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Research Article

Language of Lattice-Valued General Orthomodular Automata

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Abstract

A new theory of computation based on quantum logic is being developed as a foundational framework for quantum computation. The idea of quantum computation emerged from exploring the relationship between physics and computation, with the initial focus on understanding the thermodynamics associated with classical computation. This research represents one of the initial efforts to explore this innovative theory. In this study, quantum logic is identified as an orthomodular lattice-valued logic. The primary objective is to establish a theory of general fuzzy automata grounded in this logic, employing a semantical analysis approach. As part of this work, concepts such as lattice-valued general orthomodular automata and regular languages have been introduced, along with new ideas related to orthomodular lattice-valued regular expressions. Additionally, the Kleene theorem has been characterized within the context of quantum logic, illustrating the equivalence between general fuzzy automata and regular expressions. Furthermore, an orthomodular lattice-valued variant of the pumping lemma has also been formulated.

Keywords: Quantum logic, Orthomodular lattice, General fuzzy automata, Kleene theorem, Pumping lemma.

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1. Introduction

The idea of quantum computation emerged from exploring the relationship between physics and computation, with the initial focus on understanding the thermodynamics associated with classical computation. In 1973, Bennett [1] proposed that a reversible Turing machine is theoretically feasible. Subsequently, in 1980, Benioff [2] introduced a quantum mechanical model of a Turing machine. Although this model represented the first quantum mechanical approach to computing, it was not recognized as a true quantum computer at that time. It is important to note that in Benioff's model, the machine may exist in a fundamentally quantum state during the intermediate steps of computation. However, at the conclusion of each computational step, the machine's tape always reverts to a classical state. Quantum computers were first envisioned by Richard Feynman [3,4]. In 1982, he [3] demonstrated that certain quantum phenomena could not be simulated by any classical Turing machine without incurring an exponential slowdown. Building upon his earlier findings, Feynman then realized that quantum mechanical effects had the potential to introduce a fundamentally new paradigm in computation. Although Feynman introduced the idea of a universal quantum simulator, he did not provide a specific design for such a device. David Deutsch [5] further refined and formalized earlier concepts in a landmark paper published in 1985, where he introduced the first authentic quantum Turing machine (QTM). This machine was notable because it allowed its tape to exist in quantum states, differentiating it from Benioff's earlier model. A key innovation from Deutsch was the introduction of quantum parallelism, which enables a quantum Turing machine to simultaneously process multiple inputs encoded on the same tape. He suggested that this capability could allow quantum computers to perform certain calculations much more efficiently than classical computers. In this study, we focus on developing a theory of general fuzzy automata that is based on quantum logic. This exploration aims to integrate quantum principles into automata theory, potentially enhancing our understanding and applications of computational models.

Specifically, we utilize techniques primarily derived from the semantic analysis approach outlined in references [6,7]. The author presents a significant insight regarding automata theory, highlighting that the universal validity of several fundamental properties, including Kleene's theorem and the equivalence of deterministic and nondeterministic automata, is closely tied to the distributivity of the underlying logic. It is established that these properties hold universally only if the set of truth values in the meta-logic that underpins automata theory forms a Boolean algebra. Consequently, this indicates that such properties do not apply universally in the context of quantum logic. This leads to a critical conclusion about the limitations of applying automata theory based on quantum logic. Additionally, it highlights a significant distinction between classical computation theory and the theory of computation based on quantum logic. Fortunately, we can show that the local validity of these automata properties can be achieved by imposing specific commutativity conditions on the truth values of the statements regarding the automata in question. Remarkably, it has been demonstrated that nearly all results from classical automata theory that are not valid in a non-distributive logic can be retrieved and achieved through specific commutativity in quantum logic. In this context, the equivalence of deterministic and nondeterministic automata serves as a prime example. Another intriguing example is the pumping lemma, which is not valid for the concept of non-commutative regularity but does hold for commutative regularity. As our theory of computation based on quantum logic has progressed, the successful applications of commutativity have led us to a new question: Why does commutativity play such a crucial role in quantum automata, and is there any physical interpretation for this phenomenon? To address these questions, it is important to note that all truth values in quantum logic are derived from an orthomodular lattice. The prototype of an orthomodular lattice is represented by the collection of linear subspaces of a Hilbert space, where set inclusion serves as the ordering relation.

