + 0.5|x| & (6) \end{cases}$$

To obtain the solution of the equations, each of the them should be solved separately. Here, we solve Eq. (4) for instance. The other ones will be solved in the same manner.

**Solving equation (4):** To solve the equation, the sign of $|x|$ and $\left|\frac{y}{x}\right|$ should be determined:

$$\begin{cases} y' + 0.5\dfrac{y}{x} = x, & x \geq 0, y \geq 0 \\ y' + 0.5\dfrac{y}{x} = 2x, & x < 0, y < 0 \\ y' + 4\dfrac{y}{x} = 2x, & x < 0, y \geq 0 \\ y' + 4\dfrac{y}{x} = x, & x \geq 0, y < 0 \end{cases}$$

By solving these equations, the final solution is:

$$\begin{cases} y = \dfrac{1}{\sqrt{x}}\left(\dfrac{x^{2.5}}{2.5} + c_1\right) = \dfrac{x^2}{2.5} + \dfrac{c_1}{\sqrt{x}}, & x \geq 0, y \geq 0 \\ y = \dfrac{1}{\sqrt{x}}\left(2\dfrac{x^{2.5}}{2.5} + c_2\right) = \dfrac{x^2}{0.8} + \dfrac{c_2}{\sqrt{x}}, & x < 0, y < 0 \\ y = \dfrac{1}{x^4} \times \dfrac{2x^6}{6} + \dfrac{c_3}{x^4} = \dfrac{x^2}{3} + \dfrac{c_3}{x^4}, & x < 0, y \geq 0 \\ y = \dfrac{1}{x^4} \times \dfrac{x^6}{6} + \dfrac{c_4}{x^4} = \dfrac{x^2}{6} + \dfrac{c_4}{x^4}, & x > 0, y < 0 \end{cases}$$

**Solving equation (5):** The solution is:

$$y = \frac{1}{x^2} \times \frac{2x^4}{4} + \frac{c_5}{x^2} = \frac{x^2}{2} + \frac{c_5}{x^2}$$

**Solving equation (6):** The solution is:

$$\begin{cases} y = \dfrac{1}{x^4} \times \dfrac{2x^6}{6} + \dfrac{1}{x^4}c_6 = \dfrac{x^2}{3} + \dfrac{c_6}{x^4}, & x \geq 0, y \geq 0 \\ y = \dfrac{1}{x^4} \times \dfrac{x^6}{6} + \dfrac{1}{x^4}c_7 = \dfrac{x^2}{6} + \dfrac{c_7}{x^4}, & x < 0, y < 0 \\ y = \dfrac{1}{\sqrt{x}} \times \left(\dfrac{x^{2.5}}{2.5} + c_8\right) = \dfrac{x^2}{2.5} + \dfrac{c_8}{\sqrt{x}}, & x < 0, y \geq 0 \\ y = \dfrac{1}{\sqrt{x}} \times \left(\dfrac{2x^{2.5}}{2.5} + c_9\right) = \dfrac{2x^2}{2.5} + \dfrac{c_9}{\sqrt{x}}, & x \geq 0, y < 0 \end{cases}$$

**Example 5.** Consider the following first-order linear differential equation with fuzzy coefficients [3].

$$\frac{dy}{dx} - y = \langle 1,2,3 \rangle x$$

By multiplying the variable $x$ by the coefficient $\langle 1,2,3 \rangle$, we get:

$$\frac{dy}{dx} - y = \langle 2x - |x|, 2x, 2x + |x| \rangle$$

As a result, the following three equations are obtained:

$$\begin{cases} \dfrac{dy}{dx} - y = 2x - |x| & (7) \\ \dfrac{dy}{dx} - y = 2x & (8) \\ \dfrac{dy}{dx} - y = 2x + |x| & (9) \end{cases}$$

Next, we will solve each equation separately:

Solution of Equation (7): For ease of calculation, we consider $x$ to be positive or negative.

