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Outputs Estimation and Efficiency Improvement of Two-Stage Network Systems

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Abstract
Although data envelopment analysis (DEA) is a mathematic based technique to estimate the efficiency score of decision-making units (DMU)s with multiple inputs and outputs, inverse DEA (invDEA) is a method to estimate inputs/outputs of DMUs with respect to unchanged efficiency score. In the general point of view, DMUs are considered as black-box, however there are units, which have internal process, that are called two-stage network systems. This paper proposes method for outputs estimation and efficiency improvement of DMUs with two-stage network structure when inputs are revised. Finally, a case study is presented to show the differences between proposed methods.

Keywords: Inverse data envelopment analysis, two-stage network, unchanged efficiency score, output estimation, variable returns to scale.

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1. Introduction

Data envelopment analysis is a non-parametric technique based on mathematic programming in order to evaluate the efficiency score of DMUs with multiple inputs and multiple outputs, which proposed in [1]. They assess the efficiency score of units by CRS assumption. Then [2] did efficiency evaluation of DMUs by VRS assumption.

Inverse data envelopment analysis, which proposed by Wei et al. in [3]. They discussed such problems: if among a group of comparable DMUs we increase certain inputs to a particular DMU and assume that the DMU maintains its current efficiency level with respect to other units, how much more outputs could the unit produce? Then Yan et al. in [4] used invDEA concept to estimate inputs/outputs level of a DMU when some or all inputs/outputs entities of the unit are revised and its efficiency score stays unchanged, by preference cone constraints. Moreover in [5] the InvDEA concept was used and proposed a method to estimate outputs level of a DMU when some or all of its input's entities are revised and its current efficiency level improved. Then, Jahanshahloo et al. in [6] proposed an approach which introduced extra inputs when the outputs are estimated. Recently, many researches in InvDEA concept have been done. For example, in [7] the InvDEA and manipulating the output values of the DMU under evaluation were utilized and then the input values are estimated while ensuring constant or enhanced cost efficiency. Moreover, in [8] evaluated the sustainability of the electricity supply chain by the inverse output-oriented DEA model. In addition, a new method for considering accidents according to the environmental, traffic and geometrical conditions of the road, which considers accidents according to the interaction of the components that lead to them presented in [9].

DEA has vast applications in many fields, such as Network DEA (NDEA) concept which firstly proposed by Fare and Grosskopf [10]. Furthermore, Kao [11] proposed a relational DEA model taking into account the relationship of the process within the system, to measure the efficiency of the system and those of the processes at the same time. In addition, Kao and Hwang [12] modified the conventional DEA model by taking into account the series relationship of two sub-processes within the whole processes.

In another point of view, there are some papers which incorporate the concepts of invDEA and two-stage network DEA. These papers attempt to answer this question:

If among a group of comparable DMUs with two-stage network structure, we increase inputs/outputs of a particular DMU, how much outputs/inputs are provided in order to unchanged technical efficiency score. Shiri Daryani et al. [13] proposed a four-stage method, while the allocative efficiency scores of all the units remain stable. In this paper a method for outputs estimation and efficiency improvement of DMUs with two-stage network structure is proposed.

The rest of this paper is considered as follows:

In section 2, some basic concepts of invDEA, output estimation of a DMU in order to improvement of its efficiency score, two-stage network DEA and output estimation of a DMU with two-stage network structure are presented. Then in section 3, a model for output estimation of a two-stage network system in order to improvement of its efficiency is proposed. Finally, in section 4 a case study in banking industry presented to show the applicability of the proposed model.