Currently, several studies have been conducted on the advancement of fuzzy automata theory by researchers such as Zahedi, Abolpour, and Shamsizadeh, along with others in the field (see, for example, [8-13]).

Accordingly, this study is structured as follows: In Section 2, we introduce and elaborate on some fundamental concepts and results of quantum logic and lattice-valued general orthomodular automata. In Section 3, we introduce the concept of orthomodular lattice-valued regular expressions and generalize the Kleene theorem regarding the equivalence of regular expressions and L-valued generalized automata into the framework of quantum logic. Additionally, this section is dedicated to presenting a pumping lemma for orthomodular lattice-valued regular languages.

2. Preliminaries

In this section, the concept of lattice-valued general orthomodular automata is introduced. The understanding of related notions is therefore required in the subsequent sections.

Definition 2.1 [14] Let $\mathcal{L} = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice, and let $B \subseteq L$.

(1) If B is finite, then the commutator $\gamma(B)$ of B is defined by

$$\gamma(B) = \bigvee \left\{ \bigwedge_{b\in B} b^{f(b)} : \text{ f is a mapping from B into } \{1, -1\} \right\} = \bigvee_{b\in B} (b \wedge b^{\perp}),$$

where b^1 denotes b itself and b^{-1} denotes $b^{\perp}$.

(2) The strong commutator $\Gamma(B)$ of B is defined by

$$\Gamma(B) = \bigvee \{a: bCa \text{ for all } b \in B, \text{ and } (b_1 \wedge b)C(b_2 \wedge b) \text{ for all } b_1, b_2 \in B\}.$$

Proposition 2.2 [14] Let $\mathcal{L} = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice, and let $B \subseteq L$. Then for any $b \in B$ and $b_i \in B (i \in I)$,

$$\Gamma(B) \wedge \left(b \wedge \bigvee_{i\in I} b_i \right) \leq \bigvee_{i\in I} (b \wedge b_i),$$

$$\Gamma(B) \wedge \bigwedge_{i\in I} (b \vee b_i) \leq b \vee \bigwedge_{i\in I} b_i.$$

Lemma 2.3 [15] Let $\mathcal{L} = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice, and let $B \subseteq L$. Then for any $T \subseteq [B]$ we have

$$\Gamma(B) \leq \Gamma(T),$$

where [B] stands for the subalgebra of $\mathcal{L}$ generated by B.

Lemma 2.4 [15] Let $\mathcal{L} = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice. Then

(1) for any $x_i, y_i \in L (i = 1, \dots, n)$, let $U = \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$,

$$\Gamma(U) \wedge \bigwedge_{i=1}^{n} (x_i \rightarrow_3 y_i) \leq \bigwedge_{i=1}^{n} x_i \rightarrow_3 \bigwedge_{i=1}^{n} y_i,$$

$$\Gamma(U) \wedge \bigwedge_{i=1}^{n} (x_i \to_3 y_i) \le \bigvee_{i=1}^{n} x_i \to_3 \bigvee_{i=1}^{n} y_i,$$

(2) for any $x, y \in L$,

$$\Gamma(x,y) \wedge (x \to_3 y) \le y^{\perp} \to_3 x^{\perp}.$$

(3) for any $x, y, z \in L$,

$$\Gamma(x,y,z) \wedge (x \to_3 y) \wedge (y \to_3 z) \le x \to_3 z.$$

Definition 2.5 [8] Let $\mathcal{L} = (L, \le, \wedge, \vee, \perp, 0, 1)$ denote an orthomodular lattice. An $\mathcal{L}$-valued general orthomodular automaton, referred to as $\tilde{F}_{\mathcal{L}}$, , is defined as a 9-tuple machine represented by $\tilde{F}_{\mathcal{L}} = (Q, \Sigma, \tilde{R}_{\mathcal{L}}, \tilde{T}_{\mathcal{L}}, Z, \omega, \tilde{\delta}_{\mathcal{L}}, F_1, F_2)$, where:

the set of states, denoted as Q, is finite and can be expressed as $Q = \{p_1, p_2, \dots, p_n\}$;

the set of input symbols, represented as Σ, is finite and can be written as $\Sigma = \{x_1, x_2, \dots, x_m\}$;

