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Fixed cost allocation in two-stage system in the presence of undesirable outputs

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Abstract
Data Envelopment Analysis (DEA) is a mathematical programming-based technique used to determine the efficiency of a set of homogeneous decision-making units (DMUs). In certain cases, these units operate as a multi-stage process, where the outputs of one stage serve as inputs for the next. Such a structure is particularly common in production lines of industrial and manufacturing systems, where raw materials are used as inputs in the first stage, and the desired product is progressively completed through various stages, with the final product being the main output in the last stage. In most production activities and daily operations, besides desirable outputs, undesirable outputs such as harmful emissions, production waste, etc., are also generated. As a result, numerous studies in economics, management, and production emphasize reducing energy consumption to protect the environment and control pollution. In this paper, a new model is proposed that includes undesirable outputs and allocates fixed costs within networks. The focus of these models is to design approaches for allocating fixed costs where the reduction ratio of undesirable outputs is higher than that of desirable outputs.

Keywords: Network Data Envelopment Analysis, Fixed Cost Allocation, Undesirable Outputs.

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1. Introduction

Data envelopment analysis is a non-parametric method for evaluating the units under evaluation with several inputs and outputs, which was presented by Farrell 1957 for the first time. So, Charnes et al. presented the CCR model in 1978 [1]. Although the initial models focused only on evaluating non-negative input and output performance, but due to the wide use of DEA models in different industries, different types of DEA models were presented in different industries. It has been reported that in addition to performance evaluation, other concepts such as pattern finding, determining the eleventh type, scale, cost efficiency, etc. have been discussed [1]. Among these studies can be the research done by [2-5]. Another point to consider is that in classical DEA, the focus has always been on increasing outputs and reducing inputs. However, this approach sometimes fails to provide accurate assessments of DMUs for managers. For example, in a gas-fired power plant, electricity is a desirable final product, but alongside it, undesirable outputs such as air or environmental pollution are also generated. In this context, merely increasing outputs might not yield satisfactory results for managers. Jahanshahloo et al. (2004) introduced a multi-objective linear programming (MOLP) model to estimate the level of outputs or inputs in the presence of undesirable factors, ensuring that efficiency indices remain unchanged [6]. Jahanshahloo et al. (2005) also presented a model for evaluating the efficiency of DMUs with both desirable and undesirable outputs. Their non-radial model altered the traditional DEA view, which was based on increasing outputs and decreasing inputs [7]. Amirteimoori et al. (2006) introduced a model to improve the performance of DMUs in DEA, where undesirable inputs increased and undesirable outputs decreased [8]. Chen et al. (2012) examined undesirable factors in DEA models for companies that simultaneously produce desirable and undesirable outputs with integer values, such as the number of road accidents or human fatalities in a transportation system. Their proposed model was based on the additive DEA model [9].

The research history from the perspective of production technology shows that Hailu and Weeman (2001) proposed a technology for handling undesirable outputs by treating them as inputs. Their justification was that both undesirable inputs and outputs impose costs on DMUs, leading them to use the strong disposability principle to introduce a production possibility set (PPS). Fare and Grosskopf (2008) criticized Hailu and Weeman’s postulates for not aligning with production laws, arguing that finite inputs cannot produce infinite outputs. Later, Fare and Grosskopf (2013) introduced a PPS based on the weak disposability principle with equal contraction coefficients for both desirable and undesirable outputs [10]. Kuosmanen (2005) introduced a PPS with unequal contraction coefficients for the presence of weak disposability, suggesting a method to linearize the nonlinear PPS [11]. Kuosmanen and Podinovsky (2009) proposed a method using different contraction factors for undesirable outputs to maintain the convexity condition of the PPS, addressing issues with equal coefficients [12]. The postulates they used included: (a) inclusion of observations, (b) convexity, (c) strong disposability for desirable inputs and outputs, and (d) weak disposability for all products, both desirable and undesirable. Amirteimoori and colleagues (2017) redefined weak disposability for undesirable outputs by using a linear relationship instead of contraction coefficients to limit undesirable products [13]. They subtracted a positive value from both desirable and undesirable outputs, allowing undesirable outputs to approach zero without affecting desirable outputs. Toloo and Hanclova (2020) evaluated the efficiency of DMUs with multi-valued and undesirable outputs in DEA, where concepts like unemployment in economics, which has multiple indices, were used to define it [14]. Monzeli and colleagues (2020) introduced an appropriate production possibility set (PPS) based on the problem's assumptions, then proposed a new method for determining the undesirable performance of some inputs and outputs in DMUs [15]. Streimikis and Saraji (2022) provided a comprehensive review of DEA studies measuring efficiency in the presence of undesirable outputs, systematically reviewing 58 articles published between 2000 and 2020 [16].

The presence of undesirable factors has also been considered in Network DEA (NDEA) models. Kordrostami et al. (2005) evaluated the efficiency of a multi-stage DEA network, where undesirable output variables were introduced with a negative sign [17]. Lozano et al. (2013) used the weak disposability principle to introduce a PPS for a comprehensive network structure in the presence of undesirable products and then evaluated efficiency [18]. Maghbouli et al. (2014) examined a two-stage network structure with undesirable outputs, addressing efficiency evaluation by considering undesirable products either as final outputs or intermediate undesirable products [19]. Bian et al. (2015) used the SBM model to assess efficiency and decompose efficiency for a two-stage network with undesirable outputs [20]. Wu et al. (2015) introduced a collective model for a two-stage network where undesirable outputs were generated in the first stage [21]. Liu et al. (2015) evaluated the performance of Chinese banks using a two-stage network model with intermediate undesirable outputs [22]. Wu et al. (2016) proposed a reuse approach for intermediate undesirable outputs in a two-stage production process with a common resource [23]. Kalhor and Kazemi Matin (2018) considered a comprehensive network structure. To define the technology on this network, two approaches were examined [24]. One approach, the effect of uniform contraction coefficients of the Fare and Grosskoph model using the principle of weak Disposability, and the other approach, the effect of non-uniform contraction coefficients using the principle of weak Disposability, were considered to construct PPS. Then they checked their proposed model to evaluate the efficiency of the comprehensive network on Spanish airports. A network data envelopment analysis (NDEA) model with undesirable output was developed by Yu et al. (2020) to evaluate the environmental efficiency of 30 Chinese provinces [25]. Chen et al. (2020) discuss NDEA by considering the inputs and outputs of the production process surrounding a bank as additional adverse factors [26]. Shi et al. (2021) evaluated the efficiency of one of the bank branches in China, which were considered undesirable factors in this research. They used the SBM method to analyze the introduced series and parallel network [27]. Amirtimuri et al. (2021) evaluated the efficiency of 32 paper factories in China by using the network data overlay analysis in the presence of adverse factors. They used constants and undesirable outputs [28]. Zhou et al. (2022) evaluated the energy efficiency in different countries, they examined the three-stage network structure in the presence of adverse outputs in this research and the result was that China, Japan and Australia are on the efficiency border and the best They have energy efficiency [28]. Ma et al.