2. Preliminary

2.1. Inverse DEA

Concept of InvDEA which was proposed by Wie et al. [3] is to answer this question:

if among a group of comparable DMUs, we increase the input entities of a particular DMU, which is called DMU_o from $x_{io}\left(i=1,\ldots,m\right)$ to $x_{io}+\Delta x_{io}=\alpha_{io}\left(i=1,\ldots,m\right)$, $\Delta x_{io}\geq 0$, $\Delta x_{io}\neq 0$ $\left(i=1,\ldots,m\right)$, how much more outputs $y_{ro}+\Delta y_{ro}=\beta_{ro}\left(r=1,\ldots,s\right)$, $\Delta y_{ro}\geq 0$ the unit produces in order to unchanged efficiency score $\left(\varphi_o^{BCC}\right)$? To answer this question from VRS point of view they proposed the following MOLP model:

$$\beta=\max\left\{\beta_1,\beta_2,\ldots,\beta_s\right\} \tag{1}$$

$$\begin{aligned} s.t.\quad & \sum_{j=1}^{n}\lambda_j x_{ij}\leq x_{io}+\Delta x_{io}=\alpha_{io},\quad i=1,\ldots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj}\geq\varphi_o^{BCC}\beta_r,\quad r=1,\ldots,s \\ & \sum_{j=1}^{n}\lambda_j=1 \\ & \beta_r\geq y_{ro},\quad r=1,\ldots,s \\ & \lambda_j\geq 0,\quad j=1,\ldots,n \end{aligned}$$

Since model (1) is an MOLP, assume $p_r\left(r=1,\ldots,s\right)$ is output value (weight – Price) of rth entity. To solve model (1) we can use the following single objective model:

$$\beta=\max\sum_{r=1}^{s}p_r\beta_r \tag{2}$$

$$\begin{aligned} s.t.\quad & \sum_{j=1}^{n}\lambda_j x_{ij}\leq x_{io}+\Delta x_{io}=\alpha_{io},\quad i=1,\ldots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj}\geq\varphi_o^{BCC}\beta_r,\quad r=1,\ldots,s \\ & \sum_{j=1}^{n}\lambda_j=1 \\ & \beta_r\geq y_{ro},\quad r=1,\ldots,s \\ & \lambda_j\geq 0,\quad j=1,\ldots,n \end{aligned}$$

2.2. Output estimation of a DMU in order to improvement of its efficiency

In order to output estimation of a DMU according to improvement of its efficiency, Jahanshahloo et al. [5] proposed a method with to answer this question: if among a group of DMUs, we increase certain inputs to a particular unit and assume that its current efficiency level with respect to other units is improved, say $\eta-$ percent of φ_o^{BCC}, how much more outputs could the unit produce?

In order to answer above question, they put $\varphi_o^{BCC}-\left(\frac{\eta}{100}\right)\varphi_o^{BCC}$ and $y_{ro}+\beta'_r\left(r=1,\ldots,s\right)$ instead of φ_o^{BCC} and β_r, respectively in model (5). Then it is converted to the following form:

$$\beta'^{BCC}=\max\left\{\beta'_1,\beta'_2,\ldots,\beta'_s\right\} \tag{3}$$

$$\begin{aligned} s.t.\quad & \sum_{j=1}^{n}\lambda_j x_{ij}\leq\alpha_{io},\quad i=1,\ldots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj}\geq\left(1-\eta/100\right)\varphi_o^{BCC}\left(y_{ro}+\beta'_r\right),\quad r=1,\ldots,s \\ & \sum_{j=1}^{n}\lambda_j=1 \\ & \beta'_r\geq 0,\quad r=1,\ldots,s \\ & \lambda_j\geq 0,\quad j=1,\ldots,n. \end{aligned}$$

2.3. Two – stage network DEA

In the basic two-stage network, where all the inputs $x_{ij}\left(i=1,\ldots,m\right)$ are supplied externally and are consumed by the first stage, to produce the intermediate products $z_{gj}\left(g=1,\ldots,h\right)$ and for the second stage to produce the final outputs $y_{rj}\left(r=1,\ldots,s\right)$ [9]. The first stage does not produce final outputs and the second stage does not consume exogenous inputs. The structure of the basic two-stage system is depicted in Figure 1.