Let $\tilde{R}_{\mathcal{L}}$ be an $\mathcal{L}$-valued subset of Q those functions as a mapping from Q to L. For every element $q \in Q$, the expression $\tilde{R}_{\mathcal{L}}(q)$ represents the truth value (within the framework of quantum logic) of the assertion that q is an initial state. We define it as follows:

$$[q \in \tilde{R}_{\mathcal{L}}] = \tilde{R}_{\mathcal{L}}(q) = 1;$$

$\tilde{T}_{\mathcal{L}}$ is also considered an $\mathcal{L}$-valued subset of Q. For every element $q \in Q$, $\tilde{T}_{\mathcal{L}}(q)$ indicates the truth value (according to our quantum logic) of the proposition that q represents a final state;

$\delta_{\mathcal{L}}$ is an $\mathcal{L}$-valued subset of the Cartesian product $Q \times \Sigma \times Q$, representing a mapping from $Q \times \Sigma \times Q$ to the set L. This is referred to as the $\mathcal{L}$-valued (quantum) transition relation of $\tilde{F}_{\mathcal{L}}$. Intuitively, $\delta_{\mathcal{L}}$ acts as an $\mathcal{L}$-valued (ternary) predicate over Q, Σ and Q. For any $p, q \in Q$ and $a_k \in \Sigma$, $\delta_{\mathcal{L}}(p, a_k, q)$ indicates the truth value (according to quantum logic) of the proposition that the input a_k enables a transition from state p to state q;

the notation $\tilde{\delta}_{\mathcal{L}}$ represents an $\mathcal{L}$-valued subset of the Cartesian product $(Q \times L) \times \Sigma \times Q$. It functions as a mapping from$(Q \times L) \times \Sigma \times Q$ to the set L. This is known as the $\mathcal{L}$-valued (quantum) augmented transition relation of $\tilde{F}_{\mathcal{L}}$;

the set Z is a finite collection of output labels (symbols), defined as $Z = \{y_1, y_2, \dots, y_k\}$;

$\omega: Q \to Z$ is the output function;

the function $F_1: L \times L \to L$ is utilized via $\tilde{\delta}_{\mathcal{L}}$ to assign truth values to the active states. Consequently, it is known as the truth assignment function;

the function $F_2: L^* \to L$ is a multi-truth resolution strategy designed to address multi-truth active states by assigning a single truth value to each. Therefore, it is referred to as the multi-truth resolution function.

Definition 2.6 [8] A derivation of an input string u ($u \in \Sigma^*$) indicated as $der_i(u)$, is an ordered set of states which are passed successively upon entrance of each symbol of the string, starting from an initial state. i is an arbitrary index usually starting from 1.

Given that $u = a_1 a_2 \dots a_k a_{k+1} \dots a_m$, we have:

$$der_i(u) = \{p_{i_0} p_{i_1} \dots p_{i_k} p_{i_{k+1}} \dots p_{i_m} | p_{i_0} \in \tilde{R}_{\mathcal{L}}, p_{i_m} \in \tilde{T}_{\mathcal{L}} \wedge p_{i_0} \xrightarrow{\delta_{\mathcal{L}}(p_{i_0}, a_1, p_{i_1})} p_{i_1} \dots$$

$$p_{i_k} \xrightarrow{\delta_{\mathcal{L}}(p_{i_k}, a_{k+1}, p_{i_{k+1}})} p_{i_{k+1}} \cdots p_{i_{m-1}} \xrightarrow{\delta_{\mathcal{L}}(p_{i_{m-1}}, a_m, p_{i_m})} p_{i_m},\ 0 \le k \le m\}$$

Definition 2.7 [8] Let $\tilde{F}_{\mathcal{L}} \in \mathcal{A}(Q, \Sigma, \mathcal{L})$. The $\mathcal{L}$-valued (unary) recognizability predicate $rec_{\tilde{F}_{\mathcal{L}}}$ on Σ^* is defined as $rec_{\tilde{F}_{\mathcal{L}}} \in L^{\Sigma^*}$, for every $u \in \Sigma^*$.