$$\frac{dy}{dx} - y = 2x - |x| \rightarrow \begin{cases} \dfrac{dy}{dx} - y = x & x \geq 0 \\ \dfrac{dy}{dx} - y = 3x & x < 0 \end{cases}$$

First, we calculate the integral of $\mu = e^{\int ap(x)dx}$:

$$\begin{cases} \dfrac{dy}{dx} - y = x & \rightarrow \mu = e^{\int p(x)dx} = e^{\int -dx} = e^{-x} & x \geq 0 \\ \dfrac{dy}{dx} - y = 3x & \rightarrow \mu = e^{\int p(x)dx} = e^{\int -dx} = e^{-x} & x < 0 \end{cases}$$

Solution of Equation (7):

$$\begin{cases} y = \dfrac{1}{e^{-x}}\left[\displaystyle\int xe^{-x}dx + c_1\right] = -x - 1 + c_1e^x, & x \geq 0 \\ y = \dfrac{1}{e^{-x}}\left[\displaystyle\int 3xe^{-x}dx + c_2\right] = -3x - 3 + c_2e^x & x < 0 \end{cases}$$

Solution of Equation (8):

$$y = \frac{1}{e^{-x}}\left[\int 2xe^{-x}dx + c_3\right] = -2x - 2 + c_3e^x$$

Solution of Equation (9):

$$\begin{cases} y = -3x - 3 + c_4e^x & x \geq 0 \\ y = -x - 1 + c_5e^x & x < 0 \end{cases}$$

By considering the initial condition $y(0) = \langle 2,3,4 \rangle$, the constant values are calculated:

$$c_1 = 3 \quad c_2 = 5 \quad c_3 = 5 \quad c_4 = 5 \quad c_5 = 7$$

Finally, the general solution is obtained:

$$y = \begin{cases} \langle -x - 1 + 3e^x, -2x - 2 + 5e^x, -3x - 3 + 7e^x \rangle & x \geq 0 \\ \langle -3x - 3 + 5e^x, -2x - 2 + 5e^x, -x - 1 + 5e^x \rangle & x < 0 \end{cases}$$

Bede et al. [3] solved this example by considering the fuzzy function and derivative. Through lengthy and complex calculations, they obtained the general solution, whereas our proposed calculations are much simpler and shorter.

### 3.3. Solving exact differential equations with fuzzy coefficients

A differential equation

$$aM(x, y)dx + bN(x, y)dy = 0, \quad a, b \in \mathbb{R}$$

is called exact if there exists a continuously differentiable function $f(x,y)$, so that

$$\frac{\partial f}{\partial x}=M,\quad \frac{\partial f}{\partial y}=N.$$

To find the solution of the equation in the form $f(x,y)=c$, the following steps must be done:
*Step 1*. Initially, assume $y$ is constant and integrate $aM(x,y)$ with respect to $x$; the variable $y$ will be presented as the constant of integration, as $h(y)$, so the answer is:

$$f(x,y)=\int M(x,y)dx+h(y).$$

*Step 2*. Derive $f(x,y)$ with respect to $y$ and set it equal to $bN(x,y)$.
*Step 3* . Integrate the resulting function, $h'(y)$, and obtain $h(y)$.
*Step 4*. Finally, substitute the obtained $h(y)$ into the relationship from Step 1 and obtain $f(x,y)=c$.
Now, if the parameters $a$, $b$, and the zero on the right side of the equation be fuzzy due to the uncertainty of the problem, then we have the exact differential equation with fuzzy coefficients

$$\tilde{a}M(x,y)dx+\tilde{b}N(x,y)dy=\langle 0,0,0\rangle.$$

To solve this equation, at first, the variables are multiplied by the fuzzy parameters $\tilde{a}$ and $\tilde{b}$ according to Definition 2. After the initial calculations, the main equation with fuzzy coefficients is converted into several certain linear equations that should be solved.