(2022) investigated the efficiency of the two-stage network in the presence of adverse factors, and the purpose of the proposed model was to evaluate industrial water treatment in China. Along with economic growth, the production of industrial water waste was investigated as an undesirable output [29]. Soofizadeh and Fallahnejad (2022) presented a bargaining-based method for DEA network evaluation by considering common inputs and undesirable outputs [30]. A mixed integer network (MI-NDEA) with common inputs and undesirable outputs has been proposed by Omrani et al. (2023) to evaluate the efficiency of decision-making units [31]. In another article, Omrani et al. (2023) present a Network Data Envelopment Analysis (NDEA) model to evaluate the road transport sector by considering desirable and undesirable outputs [32]. Khoshandam and Nematizadeh (2024) presented an inverse DEA model for a production system with a two-stage network structure in the presence of adverse factors using the principle of weak Disposability [33].

In data coverage analysis to estimate the efficiency of decision-making units or production units, the ratio of the increase in outputs to the decrease in inputs is examined, but if there is an undesirable output, it is not possible to simply increase the outputs to evaluate the efficiency. In such a situation, targeting for the efficiency of the decision-making units should be carried out in such a way that the consumption of inputs and the production of undesirable products both decrease, while at the same time, the production of the final desired products increases. Different approaches have been considered for modeling decision-making units in the presence of adverse factors. One of these approaches is to build a set of production possibilities using the principle of weak Disposability.

The article is organized as follows: in the next part, the basic and preliminary concepts of DEA are given, in the third part, fixed cost allocation in the network with the presence of undesirable outputs is done. In the fourth part, an applied example with fixed cost allocation and structure A network is performed in the presence of undesirable outputs, and conclusions and suggestions are given in the fifth section.

2. Preliminaries

Weak Disposability in the Presence of Undesirable Outputs

Weak Disposability (Shephard):

Suppose desirable outputs ($\boldsymbol{y}^g$) and undesirable outputs ($\boldsymbol{y}^b$) are produced using inputs (x). Hence, the production technology is defined as follows (2015):

$$T=\left\{(\boldsymbol{y}^g,\boldsymbol{y}^b,\mathbf{x})\middle|\ \text{produce}\ \mathbf{x}\ \text{ by } (\boldsymbol{y}^g,\boldsymbol{y}^b)\right\}$$

Definition 1: Desirable and undesirable outputs have the property of weak disposability if and only if for every $(yg, yb, x) \in T$ and $0 \leq \theta \leq 10$, we have $(\theta yg, \theta yb, x) \in T$. Mathematically, it can be expressed as:

$$\forall\ (\boldsymbol{y}^g,\boldsymbol{y}^b,\mathbf{x}),\quad \forall\theta\quad \left[(\boldsymbol{y}^g,\boldsymbol{y}^b,\mathbf{x})\epsilon T\ \&0 \leq \theta \leq 1 \rightarrow (\theta\mathbf{y}^g,\theta\boldsymbol{y}^b,\mathbf{x})\epsilon T\ \right]$$

The Farrell-Grosskopf Production Possibility Set (2003) in the Presence of Undesirable Outputs

Suppose there are n $DMUs$, and for $DMUj$, $\boldsymbol{x_{ij}}$ is the input vector with m inputs, $\boldsymbol{y}^g_{(\boldsymbol{r1j})}$ is the desirable output vector with s_1 desirable outputs, and $\boldsymbol{y}^b_{(\boldsymbol{r2j})}$ is the undesirable output vector with s_2 undesirable outputs. Fare and Grosskopf proposed efficiency measures using axioms that included the weak disposability condition. The production possibility set (1) represents the production possibility set proposed by Fare and Grosskopf (2003) [34].

$$T_{Fg}=\left\{\left(y^g,y^b,x\right)\left|\begin{array}{l}\sum_{j=1}^{n}\theta\lambda_j y^g_{r_1 j}\ge y^g_{r_1},\ r_1=1,\dots,s_1,\\ \sum_{j=1}^{n}\theta\lambda_j y^b_{r_2 j}=y^b_{r_2},\ r_2=1,\dots,s_2,\\ \sum_{j=1}^{n}\lambda_j x_{ij}\le x_i,\ i=1,\dots,m,\\ \sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\ge 0,\ 0<\theta\le 1,\ j=1,\dots,n.\end{array}\right.\right\} \quad (1)$$

They introduced their production possibility set using fixed contraction coefficients θ to limit undesirable outputs as much as possible. They also considered the axioms of convexity, strong disposability for inputs and desirable outputs, and weak disposability for all outputs to propose their model.