$$rec_{\tilde{F}_{\mathcal{L}}}(u) \overset{def}{=} (\exists der_i(x) \in D_{der}(u))(B(der_i(u)) \in \tilde{R}_{\mathcal{L}} \wedge E(der_i(u)) \in \tilde{T}_{\mathcal{L}} \wedge LB(der_i(u)) = u \wedge path_{\tilde{F}_{\mathcal{L}}}(der_i(u)))$$

In other words, the truth value of the proposition that u is recognizable by $\tilde{F}_{\mathcal{L}}$ is given by

$$[rec_{\tilde{F}_{\mathcal{L}}}(u)] = \vee\{\tilde{R}_{\mathcal{L}}(B(der_i(u))) \wedge \tilde{T}_{\mathcal{L}}(E(der_i(u)) \wedge [path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]: der_i(u) \in D_{der}(u)\}.$$

Definition 2.8 [8] The $\mathcal{L}$-valued (unary and partial) predicate $CReg_{\Sigma}$ on L^{Σ^*} is referred to as commutative regularity. It is defined as $CReg_{\Sigma} \in L^{(L^{\Sigma^*})}$: for any $B \in L^{\Sigma^*}$ with finite $Range(B) = \{B(u): u \in \Sigma^*\}$,

$$CReg_{\Sigma}(B) \overset{def}{=} (\exists \tilde{F}_{\mathcal{L}} \in \mathcal{A}(Q, \Sigma, \mathcal{L}))(\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}})) \cup r(B)) \wedge (B \equiv rec_{\tilde{F}_{\mathcal{L}}})),$$

where $r(B) = \{b: b \in Range(B)\}$, and b is the nullary predicate that corresponds to the element b in L.

3. Kleene's Theorem and Pumping Lemma

A key result in classical automata theory is the Kleene theorem, which demonstrates the equivalence between finite automata and regular expressions. This section aims to introduce a generalization of the Kleene theorem in the framework of orthomodular lattice-valued structures.

Definition 3.1 Let $\tilde{F}_{\mathcal{L}} = (Q, \Sigma, \tilde{R}_{\mathcal{L}}, \tilde{T}_{\mathcal{L}}, \omega, \tilde{\delta}_{\mathcal{L}}, F_1, F_2) \in \mathcal{A}(Q, \Sigma, \mathcal{L})$ be an $\mathcal{L}$-valued general orthomodular automaton. For any $p, q \in Q$ and $Q' \subseteq Q$, $l_{pq}^{Q'}$ is defined by induction on the cardinality $|Q'|$ of Q': (i)

$$l_{pq}^{\emptyset} = \begin{cases} \sum_{a_k \in \Sigma} \delta_{\mathcal{L}}(p, a_k, q) a_k & \text{if } p \ne q, \\ \varepsilon + \sum_{a_k \in \Sigma} \delta_{\mathcal{L}}(p, a_k, q) a_k & \text{if} p = q. \end{cases}$$

(ii) if $Q' \ne \emptyset$, then we choose any $r \in Q$ and define

$$l_{pq}^{Q'} = l_{pq}^{Q'-\{r\}} + l_{pr}^{Q'-\{r\}}.\left(l_{rr}^{Q'-\{r\}}\right)^*.l_{rq}^{Q'-\{r\}}.$$

Then the $\mathcal{L}$-valued regular expression

$$\mathcal{K}(\tilde{F}_{\mathcal{L}}) = \sum_{q_0, q \in Q} (\tilde{R}_{\mathcal{L}}(q_0) \wedge \tilde{T}_{\mathcal{L}}(q)) l_{q_0 q}^{Q}$$

is referred to as a Kleene representation of $\tilde{F}_{\mathcal{L}}$. The subsequent theorem effectively illustrates the connection between the language recognized by an L-valued Generalized Automaton (GoA) and the language represented by its Kleene representation.

Theorem 3.2 Let $\mathcal{L} = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice, and let $\rightarrow$ satisfy the Birkhoff-von Neumann condition.