**Example 6.** Consider the following first-order exact differential equation with fuzzy coefficients*:*

$$(\langle 1,2,3\rangle x^2+\langle 1,2,3\rangle xy^2+\langle 4,5,6\rangle y)dx+(\langle 1,2,3\rangle x^2y+\langle 4,5,6\rangle x+\langle 4,5,5\rangle y^4)dy=\langle 0,0,0\rangle$$

Initially, the variables are multiplied by the fuzzy coefficients. The result is as follows:

$$\begin{aligned}(\langle 2x^2-|x^2|,2x^2,2x^2+|x^2|\rangle+\langle 2xy^2-|xy^2|,2xy^2,2xy^2+|xy^2|\rangle+\langle 5y-|y|,5y,5y+|y|\rangle)dx\\+(\langle 2x^2y-|x^2y|,2x^2y,2x^2y+|x^2y|\rangle\\+\langle 5x-|x|,5x,5x+|x|\rangle+\langle 4.5y^4-0.5|y^4|,5y^4,4.5y^4+0.5|y^4|\rangle)dy=\langle 0,0,0\rangle\end{aligned}$$

The terms with even power are positive, so the equation can be simplified as follows:

$$(\langle x^2,2x^2,3x^2\rangle+\langle 2xy^2-|x|y^2,2xy^2,2xy^2+|x|y^2\rangle+\langle 5y-|y|,5y,5y+|y|\rangle)dx+$$

$$+(\langle 2x^2y-x^2|y|,2x^2y,2x^2y+x^2|y|\rangle+\langle 5x-|x|,5x,5x+|x|\rangle+\langle 4y^4,5y^4,5y^4\rangle)dy=\langle 0,0,0\rangle$$

After the initial computations, the following three equations are obtained:

$$\begin{cases}(x^2+\ 2xy^2-|x|y^2+5y-|y|)dx+(2x^2y-x^2|y|+5x-|x|+4y^4)dy=0 & (10)\\(2x^2+2xy^2+5y)dx+(2x^2y+5x+5y^4)dy=0 & (11)\\(3x^2+2xy^2+|x|y^2+5y+|y|)dx+(2x^2y+x^2|y|+5x+|x|+5y^4)dy=0 & (12)\end{cases}$$

Each equation is an exact differential equation with certain coefficients. The solutions of these equations are as follows:

$$f_1(x,y)=\begin{cases}\dfrac{x^3}{3}+\dfrac{1}{2}x^2y^2+4xy+\dfrac{4}{5}y^5+c_1 & x\ge 0,y\ge 0\\ \dfrac{x^3}{3}+\dfrac{3}{2}x^2y^2+6xy+\dfrac{4}{5}y^5+c_2 & x<0,y<0\\ \dfrac{x^3}{3}+\dfrac{3}{2}x^2y^2+4xy+\dfrac{4}{5}y^5+c_3 & x\ge 0,y<0\\ \dfrac{x^3}{3}+\dfrac{1}{2}x^2y^2+6xy+\dfrac{4}{5}y^5+c_4 & x<0,y\ge 0\end{cases}$$

$$f_2(x,y)=\frac{2}{3}x^3+x^2y^2+5xy+y^5+c_5$$

$$f_3(x,y)=\begin{cases} x^3+\frac{3}{2}x^2y^2+6xy+y^5+c_6 & x\geq 0, y\geq 0\\ x^3+\frac{1}{2}x^2y^2+4xy+y^5+c_7 & x<0, y<0\\ x^3+\frac{1}{2}x^2y^2+6xy+y^5+c_8 & x\geq 0, y<0\\ x^3+\frac{3}{2}x^2y^2+4xy+y^5+c_9 & x<0, y\geq 0\end{cases}$$

**Example 7.** Consider the following differential equation:

$$(\langle 1,2,3\rangle x+\langle 1,1,1\rangle y)dx+(\langle 1,1,2\rangle x+\langle 2,2,3\rangle y+\langle 0,1,1\rangle)dy=\langle 0,0,0\rangle$$