The Kuosmanen Production Possibility Set (2005) in the Presence of Undesirable Outputs

Since the set proposed by Fare and Grosskopf (2003) might not fully exhibit convexity, Kuosmanen later proposed relations that better represented the convexity property (2005). Kuosmanen constructed the PPS using the axioms of convexity, strong disposability for all inputs and desirable outputs, and weak disposability for all outputs. In Kuosmanen's technology, unequal contraction coefficients were used to better exhibit the convexity property. This production possibility set is expressed as (2).

$$T_{k}=\left\{\left(y^g,y^b,x\right)\left|\begin{array}{l}\sum_{j=1}^{n}\theta_j\lambda_j y^g_{r_1 j}\ge y^g_{r_1},\ r_1=1,\dots,s_1,\\ \sum_{j=1}^{n}\theta_j\lambda_j y^b_{r_2 j}=y^b_{r_2},\ r_2=1,\dots,s_2,\\ \sum_{j=1}^{n}\lambda_j x_{ij}\le x_i,\ i=1,\dots,m,\\ \sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\ge 0,\ 0<\theta_j\le 1,\ j=1,\dots,n.\end{array}\right.\right\} \quad (2)$$

For linearizing the relations in (2), the following variable transformation was used:

$$\begin{cases}\theta_j\lambda_j=p_j\\ (1-\theta_j)\lambda_j=\mu_j\\ \mu_j+p_j=\lambda_j\end{cases} \quad (3)$$

For linearizing the relations in (2), the following variable transformation was used:

$$T_{L}^{k}=\left\{\left(y^g,y^b,x\right)\left|\begin{array}{l}\sum_{j=1}^{n}\rho_j y^g_{r_1 j}\ge y^g_{r_1},\ r_1=1,\dots,s_1,\\ \sum_{j=1}^{n}\rho_j y^b_{r_2 j}=y^b_{r_2},\ r_2=1,\dots,s_2,\\ \sum_{j=1}^{n}\left(\mu_j+\rho_j\right)x_{ij}\le x_i,\ i=1,\dots,m,\\ \sum_{j=1}^{n}\left(\mu_j+\rho_j\right)=1,\ \rho_j\ge 0,\ \mu_j\ge 0,\ j=1,\dots,n\end{array}\right.\right\} \quad (4)$$

Model (4) is the linearized version of Kuosmanen’s equations, which successfully demonstrates the convexity property using unequal contraction coefficients in the production possibility set.

3. Fixed Cost Allocation in Networks with Undesirable Outputs

Suppose there are n units under evaluation with a two-stage network structure similar to Figure (1). In Figure (1), $X=(x_1,\dots,x_m)$ are the inputs for the first stage, and $Z=(z_1,\dots,z_h)$ are the outputs of the first stage, which serve as inputs to the second stage. $Y^D=(y_1^D,\dots,y_{S_1}^D)$ are the desirable outputs for the second stage, and $Y^{UD}=(y_1^{UD},\dots,y_{S_2}^{UD})$are the undesirable outputs for the second stage.

To evaluate the efficiency of the network shown in Figure (1), Kuosmanen’s technology is used. Kuosmanen utilized the axioms of convexity, strong disposability for all inputs and desirable outputs, and weak disposability for all outputs to construct the PPS. Kuosmanen's technology employed unequal contraction coefficients to better demonstrate the convexity property. The production possibility set, along with the weak disposability condition using contraction coefficients for the two-stage network, is defined in relation (5):

$$T=\left\{\left(x,z,y^D,y^{UD}\right)\left|\begin{array}{l}\sum_j\lambda_j^1x_j\le x,\sum_j\lambda_j^1z\ge z,\sum_j\lambda_j^2z\le z,\\ \sum_j\lambda_j^1z\ge\sum_j\lambda_j^2z,\sum_j\theta_j\lambda_j^2y_j^D\ge y^D,\\ \sum_j\theta_j\lambda_j^2y_j^{UD}=y^{UD},\sum_j\lambda_j^1=1,\sum_j\lambda_j^2=1,\\ \lambda_j^1\ge0,\ \lambda_j^2\ge0,1\ge\theta_j\ge0,\ j=1,\dots,n\end{array}\right.\right\} \tag{5}$$

After considering the production possibility set (5) shown in Figure (1), model (6) is proposed for evaluating the performance of two-stage systems, as depicted in Figure (1).

$$\begin{aligned}
e_T&=\min\theta\\
s.t\quad&\sum_j\lambda_j^1X_j\le\theta X_p\\
&\sum_j\lambda_j^1Z_j\ge\sum_j\lambda_j^2Z_j\\
&\sum_j\theta_j\lambda_j^2Y_j^D\ge Y_p^D\\
&\sum_j\theta_j\lambda_j^2Y_j^{UD}=Y_p^{UD}\\
&\sum_j\lambda_j^1=1\\
&\sum_j\lambda_j^2=1\\
&\lambda_j^1\ge0,\lambda_j^2\ge0,1\ge\theta_j\ge0,\ j=1,\dots,n
\end{aligned} \tag{6}$$

To linearize model (6), variable transformations (7) are used.

$$\begin{aligned}
&\lambda_j^1=\lambda_j, && j=1,\dots,n\\
&\lambda_j^2=\alpha_j+\beta_j, && j=1,\dots,n\\
&\theta_j\lambda_j^2=\beta_j, && j=1,\dots,n\\
&\left(1-\theta_j\right)\lambda_j^2=\alpha_j, && j=1,\dots,n
\end{aligned} \tag{7}$$

To linearize model (6), variable transformations (7) are used.

$$\begin{aligned}
e_T&=\min\theta\\
s.t\quad&\sum_j\lambda_jX_j\le\theta X_p\\
&\sum_j\lambda_jZ_j\ge\sum_j\left(\alpha_j+\beta_j\right)Z_j\\
&\sum_j\beta_jY_j^D\ge Y_p^D\\
&\sum_j\beta_jY_j^{UD}=Y_p^{UD}\\
&\sum_j\lambda_j=1\\
&\sum_j\left(\alpha_j+\beta_j\right)=1\\
&\lambda_j\ge0,\alpha_j\ge0,\beta_j\ge0,\ j=1,\dots,n
\end{aligned} \tag{8}$$

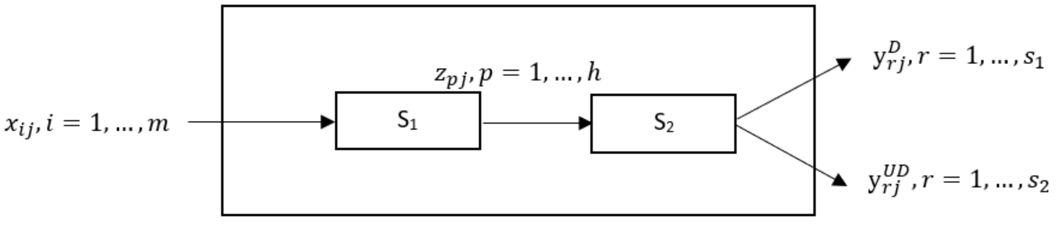

Figure (1): Network with two-stage DMUj structure in the presence of undesirable output