(i) For any $\tilde{F}_{\mathcal{L}} \in \mathcal{A}(Q, \Sigma, \mathcal{L})$ and $u \in \Sigma^*$, if $\mathcal{K}(\tilde{F}_{\mathcal{L}})$ is a Kleene representation of $\tilde{F}_{\mathcal{L}}$, then

$$\overset{\mathcal{L}}{\vDash} \mathrm{rec}_{\tilde{F}_{\mathcal{L}}}(u) \rightarrow u \in L(\mathcal{K}(\tilde{F}_{\mathcal{L}})).$$

(ii) For any $\tilde{F}_{\mathcal{L}} \in \mathcal{A}(Q, \Sigma, \mathcal{L})$ and $u \in \Sigma^*$, and for any Kleene representation $\mathcal{K}(\tilde{F}_{\mathcal{L}})$ of $\tilde{F}_{\mathcal{L}}$, we have

$$\overset{\mathcal{L}}{\vDash} \gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}})) \wedge u \in L(\mathcal{K}(\tilde{F}_{\mathcal{L}})) \rightarrow \mathrm{rec}_{\tilde{F}_{\mathcal{L}}}(u),$$

and especially if $\rightarrow = \rightarrow_3$, then

$$\overset{\mathcal{L}}{\vDash} \gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}})) \rightarrow (\mathrm{rec}_{\tilde{F}_{\mathcal{L}}}(u) \leftrightarrow u \in L(\mathcal{K}(\tilde{F}_{\mathcal{L}}))).$$

(iii) The following two statements are equivalent:

(1) $\mathcal{L}$ is a Boolean algebra.

(2) For any $\tilde{F}_{\mathcal{L}} \in \mathcal{A}(Q, \Sigma, \mathcal{L})$ and $u \in \Sigma^*$, and for any Kleene representation $\mathcal{K}(\tilde{F}_{\mathcal{L}})$ of $\tilde{F}_{\mathcal{L}}$,

$$\overset{\mathcal{L}}{\vDash} \mathrm{rec}_{\tilde{F}_{\mathcal{L}}}(u) \leftrightarrow u \in L(\mathcal{K}(\tilde{F}_{\mathcal{L}})).$$

Proof. We prove (i) and (ii) together. Accordingly, we have to express that for any $p, q \in Q$, $Q' \subseteq Q$ and $u \in \Sigma^*$,

(a) $\bigvee\{[\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der}_i(u))]: \mathrm{der}_i(u) \in D_{\mathrm{der}}(u), B(\mathrm{der}_i(u)) = p, E(\mathrm{der}_i(u)) = q, \mathcal{M}(\mathrm{der}_i(u)) \subseteq Q', LB(\mathrm{der}_i(u)) = u\} \leq L(l_{pq}^{Q'}(u))$,

(b) $[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pq}^{Q'})(u)$

$$\leq \bigvee\{[\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der}_i(u))]: \mathrm{der}_i(u) \in D_{\mathrm{der}}(u), B(\mathrm{der}_i(u)) = p, E(\mathrm{der}_i(u)) = q,$$

$$\mathcal{M}(\mathrm{der}_i(u)) \subseteq Q', LB(\mathrm{der}_i(u)) = u\},$$

where $\mathcal{M}(\mathrm{der_i(u)})$ stands for the set of states along $\mathrm{der_i(u)}$ except p and q; more exactly, $\mathcal{M}(\mathrm{der_i(u)}) = \{q_1, q_2, \ldots, q_{k-1}\}$ if $\mathrm{der_i(u)} = pa_1q_1 \ldots q_{k-1}a_kq$. This claim may be established by induction on $|Q'|$.

For the case of $Q' = \emptyset$, it is easy. We now suppose that $r \in Q' \neq \emptyset$ and

$$l_{pq}^{Q'} = l_{pq}^{Q'-\{r\}} + [l_{pr}^{Q'-\{r\}}\left(l_{rr}^{Q'-\{r\}}\right)^*]l_{rq}^{Q'-\{r\}}.$$

We first assert that (a) is valid in this case. From the induction hypothesis we have

$$(c)\bigvee\{[\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der_i(u)})]: \mathrm{der_i(u)} \in D_{\mathrm{der}}(u), B(\mathrm{der_i(u)}) = E(\mathrm{der_i(u)}) = r$$
$$\mathcal{M}(\mathrm{der_i(x)}) \subseteq Q' - \{r\}, \mathrm{LB}(\mathrm{der_i(x)}) = u\} \leq L(l_{rr}^{Q'-\{r\}})(u)$$

for each $u \in \Sigma^*$. Then we emphasize that for all $u \in \Sigma^*$,

$$(d)\bigvee\{[\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der_i(u)})]: \mathrm{der_i(u)} \in D_{\mathrm{der}}(u), B(\mathrm{der_i(u)}) = E(\mathrm{der_i(u)}) = r,$$
$$\mathcal{M}(\mathrm{der_i(u)}) \subseteq Q', \mathrm{LB}(\mathrm{der_i(u)}) = u\} \leq L((l_{rr}^{Q'-\{r\}})^*)(u).$$