Multiplying the variables by the coefficients, the result is follows:

$$(\langle 2x-|x|,2x,2x+|x|\rangle+\langle y,y,y\rangle)dx+(\langle 1.5x-0.5|x|,x,1.5x+0.5|x|\rangle \\ +\langle 2.5y-0.5|y|,2y,2.5y+0.5|y|\rangle+\langle 0,1,1\rangle)dy=\langle 0,0,0\rangle$$

Finally, the following three equations should be computed:

$$\begin{cases}(2x-|x|+y)dx+\left(\frac{3}{2}x-\frac{1}{2}|x|+\frac{5}{2}y-\frac{1}{2}|y|\right)dy=0 & (13)\\ (2x+y)dx+(x+2y+1)dy=0 & (14)\\ (2x+|x|+y)dx+\left(\frac{3}{2}x+\frac{1}{2}|x|+\frac{5}{2}y+\frac{1}{2}|y|+1\right)dy & (15)\end{cases}$$

By solving each equation separately, we have:

$$f_1(x,y)=\begin{cases}\frac{1}{2}x^2+xy+y^2 & x\geq 0, y\geq 0\\ \frac{3}{2}x^2+2xy+\frac{3}{2}y^2 & x<0, y<0\\ \frac{1}{2}x^2+xy+\frac{3}{2}y^2 & x\geq 0, y<0\\ \frac{3}{2}x^2+2xy+y^2 & x<0, y\geq 0\end{cases}$$

$$f_2(x,y)=x^2+xy+y^2+y+c_1$$

$$f_3(x,y)=\begin{cases}\frac{3}{2}x^2+2xy+\frac{3}{2}y^2+y & x\geq 0, y\geq 0\\ \frac{1}{2}x^2+xy+y^2+y & x<0, y<0\\ \frac{3}{2}x^2+2xy+y^2+y & x\geq 0, y<0\\ \frac{1}{2}x^2+xy+\frac{3}{2}y^2+y & x<0, y\geq 0\end{cases}$$

### 3.4. Bernoulli Differential Equations with Fuzzy Coefficients

Bernoulli differential equations have the form $y' + ap(x)y = bq(x)y^n$, where $p(x)$ and $q(x)$ are continuous functions of $x$ and $n$ is a real number. In the case of $n = 0$ and $n = 1$, the equation becomes a first-order linear differential equation. This equation is recognized as a type of nonlinear differential equation that can be transformed into linear equations through suitable variable changes. This special feature aids in solving these equations. To solve them, both sides of the equation are divided by $y^n$. Then, by changing the variable $u = y^{1-n}$, the equation is transformed into a first-order linear differential equation in terms of $u$. We will then solve the resulting linear equation according to the specified method.

Now, if the parameters $a$ and $b$ are presented in a fuzzy form due to the uncertainty of the problem, then Bernoulli differential equations with fuzzy coefficients are displayed as $y' + \tilde{a}p(x)y = \tilde{b}q(x)y^n$. To solve this, first, the fuzzy parameters $\tilde{a}$ and $\tilde{b}$ are multiplied into the variables according to the definition 2, After performing the initial calculations, the equation with fuzzy coefficients will be transformed into several deterministic Bernoulli equations and solved.

**Example 8.** Consider the Bernoulli equation below:

$$\frac{dy}{dx} + \langle 1,2,2.6\rangle y = \langle 1.4,2,2.8\rangle x y^{\frac{3}{2}}$$

First, by multiplying the fuzzy coefficients into the variables, we get the following equation:

$$\frac{dy}{dx} + \langle 1.8y - 0.8|y|, 2y, 1.8y + 0.8|y|\rangle = \langle 2.1xy^{\frac{3}{2}} - 0.7\left|xy^{\frac{3}{2}}\right|, 2xy^{\frac{3}{2}}, 2.1xy^{\frac{3}{2}} + 0.7\left|xy^{\frac{3}{2}}\right|\rangle$$