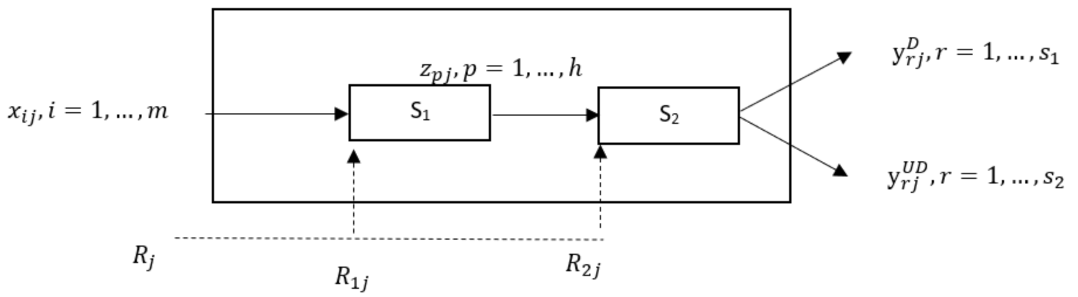

Figure 2: DMUj two-stage network in the presence of undesirable output and fixed cost allocation

Model (9) is the arranged version of model (8):

$$
\begin{aligned}
& e_T = \min \theta \\
& s.t \quad -\sum_{j=1}^{n} \lambda_j X_j + \theta X_p \geq 0 \\
& \sum_{j=1}^{n} \lambda_j Z_j - \sum_{j=1}^{n} \alpha_j Z_j - \sum_{j=1}^{n} \beta_j Z_j \geq 0 \\
& \sum_{j=1}^{n} \beta_j Y_j^{D} \geq Y_p^{D} \\
& \sum_{j=1}^{n} \beta_j Y_j^{UD} = Y_p^{UD} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_j = 1 \\
& \lambda_j \geq 0, \alpha_j \geq 0, \beta_j \geq 0, \ j = 1, \ldots, n
\end{aligned}
\tag{9}
$$

Also, the dual of model (9) is written as model (10):

$$
\begin{aligned}
& Max \ \ U^1 Y_P^{D} + U^2 Y_P^{UD} + \mu^1 + \mu^2 \\
& s.t \quad -VX_j + WZ_j + \mu^1 \leq 0, \qquad j = 1, \ldots, n \\
& -WZ_j + U^1 Y_j^{D} + U^2 Y_j^{UD} + \mu^2 \leq 0, \ j = 1, \ldots, n \\
& -WZ_j + \mu^2 \leq 0, \qquad j = 1, \ldots, n \\
& VX_p = 1 \\
& (V, W, U^1) \geq 0, U^2, \ \mu^1, \ \mu^2 \ free
\end{aligned}
\tag{10}
$$

By solving model (10) for each DMU, the maximum efficiency for the unit under evaluation is obtained.

Efficiency Evaluation with Fixed Cost Allocation in Network Structures with Undesirable Outputs

One of the issues in data envelopment analysis (DEA) is how to allocate fixed costs among decision-making units (DMUs). DEA techniques propose methods for fair and equitable cost allocation to meet the organization's overall objective. The management's goal is to allocate these costs in a way that does not alter or improves relative efficiency.

$$
Max \frac{U^1 Y_P^{D} + U^2 Y_P^{UD} + \mu^1 + \mu^2}{VX_p} \tag{11}
$$

$$
\begin{aligned}
& s.t \quad \frac{WZ_j + \mu^1}{VX_j} \leq 1, \qquad j = 1, \ldots, n \\
& \frac{U^1 Y_j^{D} + U^2 Y_j^{UD} + \mu^2}{WZ_j} \leq 1, \qquad j = 1, \ldots, n \\
& \frac{\mu^2}{WZ_j} \leq 1, \qquad j = 1, \ldots, n \\
& (V, W, U^1) \geq 0, U^2, \ \mu^1, \ \mu^2 \ free
\end{aligned}
$$

Using model (11), which is the fractional form of model (10), fixed cost allocation in a two-stage network with undesirable outputs will be carried out. Consider the structure shown in Figure (2). It is assumed that there is a fixed cost R that must be allocated.

As shown in Figure (2), each unit receives a non-negative cost allocation R_j , such that:

$$\sum_{j=1}^{n} R_j = R,\ R_j \ge 0,\ \ j=1,...,n \tag{12}$$

Here, the fixed cost allocation in the first and second stages is represented by R_{1j} and R_{2j}, respectively, with relation (13) holding true:

$$R_j = R_{1j} + R_{2j},\quad R_{1j}\ ,R_{2j} \ge 0,\ j=1,...,n \tag{13}$$

Therefore, for fixed cost allocation in the two-stage network of Figure (2), model (14) is proposed:

$$\begin{aligned}
&Max\ \frac{U^1Y_P^D + U^2Y_P^{UD} + \mu^1 + \mu^2}{VX_p + v_{m+1}R_{1P} + v_{m+1}R_{2P}} && (14)\\
&s.t\ \ \frac{WZ_j + \mu^1}{VX_j + v_{m+1}R_{1j}} \le 1, && j=1,...,n\\
&\qquad \frac{U^1Y_j^D + U^2Y_j^{UD} + \mu^2}{WZ_j + v_{m+1}R_{2j}} \le 1, && j=1,...,n\\
&\qquad \frac{\mu^2}{WZ_j} \le 1, && j=1,...,n\\
&\sum_{j=1}^{n}\left(R_{1j} + R_{2j}\right) = R\quad R_{1j}, R_{2j} \ge 0,\ j=1,\dots,n\\
&\qquad (V,W,U^1) \ge 0
\end{aligned}$$

Model (14) is a fractional programming model, which, by applying the Charnes-Cooper transformation (1962), is converted to the linear form in model (15).