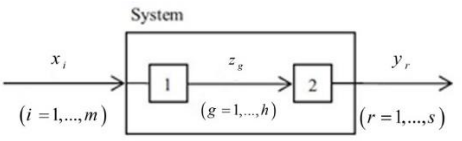

Figure 1. Structure of the basic two-stage network

There are some perspectives to evaluate the efficiency score of network systems [14]. Based on the structure of the basic two-stage system shown in Figure1, the input-oriented model proposed by Kao and Hwang [12] under constant returns to scale (CRS) in a multiplier form is:

$$E_o^{input} = \max \sum_{r=1}^{s} u_r y_{ro} \quad (4)$$
$$s.t. \sum_{i=1}^{m} v_i x_{io} = 1$$
system constraints:
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j=1,\dots,n$$
division constraints:
$$\sum_{g=1}^{h} w_g z_{gj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j=1,\dots,n$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{g=1}^{h} w_g z_{gj} \le 0, \quad j=1,\dots,n$$
$$u_r \ge 0, \quad r=1,\dots,s$$
$$v_i \ge 0, \quad i=1,\dots,m$$
$$w_g \ge 0, \quad g=1,\dots,h.$$

As for optimality, the system efficiency in the input-oriented form $\left(E_o^{input}\right)$ and the stage efficiencies ($E_o^{(1)}$ and $E_o^{(2)}$) are based on constrains of model (4) and can be expressed as:

$$E_o^{input} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad (5)$$
$$E_o^{(1)input} = \frac{\sum_{g=1}^{h} w_g^* z_{go}}{\sum_{i=1}^{m} v_i^* x_{io}} \quad \text{and} \quad E_o^{(2)input} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{g=1}^{h} w_g^* z_{go}}$$

The system efficiency is the product of the two stage efficiencies. The dual of model (4) is as follows:

$$E_o^{input} = \min \quad \theta_o \quad (6)$$
$$s.t. \sum_{j=1}^{n} \mu_j x_{ij} \le \theta_o x_{io}, \quad i=1,\dots,m$$
$$\sum_{j=1}^{n} \left(\mu_j - \lambda_j\right) z_{gj} \ge 0, \quad g=1,\dots,h$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r=1,\dots,s$$
$$\lambda_j \ge 0, \quad \mu_j \ge 0, \quad j=1,\dots,n.$$

The output-oriented version of model (4) is:

$$E_o^{output} = \min \sum_{r=1}^{s} v_i x_{io} \quad (7)$$
$$s.t. \sum_{r=1}^{s} u_r y_{ro} = 1$$
system constraints:
$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \ge 0, \quad j=1,\dots,n$$
division constraints:
$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{g=1}^{h} w_g z_{gj} \ge 0, \quad j=1,\dots,n$$
$$\sum_{g=1}^{h} w_g z_{gj} - \sum_{r=1}^{s} u_r y_{rj} \ge 0, \quad j=1,\dots,n$$
$$u_r \ge 0, \quad r=1,\dots,s$$
$$v_i \ge 0, \quad i=1,\dots,m$$
$$w_g \ge 0, \quad g=1,\dots,h.$$

The dual of model (7) is as follows:

$$
\begin{aligned}
E_o^{output} &= \max \quad \varphi_o \qquad (8)\\
s.t. \; &\sum_{j=1}^{n} \mu_j x_{ij} \le x_{io}, \quad i=1,\dots,m\\
&\sum_{j=1}^{n} \left(\mu_j - \lambda_j\right) z_{gj} \ge 0, \quad g=1,\dots,h\\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi_o y_{ro}, \quad r=1,\dots,s\\
&\lambda_j \ge 0, \quad \mu_j \ge 0, \quad j=1,\dots,n,
\end{aligned}
$$

which evaluates the output-oriented technical efficiency score of the two-stage network system in the constant returns to scale (CRS) point of view. To evaluate the variable returns to scale (VRS) efficiency score of the DMUs, we can add constraints $\sum_{j=1}^{n} \lambda_j = 1$ and $\sum_{j=1}^{n} \mu_j = 1$ into model (8).