Then, by performing fuzzy addition, the equation transforms as follows:

$$\langle\frac{dy}{dx} + 1.8y - 0.8|y|, \frac{dy}{dx} + 2y, \frac{dy}{dx} + 1.8y + 0.8|y|\rangle = \langle 2.1xy^{\frac{3}{2}} - 0.7\left|xy^{\frac{3}{2}}\right|, 2xy^{\frac{3}{2}}, 2.1xy^{\frac{3}{2}} + 0.7\left|xy^{\frac{3}{2}}\right|\rangle$$

After performing the initial calculations, the following equations are obtained:

$$\begin{cases} \dfrac{dy}{dx} + 1.8y - 0.8|y| = 2.1xy^{\frac{3}{2}} - 0.7\left|xy^{\frac{3}{2}}\right| & (16) \\ \dfrac{dy}{dx} + 2y = 2xy^{\frac{3}{2}} & (17) \\ \dfrac{dy}{dx} + 1.8y + 0.8|y| = 2.1xy^{\frac{3}{2}} + 0.7\left|xy^{\frac{3}{2}}\right| & (18) \end{cases}$$

Solution of the Equation (16)

$$\frac{dy}{dx} + 1.8y - 0.8|y| = 2.1xy^{\frac{3}{2}} - 0.7\left|xy^{\frac{3}{2}}\right|$$

For ease of calculation, $x$ and $y$ will be taken out of the absolute value:

$$\begin{cases} \dfrac{dy}{dx} + y = 1.4xy^{\frac{3}{2}} & x \geq 0, y \geq 0 \\ \dfrac{dy}{dx} + y = 2.8xy^{\frac{3}{2}} & x < 0, y \geq 0 \end{cases}$$

With the help of the variable change $u = y^{\frac{-1}{2}}$ and$u' = \frac{-1}{2}y^{\frac{-3}{2}}y'$, the equations transform into the following linear forms:

$$\begin{cases} u' - 0.5u = -0.7x & x \geq 0, y \geq 0 \\ u' - 0.5u = -1.4x & x < 0, y \geq 0 \end{cases}$$

As a result, the general solution of the equation (16) is as follows:

$$\begin{cases} y = \left(\dfrac{1}{1.4x + 2.8 + c_1 e^{\frac{1}{2}x}}\right)^2 & x \geq 0, y \geq 0 \\ y = \left(\dfrac{1}{2.8x + 5.6 + c_2 e^{\frac{1}{2}x}}\right)^2 & x < 0, y \geq 0 \end{cases}$$

With the help of the variable change $u = y^{\frac{-1}{2}}$ and $u' = \frac{-1}{2} y^{\frac{-3}{2}} y'$, Equation (17) also transforms into the following linear forms:

$$u' - u = -x$$

The final answer of the above equation is as follows:

$$y = \left(\frac{1}{x + 1 + c_3 e^x}\right)^2$$

The general solution of the Equation (18)

$$\begin{cases} y = \left(\dfrac{1}{1.076x + 0.82 + c_4 e^{1.38x}}\right)^2 & x \geq 0, y \geq 0 \\ y = \left(\dfrac{1}{0.538x + 0.41 + c_5 e^{1.38x}}\right)^2 & x < 0, y \geq 0 \end{cases}$$

## 4. Application in Electrical Circuits

Figure 2 shows an electrical circuit that includes a resistance $R$ (ohms), a coil with self-inductance $L$ (henries), and a battery $E$ (volts), where these three electrical elements are connected in series. In general, $R$ and $L$ are assumed to be constant and positive, but $E = E(t)$ is regarded to be a function of time, $t$. When the switch is closed, the circuit is said to be closed, the battery delivers an electric current with intensity $I(t)$ amperes to the circuit. We recall that:

- The voltage drop across the resistance is $E_R = RI$
- The voltage drop across the coil is $E_L = L\frac{dI}{dt}$