$$\begin{aligned}
&Max\ U^1Y_P^D + U^2Y_P^{UD} + \mu^1 + \mu^2 && (15)\\
&s.t\ VX_p + v_{m+1}R_{1P} + v_{m+1}R_{2P} = 1\\
&WZ_j + \mu^1 - VX_j - v_{m+1}R_{1j} \le 0,\ j=1,...,n\\
&U^1Y_j^D + U^2Y_j^{UD} + \mu^2 - WZ_j - v_{m+1}R_{2j} \le 0,\ j=1,...,n\\
&\mu^2 - WZ_j \le 0, && j=1,...,n\\
&\sum_{j=1}^{n}\left(R_{1j} + R_{2j}\right) = R\quad R_{1j}, R_{2j} \ge 0,\quad j=1,\dots,n\\
&\qquad (V,W,U^1) \ge 0
\end{aligned}$$

Since the constraints contain the product of variables, model (15) is a nonlinear programming model. To solve this issue, variable transformations (for each) are used, and model (15) is rewritten as model (16):

$$\begin{aligned}
&Max\ U^1Y_P^D + U^2Y_P^{UD} + \mu^1 + \mu^2 && (16)\\
&s.t\,VX_p + r_{1P} + r_{2P} = 1\\
&WZ_j + \mu^1 - VX_j - r_{1j} \le 0, && j=1,...,n\\
&U^1Y_j^D + U^2Y_j^{UD} + \mu^2 - WZ_j - r_{2j} \le 0,\ j=1,...,n\\
&\mu^2 - WZ_j \le 0, && j=1,...,n\\
&\sum_{j=1}^{n}\left(r_{1j} + r_{2j}\right) = v_{m+1}R\quad r_{1j}, r_{2j} \ge 0,\quad j=1,\dots,n\\
&\qquad (V,W,U^1) \ge 0
\end{aligned}$$

By solving model (16) for each DMU, the maximum efficiency with feasible fixed cost allocation for the evaluated unit is obtained. It is assumed that the optimal solution is derived using model (16); then, the relative efficiency after cost allocation for unit P is given by equation (17):

$$e_P^* = U^1Y_P^D + U^2Y_P^{UD} + \mu^1 + \mu^2 \tag{17}$$

Additionally, the fixed cost allocation scheme can be determined using equation (18):

$$\begin{aligned}
&R_j^{p*} = R_{1j}^{p*} + R_{2j}^{p*},\quad j=1,\dots,n\\
&R_{1j}^{p*} = \frac{r_{1j}^{p*}}{v_{m+1}^{p*}},\qquad j=1,\dots,n && (18)\\
&R_{2j}^{p*} = \frac{r_{2j}^{p*}}{v_{m+1}^{p*}},\qquad j=1,\dots,n
\end{aligned}$$

Thus, by solving model (16), which is a linear programming model, one can easily calculate the relative efficiency and optimal stage allocation for the network presented in Figure (2). Next, some properties of the proposed model will be discussed.

Lemma 1: The optimal value of the objective function in model (16) is always less than or equal to one. **Proof:** By summing the p-th index of the second and third constraints in model (16), the following inequality is obtained:

$$WZ_p + \mu^1 - VX_p - r_{1p} + U^1 Y_p^D + U^2 Y_p^{UD} + \mu^2 - WZ_p - r_{2p} \le 0$$

After simplifying and rearranging some terms, the following inequality results:

$$U^1 Y_p^D + U^2 Y_p^{UD} + \mu^1 + \mu^2 \le VX_p + r_{1p} + r_{2p}$$

The right-hand side of the above inequality is the first constraint of model (16), which is equal to one. Hence, the inequality can be rewritten as follows:

$$U^1 Y_p^D + U^2 Y_p^{UD} + \mu^1 + \mu^2 \le 1$$

This means that the objective function and the optimal value of the model are always less than or equal to one, and the lemma is proven.

Theorem 1: The optimal value of the objective function in model (16) is always equal to one.
Proof: First, the dual of the linear programming model (16) is written as follows:

$$\begin{aligned}
& e_p = \min \theta \\
& s.t \quad -\sum_{j=1}^{n} \lambda_j X_j + \theta X_p \ge 0 \\
& \sum_{j=1}^{n} \lambda_j Z_j - \sum_{j=1}^{n} \alpha_j Z_j - \sum_{j=1}^{n} \beta_j Z_j \ge 0 \\
& \sum_{j=1}^{n} \beta_j Y_j^D \ge Y_p^D \\
& \sum_{j=1}^{n} \beta_j Y_j^{UD} = Y_p^{UD} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_j = 1 \\
& -\lambda_p + \gamma + \theta \ge 0 \\
& -\lambda_j + \gamma \ge 0, \qquad j \ne p \\
& -\beta_p + \gamma + \theta \ge 0 \\
& -\beta_j + \gamma \ge 0, \qquad j \ne p \\
& -\gamma R \ge 0 \\
& \lambda_j \ge 0,\ \alpha_j \ge 0,\ \beta_j \ge 0, \qquad j = 1,\dots,n
\end{aligned}$$

Given the definition of $R \ge 0$ and the constraint $-\gamma R \ge 0$, it follows that $\gamma \le 0$. From the eighth constraint above, where $0 \ge \gamma \ge \lambda_j\ (for\ j \ne p)$, it results that $\lambda_j = 0\ (for\ j \ne p)$. From the constraint, it follows that $\lambda_p = 1$, and from the constraint, it is concluded that the optimal value of model (16) is always equal to one, thus proving the theorem. ■

Definition 2: After fixed cost allocation, a DMU is considered efficient if its efficiency is equal to one.

Theorem 2: In model (16), a DMU will be efficient if and only if each of its sub-stages is efficient.
Proof: The sufficiency condition is evident based on the definition of DMU efficiency and the efficiency of ub-stages. Now, the necessity condition is examined, meaning that if DMUs are efficient, their sub-stages are also efficient. Given that, it is assumed that is efficient; hence,

$$e_j = we_{1j} + (1-w)e_{2j} = \frac{U^1 Y_P^D + U^2 Y_P^{UD} + \mu^1 + \mu^2}{VX_p + v_{m+1} R_{1P} + v_{m+1} R_{2P}} = 1$$

Note that $0 < w < 1$, $0 < e_{1j}, e_{2j} \le 1$

By contradiction, assume there exists an index such that. Without loss of generality, assume $t = 1$, so: $e_{1j} < 1,\ \ e_{2j} \le 1$

Multiplying both sides of the inequalities by the weights and will result in: Thus, the equation will not hold, leading to a contradiction.