2.4. Output estimation of a two-stage network system when inputs are revised

Shiri Daryani et al. [13] proposed the following model to answer this question: if among a group of comparable DMUs with two-stage network structure, we increase inputs of DMU_o from $x_{io}\left(i=1,\dots,m\right)$ to $x_{io} + \Delta x_{io} = \alpha_{io}\left(i=1,\dots,m\right)$, $\Delta x_{io} \ge 0$, $\Delta x_{io} \ne 0$ $\left(i=1,\dots,m\right)$, how much more outputs $y_{ro} + \Delta y_{ro} = \beta_{ro}\left(r=1,\dots,s\right)$ the unit would produce in order to unchanged efficiency score $\left(E_o^{output}\right)$?

$$
\begin{aligned}
\beta &= \max\left(\beta_1,\dots,\beta_s\right) \qquad (9)\\
s.t. \; &\sum_{j=1}^{n} \lambda_j x_{ij} \le \alpha_{io}, \quad i=1,\dots,m\\
&\sum_{j=1}^{n} \left(\mu_j - \lambda_j\right) z_{gj} \ge 0, \quad g=1,\dots,h\\
&\sum_{j=1}^{n} \mu_j y_{rj} \ge E_o^{output} \beta_{ro}, \quad r=1,\dots,s\\
&\sum_{j=1}^{n} \lambda_j = 1\\
&\sum_{j=1}^{n} \mu_j = 1\\
&\lambda_j \ge 0, \quad \mu_j \ge 0, \quad j=1,\dots,n,
\end{aligned}
$$

3. Output estimation of a two-stage network system in according to improvement of its efficiency

In this section, we integrate the proposed methods by Jahanshahloo et al. [5] and Shiri Daryani et al. [13] for output estimation of a DMU with two-stage network structure when inputs are revised and the efficiency scores of the whole system is improved. Assume that the inputs of DMU_o with two-stage network structure which is depicted in Figure1 is increased from $x_{io}\left(i=1,\dots,m\right)$ to $x_{io} + \Delta x_{io} = \alpha_{io}\left(i=1,\dots,m\right)$, $\Delta x_{io} \ge 0$, $\Delta x_{io} \ne 0$ $\left(i=1,\dots,m\right)$ and the decision maker wants to improve the efficiency score of each unit $\left(E_o^{output}\right)$ by -percent. Now we would like to estimate the produced outputs level of DMUs by the following model:

$$\beta' = \max\left\{\beta_1', \beta_2', ..., \beta_s'\right\}$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \alpha_{io}, \quad i = 1,...,m$$
$$\sum_{j=1}^{n} \left(\mu_j - \lambda_j\right) z_{gj} \ge 0, \quad g = 1,...,h$$
$$\sum_{j=1}^{n} \mu_j y_{rj} \ge \left(1 - \eta/100\right) E_o^{output}\left(y_{ro} + \beta_r'\right), \quad r = 1,...,s$$
$$\sum_{j=1}^{n} \lambda_j = 1 \qquad (10)$$
$$\sum_{j=1}^{n} \mu_j = 1$$
$$\beta_r' \ge 0, \quad r = 1,...,s$$
$$\lambda_j \ge 0, \; \mu_j \ge 0, \quad j = 1,...,n.$$

4. Case Study

In this section, we examine our proposed model for a data set of 27 banks from [15]. The data are given in Table 1. Fixed assets, IT budget and number of employees are three indicators which are considered as inputs (second, third and fourth column) to produce deposits as intermediate product (fifth column). Profit and fraction of loans recovered are final outputs (sixth and seventh column).