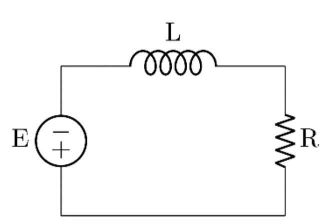



**Figure 2.** Electrical Circuit

Now, we recall Kirchhoff's laws:
In an electrical circuit, the sum of current intensities entering a node is equal to the sum of current intensities leaving that node.
In a closed loop, the sum of voltage drops across resistances and coils is equal to the total electromotive force.
According to the Kirchhoff's second law, in the closed circuit, as shown in Figure 1, we can write $E_R + E_L = E$.
Thus, we have the following first-order linear differential equation

$$L\frac{dI}{dt} + RI = E \tag{19}$$

From which, $I = I(t)$ is obtained. In Equation (19), $R$ and $E$ are not definitive and they have uncertainty due to heat, wire thickness, and other circuit parameters. Therefore, by considering these parameters as fuzzy values, the differential equation will be converted to a fuzzy differential equation.

**Example 9.** Consider an electrical circuit with $L = 1\,H$, $R = 5 \pm 0.2\,\Omega$ and $E = 25\,V$. Exerting uncertain circuit parameters, we have the following first-order linear differential equation with fuzzy coefficients:

$$\frac{dI}{dt} + \langle 4.8,5,5.2 \rangle I = \langle 24.5,25,25.5 \rangle$$

Initially, by multiplying the fuzzy coefficients by the variable, we will have:

$$\frac{dI}{dt} + \langle 5I - 0.2|I|, 5I, 5I + 0.2|I| \rangle = \langle 24.5,25,25.5 \rangle$$

By performing fuzzy addition, the equation is transformed as follows:

$$\left\langle \frac{dI}{dt} + 5I - 0.2|I|, \frac{dI}{dt} + 5I, \frac{dI}{dt} + 5I + 0.2|I| \right\rangle = \langle 24.5,25,25.5 \rangle$$

After the initial computations, the following equations are obtained:

$$\begin{cases} \dfrac{dI}{dt} + 5I - 0.2|I| = 24.5, & (20) \\ \dfrac{dI}{dt} + 5I = 25, & (21) \\ \dfrac{dI}{dt} + 5I + 0.2|I| = 25.5 & (22) \end{cases}$$

Each of the equations will be solved separately.

**Solving equation (20)**: By separating the equation for positive and negative $I$, we have:

$$\frac{dI}{dt} + 5I - 0.2|I| = 24.5 \rightarrow \begin{cases} \dfrac{dI}{dt} + 4.8I = 24.5 & I \geq 0 \\ \dfrac{dI}{dt} + 5.2I = 24.5 & I < 0 \end{cases}$$

The obtained equations are first-oreder liner differential equations and can be solved as mentioned before. At first, $\mu(t) = \int p(t)dt$ is computed:

$$\begin{cases} \mu(t) = e^{\int 4.8dt} = e^{4.8t} & I \geq 0 \\ \mu(t) = e^{\int 5.2dt} = e^{5.2t} & I < 0 \end{cases}$$

Finally, the general solution will be obtained as follows:

$$\begin{cases} I = \dfrac{1}{e^{4.8t}}\left(\displaystyle\int 24.5e^{4.8t}dt + c_1\right) = e^{-4.8t}\left(\dfrac{24.5}{4.8}e^{4.8t} + c_1\right) = 5.1 + c_1e^{-4.8t} & I \geq 0 \\ I = \dfrac{1}{e^{5.2t}}\left(\displaystyle\int 24.5e^{5.2t}dt + c_2\right) = e^{-5.2t}\left(\dfrac{24.5}{5.2}e^{5.2t} + c_2\right) = 4.7 + c_2e^{-5.2t} & I < 0 \end{cases}$$

**Solving equation (21)**: The obtained solution is:

$$I = \frac{1}{e^{5t}}\left(\int 25 \times e^{5t}dt + c_3\right) = 5 + c_3e^{-5t}$$