Next, it will be shown that all DMUs and their sub-stages can simultaneously be efficient under a common set of weights. Therefore, an efficient common set of weights with cost allocation can be proposed for DMUs.

Theorem 3: There exists at least one optimal allocation from model (16) such that the unit under consideration will be efficient.

Proof: According to Theorem 1, the optimal value of the objective function in the model is always equal to one, so the total efficiency is always one. Hence, after allocation, the total efficiency is always equal to one, and according to Theorem 2, all its sub-components are also efficient.

The allocation using a common set of weights will be as in (19):

$$Max\left\{\begin{array}{l}\dfrac{U^{11}Y_1^{D}+U^{21}Y_1^{UD}+\mu^{11}+\mu^{21}}{V^1X_1+v_{m+1}^{1}(R_{11}+R_{21})},\\ \dfrac{U^{12}Y_2^{D}+U^{22}Y_2^{UD}+\mu^{12}+\mu^{22}}{V^2X_2+v_{m+1}^{2}(R_{12}+R_{22})},\\ \dots,\dfrac{U^{1n}Y_n^{D}+U^{2n}Y_n^{UD}+\mu^{1n}+\mu^{2n}}{V^nX_n+v_{m+1}^{n}(R_{1n}+R_{2n})}\end{array}\right\} \tag{19}$$

$$s.t\ \frac{W^pZ_j+\mu^{1p}}{V^pX_j+v_{m+1}^pR_{1j}}\le 1,\quad j=1,\dots,n,\ p=1,\dots,n$$

$$\frac{U^{1p}Y_j^D+U^{2p}Y_j^{UD}+\mu^{2p}}{W^pZ_j+v_{m+1}^pR_{2j}}\le 1,\quad j=1,\dots,n,\ p=1,\dots,n$$

$$\frac{\mu^{2p}}{W^pZ_j}\le 1,\qquad j=1,\dots,n,\ p=1,\dots,n$$

$$\sum_{j=1}^{n}\left(R_{1j}+R_{2j}\right)=R\quad R_{1j},R_{2j}\ge 0,\ j=1,\dots,n$$

$$(V^p,W^p,U^{1p})\ge 0\ ,\qquad p=1,\dots,n$$

Model (19) is converted into model (20) using the GP method:

$$min\sum_{j=1}^{n}(\varphi_j^-+\varphi_j^+) \tag{20}$$

$$s.t\ \frac{W^pZ_j+\mu^{1p}+\varphi_j^+}{V^pX_j+v_{m+1}^pR_{1j}-\varphi_j^-}=1,\ j=1,\dots,n,\ p=1,\dots,n$$

$$\frac{U^{1p}Y_j^D+U^{2p}Y_j^{UD}+\mu^{2p}+\varphi_j^+}{W^pZ_j+v_{m+1}^pR_{2j}-\varphi_j^-}=1,\ j=1,\dots,n,\ p=1,\dots,n$$

$$\frac{\mu^{2p}+\varphi_j^+}{W^pZ_j-\varphi_j^-}=1,\quad j=1,\dots,n,\ p=1,\dots,n$$

$$\sum_{j=1}^{n}\left(R_{1j}+R_{2j}\right)=R\ \ R_{1j},R_{2j}\ge 0,\ j=1,\dots,n$$

$$(V^p,W^p,U^{1p})\ge 0\qquad p=1,\dots,n$$

$$\varphi_j^+,\varphi_j^-\ge 0,\qquad j=1,\dots,n$$

In DEA models, the target for DMU_j is set to one, and ϕ_j^- and ϕ_j^+ represent the negative and positive deviations from the target, respectively. Adding ϕ_j^+ and subtracting ϕ_j^- from the denominator of the constraints aims to minimize the sum of gaps with the benchmark. Model (20) is converted into model (21) as follows:

$$min\sum_{j=1}^{n}(\varphi_j^-+\varphi_j^+) \tag{21}$$

$$s.t\ \begin{pmatrix}W^pZ_j+\mu^{1p}+\varphi_j^+\\ -V^pX_j-v_{m+1}^pR_{1j}+\varphi_j^-=0,\end{pmatrix},\ j=1,\dots,n,\ p=1,\dots,n$$

$$\begin{pmatrix}U^{1p}Y_j^D+U^{2p}Y_j^{UD}+\mu^{2p}+\varphi_j^+\\ -W^pZ_j-v_{m+1}^pR_{2j}+\varphi_j^-\ =0\end{pmatrix},\ \ j=1,\dots,n,\ p=1,\dots,n$$

$$\mu^{2p}+\varphi_j^+-W^pZ_j+\varphi_j^-=0,\qquad j=1,\dots,n,\ p=1,\dots,n$$

$$\sum_{j=1}^{n}\left(R_{1j}+R_{2j}\right)=R\ \ R_{1j},R_{2j}\ge 0,\ \ j=1,\dots,n$$

$$(V^p,W^p,U^{1p})\ge 0\qquad p=1,\dots,n$$

$$\varphi_j^+,\varphi_j^-\ge 0,\qquad j=1,\dots,n$$

Since the constraints involve the product of variables and, model (21) is a nonlinear programming model. To address this issue, variable transformations (for each) are used, and model (21) is rewritten as model (22):

$$
\begin{aligned}
& min \sum_{j=1}^{n} (\varphi_j^- + \varphi_j^+) \\
& s.t \begin{pmatrix} W^p Z_j + \mu^{1p} + \varphi_j^+ \\ -V^p X_j - r_{1j} + \varphi_j^- = 0 \end{pmatrix}, \ j=1,\dots,n, \ p=1,\dots,n \\
& \begin{pmatrix} U^{1p} Y_j^D + U^{2p} Y_j^{UD} + \mu^{2p} \\ +\varphi_j^+ - W^p Z_j - r_{2j} + \varphi_j^- \ = 0 \end{pmatrix}, j=1,\dots,n, \ p=1,\dots,n \\
& \mu^{2p} + \varphi_j^+ - W^p Z_j + \varphi_j^- = 0, \quad j=1,\dots,n, \ p=1,\dots,n \\
& \sum_{j=1}^{n} \left(r_{1j} + r_{2j} \right) = v_{m+1}^p R \quad r_{1j}, r_{2j} \geq 0, \ j=1,\dots,n, \ p=1,\dots,n \\
& (V^p, W^p, U^{1p}) \geq 0 \qquad\qquad p=1,\dots,n \\
& \varphi_j^+, \varphi_j^- \geq 0, \qquad\qquad j=1,\dots,n
\end{aligned}
$$