Table 1: Data set of Chen and Zhu (2004)

Banks	Fixed assets ($ billion) (x_1)	IT budget ($ billion) (x_2)	# of employees (thousand) (x_3)	Deposits ($ billion) (z_1)	Profit ($ billion) (y_1)	Fraction of loans recovered (y_2)
1	0.713	0.15	13.3	14.478	0.232	0.986
2	1.071	0.17	16.9	19.502	0.34	0.986
3	1.224	0.235	24	20.952	0.363	0.986
4	0.363	0.211	15.6	13.902	0.211	0.982
5	0.409	0.133	18.485	15.206	0.237	0.984
6	5.846	0.497	56.42	81.186	1.103	0.955
7	0.918	0.06	56.42	81.186	1.103	0.986
8	1.235	0.071	12	11.441	0.199	0.985
9	18.12	1.5	89.51	124.072	1.858	0.972
10	1.821	0.12	19.8	17.425	0.274	0.983
11	1.915	0.12	19.8	17.425	0.274	0.983
12	0.874	0.05	13.1	14.342	0.177	0.985
13	6.918	0.37	12.5	32.491	0.648	0.945
14	4.432	0.44	41.9	47.653	0.639	0.979
15	4.504	0.431	41.1	52.63	0.741	0.981
16	1.241	0.11	14.4	17.493	0.243	0.988
17	0.45	0.053	7.6	9.512	0.067	0.98
18	5.892	0.345	15.5	42.469	1.002	0.948
19	0.973	0.128	12.6	18.987	0.243	0.985
20	0.444	0.055	5.9	7.546	0.153	0.987
21	0.508	0.057	5.7	7.595	0.123	0.987
22	0.37	0.098	14.1	16.906	0.233	0.981
23	0.395	0.104	14.6	17.264	0.263	0.983
24	2.68	0.206	19.6	36.43	0.601	0.982
25	0.781	0.067	10.5	11.581	0.12	0.987

26	0.872	0.1	12.1	22.207	0.248	0.972
27	1.757	0.0106	12.7	20.67	0.253	0.988

Assume that $(\rho_1, \rho_2) = (1,1)$ are the output prices and the decision maker wants to increase all the inputs by %10 and improves the efficiencies of all the units by %25. First, we evaluate the efficiency score of all the DMUs by model (8) and in order to determine the outputs of all the DMUs, we solve model (14). The results are depicted in Table2 as follows:

Table 2: The VRS efficiency score and output increment of banks

Banks	E_o^{output}	β_1'	β_2'
1	1.00	1.21	0.30
2	1.01	1.14	0.30
3	1.01	1.13	0.30
4	1.01	1.20	0.30
5	1.01	1.19	0.30
6	1.03	0.70	0.30
7	1.00	0.74	0.31
8	1.01	1.21	0.30
9	1.00	0.33	0.32
10	1.01	1.05	0.30
11	1.01	1.05	0.30
12	1.01	1.14	0.30
13	1.04	0.63	0.30
14	1.01	0.69	0.31
15	1.00	0.59	0.32
16	1.01	1.08	0.30
17	1.02	1.25	0.30
18	1.00	0.33	0.35
19	1.01	1.08	0.30
20	1.01	1.17	0.30
21	1.01	1.20	0.30
22	1.01	1.08	0.30
23	1.01	1.06	0.30
24	1.00	0.73	0.31
25	1.01	1.20	0.30
26	1.02	1.06	0.30
27	1.00	1.08	0.31

Consider DMU13 with the highest technical efficiency score of 1.04. By 10 percent perturbation in its input levels and %25 of efficiency improvement, its first and second output levels will increase by 0.63 and 0.30 units, respectively.

5. Conclusion

In this paper the inverse DEA model was generalized for outputs estimation to the two-stage network systems when the efficiency levels of units are improved by the decision maker. The proposed models deal with inputs revision and estimate output levels of the unit under evaluation in order to improve VRS technical efficiency scores of the unit.

In the proposed MOLP model, the decision maker's preferences can be considered in inputs weights in the output estimation procedure. The proposed method was applied to an empirical example in the banking industry.

A stream of future research can extend our framework to other network DEA structures, two-stage structures with negative or imprecise data.

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