**Solving equation (22)**: Using the same scheme as (20), we have:

$$\begin{cases} I = \dfrac{1}{e^{5.2t}}\left(\displaystyle\int 25.5e^{5.2t}dt + c_4\right) = e^{-5.2t}\left(\dfrac{25.5}{5.2}e^{5.2t} + c_4\right) = 4.9 + c_4e^{-5.2t} & I \geq 0 \\ I = \dfrac{1}{e^{4.8t}}\left(\displaystyle\int 25.5e^{4.8t}dt + c_5\right) = e^{-4.8t}\left(\dfrac{25.5}{4.8}e^{4.8t} + c_5\right) = 5.3 + c_5e^{-4.8t} & I < 0 \end{cases}$$

Since the problem has an initial condition as following:

$$I(0) = 0, \tag{23}$$

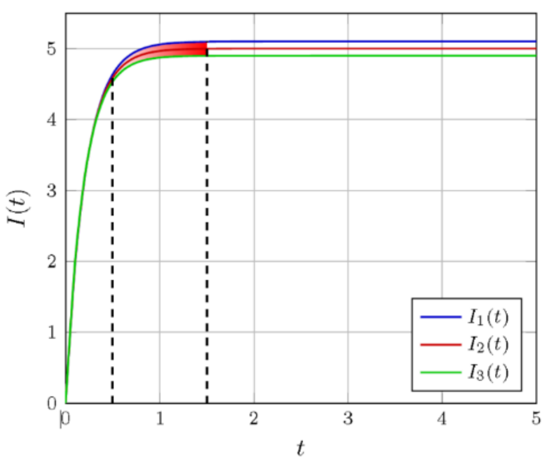
the constant parameters are obtained as follows:

$$c_1 = -5.1, c_2 = -4.7, c_3 = -5, c_4 = -4.9, c_5 = -5.3. \tag{24}$$

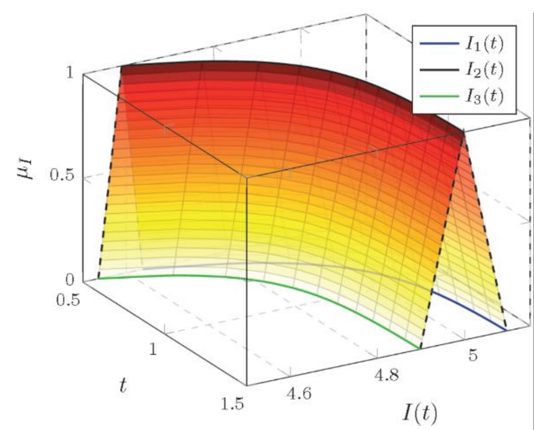
Besides, since time is a positive quantity, $t \geq 0$, all of the obtained solutions should be positive, so the final solution is:

$$I(t) = \langle 5.1 - 5.1e^{-4.8t}, 5 - 5e^{-5t}, 4.9 - 4.9e^{-5.2t} \rangle \tag{25}$$

The results are given in Figure 3. Figure 3a shows the graphs of current intnsity for $t \in [0,5]$. In Figure 3b, the fuzzy form of the results is shown for $t \in [0.5,1.5]$.




(a): Current intensity




(b): Fuzzy form of current intensity

**Figure 3.** Obtained results for current intensity in Example 7.

## 5. Conclusion and Summary

Classical mathematics eliminates the uncertainty in the real world, because it lacks the ability to model it. Fuzzy theory is a framework that has the capability to model reality as it is and accepts uncertainty and imprecision as an important part of the real world. In this paper, due to the uncertainty of problems in the real world, we used fuzzy theories to solve differential equations. In this paper, numerous examples of differential equations with fuzzy coefficients are examined and solved. The equations are of different types: separable, linear, and exact differential equations. The process of solving such equations is such that by performing initial calculations, the equation with fuzzy coefficients will be converted into several definite equations, each of which is examined and solved, and finally, the solution is presented in a fuzzy form.

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