The fixed cost allocation scheme can be determined as follows:

$$
\begin{aligned}
& R_j^* = R_{1j}^* + R_{2j}^*, \qquad j=1,\dots,n \\
& R_{1j}^* = \frac{r_{1j}^*}{v_{m+1}^{j*}}, \qquad j=1,\dots,n \quad (23) \\
& R_{2j}^* = \frac{r_{2j}^*}{v_{m+1}^{j*}}, \qquad j=1,\dots,n
\end{aligned}
$$

Thus, by solving model (22), which is a linear programming model, the relative efficiency and optimal allocation of stages for the network shown in Figure (2) can be easily calculated.

4. Practical Example

To evaluate the effect of the model for efficiency assessment and fixed cost allocation, 37 bank branches have been studied. Selection of Decision-Making Units for the Practical Example A total of 37 branches from commercial banks in Iran, located in one geographical area of Tehran, were selected. The system of these branches operates as a network with two inputs, two intermediate outputs, and three final outputs—two desirable and one undesirable output. The personnel score and interest paid are considered inputs for the two-stage network depicted in Figure (1). The intermediate components between the first and second stages include four types of deposits and other resources. Interest received and fees collected are the final desirable outputs, while overdue loans are considered the undesirable output.

Calculation of Efficiency and Fixed Cost Allocation

As mentioned earlier, the efficiency of 37 bank branches will be assessed. Thus, the total number of DMUs is 37. The overall efficiency is analyzed using data from the 37 bank branches. The overall efficiency, calculated using model (9), and the fixed cost allocation values, calculated using model (22), are shown in Table (1).

5. Conclusion

In many managerial applications, decision-makers of large organizations often face the significant issue of how to allocate or charge a shared cost across a set of entities. Fixed cost allocation has become one of the most important applications of the DEA method. However, only a few existing approaches address the fixed cost allocation problem in a network environment. This paper allocates a fixed cost to all DMUs with a network structure that includes undesirable outputs. To this end, we evaluate the relative efficiency of network processes while considering the allocated costs.

This paper can be generalized in several directions. Since different types of two-stage network structures exist, similar methods can be applied to allocate fixed costs to various network structures, including those with undesirable inputs and outputs. Additionally, these methods can be employed to allocate fixed costs for parallel production systems**.**

Table (1): Efficiency and Fixed Cost Allocation Values for the First Stage and Intermediate Outputs of the Bank Branches in Iran

DMUs	e_T	R_1	R_2
DMU1	0.40764	539.4811715	1038.242678
DMU2	0.57804	0	0
DMU3	1	0	492.58159
DMU4	0.88177	418.4100418	81.0460251
DMU5	0.492	300.3891213	345.6192469
DMU6	0.55129	0	462.9874477
DMU7	0.38314	503.9539749	1292.799163
DMU8	0.47429	638.9497908	359.58159
DMU9	0.33173	787.9665272	4282.221757
DMU10	0.46407	1565.200837	3056.054393
DMU11	0.52936	244.7196653	3909.430962
DMU12	0.53374	0	50.53974895
DMU13	0.67534	1199.129707	9986.644351
DMU14	1	0	619.4100418
DMU15	0.43745	571.125523	6548.025105
DMU16	0.30063	392.6527197	4755.384937
DMU17	1	655.8870293	9050.677824
DMU18	0.5625	445.707113	8553.577406
DMU19	0.27489	476.0041841	1021.414226
DMU20	0.30485	648.0669456	3277.263598
DMU21	0.46083	290.2887029	1183.364017
DMU22	0.43796	449.0794979	3342.803347
DMU23	0.293	393.9037657	1823.761506
DMU24	0.24584	837.7573222	1021.732218
DMU25	0.32792	482.4393305	1349.096234
DMU26	0.72772	157.2970711	442.3263598
DMU27	0.83324	523.916318	0
DMU28	0.50475	334.4811715	5202.075314
DMU29	0.40172	103.8368201	762.5523013
DMU30	0.98136	0	411.7824268
DMU31	0.36962	469.5732218	1224.58159
DMU32	0.42698	0	368.7698745
DMU33	0.3267	678.1129707	1828.878661
DMU34	0.60554	418.4100418	1455.677824
DMU35	0.39326	31.12133891	1741.497908
DMU36	0.44906	0	4098.757322
DMU37	0.85616	0	0

References

[1] Charnes A, Cooper WW, Rhodes E. Measuring the efficiency of decision-making units. European Journal of Operational Research 1978;2(6):429–44.

[2] Rostamy-Malkhalifeh, M., Sanei, M., & Saleh, H. (2009). A New Method for Solving Fuzzy DEA Models by Trapeziodal Approximation. *Journal of Mathematical Extension*.

[3] Saleh, H. I. L. D. A., Hosseinzadeh Lotfi, F., Rostmay-Malkhalifeh, M., & Shafiee, M. (2021). Provide a mathematical model for selecting suppliers in the supply chain based on profit efficiency calculations. Journal of New Researches in Mathematics, 7(32), 177-186.

[4] Shafiee, M., Hosseinzade Lotfi, F., & Saleh, H. (2021). Benchmark forecasting in data envelopment analysis for decision making units. *International Journal of Industrial Mathematics*, *13*(1), 29-42.

[5] Shafiee, M., Saleh, H., & Ziyari, R. (2022). Projects efficiency evaluation by data envelopment analysis and balanced scorecard. *Journal of decisions and operations research*, *6*(Special Issue), 1-19.

[6] Jahanshahloo, G.R., Hadi Vencheh, A., Foroughi, A.A., & Kazemi Matin, R. (2004). Inputs/outputs estimation in DEA when some factors are undesirable, Applied Mathematics and Computation, 156, 19-32.

[7] Jahanshahloo, G.R., Hosseinzadeh Lotfi, F., Shoja, N., Tohidi, G., & Razavyan, S. (2005). Undesirable inputs and outputs in DEA models, Applied Mathematics and Computation, 169, 917-925.

[8] Amirteimoori, A., Kordrostami, S., & Sarparast, M. (2006). Modeling undesirable factors in data envelopment analysis. *Applied Mathematics and Computation*, *180*(2), 444-452.

[9] Chen, C. M., Du, J., Huo, J., & Zhu, J. (2012). Undesirable factors in integer-valued DEA: Evaluating the operational efficiencies of city bus systems considering safety records. *Decision Support Systems*, *54*(1), 330-335.

[10] Fare, R., Grosskopf, S., & pasurka, C. (2013). Joint production of Good and Bad Outputs with a Network Application. Encyclopedia of Energy, Natural Resource, and Environmental Economics, 2, 109-118.

[11] Kuosmanen, T. (2005). Weak disposability in nonparametric production analysis with undesirable outputs. *American journal of agricultural economics*, *87*(4), 1077-1082.

[12] Kuosmanen, T., & Podinovski, V. (2009). Weak disposability in nonparametric production analysis: reply to Färe and Grosskopf. *American journal of agricultural economics*, *91*(2), 539-545.

[13] Amirteimoori, A., Fouladvand, M., & Kordrostami, S. (2017). Efficiency measurement using nonparametric production analysis in the presence of undesirable outputs. Application to power plants. *Operations research and decisions*, *27*(3), 5-20.

[14] Toloo, M., & Hančlová, J. (2020). Multi-valued measures in DEA in the presence of undesirable outputs. *Omega*, *94*, 102041.

[15] Monzeli, A., Daneshian, B., Tohidi, G., Sanei, M., & Razavian, S. (2020). Efficiency study with undesirable inputs and outputs in DEA. Journal of fuzzy extension and applications, 1(1), 73-80.

[16] Streimikis, J., & Saraji, M. K. (2022). Green productivity and undesirable outputs in agriculture: A systematic review of DEA approach and policy recommendations. Economic research-Ekonomska istraživanja, 35(1), 819-853.

[17] Kordrostami, S., & Amirteimoori, A. (2005). Un-desirable factors in multi-component performance measurement. *Applied Mathematics and Computation*, *171*(2), 721-729.

[18] Lozano, S., Gutiérrez, E., & Moreno, P. (2013). Network DEA approach to

airports performance assessment considering undesirable outputs. *Applied Mathematical Modelling*, *37*(4), 1665-1676.

[19] Maghbouli, M., Amirteimoori, A., & Kordrostami, S. (2014). Two-stage network structures with undesirable outputs: A DEA based approach. *Measurement*, *48*, 109-118.

[20] Bian, Y., Liang, N., & Xu, H. (2015). Efficiency evaluation of Chinese regional industrial systems with undesirable factors using two-stage slacks-based measure approach. *Journal of Cleaner Production*, *87*, 348-356.

[21] Wu, J., Zhu, Q., Chu, J., & Liang, L. (2015). Two-stage network structures with undesirable intermediate outputs reused: a DEA based approach. *Computational Economics*, *46*, 455-477.

[22] Liu, W., Zhou, Z., Ma, C., Liu, D., & Shen, W. (2015). Two-stage DEA models with undesirable input-intermediate-outputs. *Omega*, *56*, 74-87.

[23] Wu, J., Zhu, Q., Ji, X., Chu, J., & Liang, L. (2016). Two-stage network processes with shared resources and resources recovered from undesirable outputs. *European journal of operational research*, *251*(1), 182-197.

[24] Kalhor, A., & KazemiMatin, R. (2018). Study the effects of abatement factors of weak disposability in network data envelopment analysis with undesirable outputs, Journal of operational reearch and its applications, 15, 103-121.

[25] Yu, S., Liu, J., & Li, L. (2020). Evaluating provincial eco-efficiency in China: an improved network data envelopment analysis model with undesirable output. Environmental Science and Pollution Research, 27, 6886-6903.

[26] Chen, Y. C., Chiu, Y. H., & Chiu, C. J. (2020). The performance evaluation of banks considering risk: an application of undesirable relation network DEA. International transactions in operational research, 27(2), 1101-1120.

[27] Shi, X., Emrouznejad, A., & Yu, W. (2021). Overall efficiency of operational process with undesirable outputs containing both series and parallel processes: A SBM network DEA model. *Expert Systems with Applications*, *178*, 115062.

[28] Amirteimoori, A., & Masrouri, S. (2021). DEA-based competition strategy in the presence of undesirable products: an application to paper mills. *Operations Research and Decisions*, *31*(2).

[29] Ma, H., Geng, B., Fu, Y., Sun, Y., & Sun, Z. (2022). Efficiency analysis of industrial water treatment in China based on two-stage undesirable fixed-sum output DEA model. *Journal of Systems Science and Information*, *9*(6), 660-680.

[30] Soofizadeh, S., & Fallahnejad, R. (2022). A bargaining game model for performance evaluation in network DEA considering shared inputs in the presence of undesirable outputs. Journal of Mathematical Modeling, 10(2), 227-245.

[31] Omrani, H., Oveysi, Z., Emrouznejad, A., & Teplova, T. (2023). A mixed-integer network DEA with shared inputs and undesirable outputs for performance evaluation: Efficiency measurement of bank branches. Journal of the Operational Research Society, 74(4), 1150-1165.

[32] Omrani, H., Yang, Z., Karbasian, A., & Teplova, T. (2023). Combination of top-down and bottom-up DEA models using PCA: A two-stage network DEA with shared input and undesirable output for evaluation of the road transport sector. Socio-Economic Planning Sciences, 89, 101706.

[33] Khoshandam, L., & Nematizadeh, M. (2024). An inverse network DEA model for two-stage processes in the presence of undesirable factors. Journal of Applied Research on Industrial Engineering, 11(2), 179-194.

[34] Färe, R., & Grosskopf, S. (2003). Nonparametric productivity analysis with undesirable outputs: comment. *American journal of agricultural economics*, *85*(4), 1070-1074.