In fact, for any $\mathrm{der_i(u)} \in D_{\mathrm{der}}(u)$, if $B(\mathrm{der_i(u)}) = E(\mathrm{der_i(u)}) = r, \mathcal{M}(\mathrm{der_i(u)}) \subseteq Q'$ and $\mathrm{LB}(\mathrm{der_i(u)}) = u$, we write $\mathrm{der}(u_i)$ for the substring of $\mathrm{der_i(u)}$ begining with the ith r and ending at the $(i+1)$th r. If the number of occurrences of r in $\mathrm{der_i(u)}$ is $m+1$, then

$$[\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der_i(u)})] = \bigwedge_{i=1}^{m} [\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der}(u_i))].$$

Additionally, by employing (c) and indicating that $u = \mathrm{LB}(\mathrm{der}(u_1)) \ldots \mathrm{LB}(\mathrm{der}(u_m))$ we achieve $[\mathrm{path}_{\tilde{F}_{\mathcal{L}}}(\mathrm{der_i(u)})] = \bigwedge_{i=1}^{m} L(l_{rr}^{Q'-\{r\}})(\mathrm{LB}(\mathrm{der}(u_i)))$

$$\leq \bigvee\{\bigwedge_{i=1}^{n} L(l_{rr}^{Q'-\{r\}})(a_i): n \geq 0, a_1, a_2, \ldots, a_n \in \Sigma^*, u = a_1a_2 \ldots a_n\}$$
$$= (L(l_{rr}^{Q'-\{r\}}))^*(u)$$
$$= L((l_{rr}^{Q'-\{r\}})^*)(u).$$

Let $\mathrm{der_i(u)}$ range over $\{\mathrm{der_i(u)} \in D_{\mathrm{der}}(u): B(\mathrm{der_i(u)}) = E(\mathrm{der_i(u)}) = r, \mathcal{M}(\mathrm{der_i(u)}) \subseteq Q', \mathrm{LB}(\mathrm{der_i(u)}) = u\}$. Then (d) is proved.

Moreover, from the induction hypothesis and (d) have

$$([L(l_{pr}^{Q'-\{r\}})L((l_{rr}^{Q'-\{r\}})^*)]L(l_{rq}^{Q'-\{r\}}))(u)$$
$$= \bigvee\{[L(l_{pr}^{Q'-\{r\}})L((l_{rr}^{Q'-\{r\}})^*)](\alpha) \wedge L(l_{rq}^{Q'-\{r\}})(\beta): u = \alpha\beta\}$$

$= \vee\{\vee\{L(l_{pr}^{Q'-\{r\}})(b_1) \wedge L((l_{rr}^{Q'-\{r\}})^*)(b_2): \alpha = b_1 b_2\} \wedge L(l_{rq}^{Q'-\{r\}})(\beta): u = \alpha\beta\}$

$\geq \vee\{L(l_{rq}^{Q'-\{r\}})(b_1) \wedge L((l_{rr}^{Q'-\{r\}})^*)(b_2) \wedge L(l_{rq}^{Q'-\{r\}})(\beta): u = b_1 b_2 \beta\}$

$\geq \vee\{[path_{\tilde{F}_{\mathcal{L}}}(der(u_1))] \wedge [path_{\tilde{F}_{\mathcal{L}}}(der(u_2))] \wedge [path_{\tilde{F}_{\mathcal{L}}}(der(u_3))]:$

$der(u_1), der(u_2), der(u_3) \in D_{der}(u), B(der(u_1)) = p,$

$E(der(u_1)) = B(der(u_2)) = E(der(u_2)) = B(der(u_3)) = r,$

$E(der(u_3)) = q, u = LB(der(u_1))LB(der(u_2))LB(der(u_3))\}$

$= \vee\{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]: der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q, r \in$

$\mathcal{M}(der_i(u))\}.$

This yields further

$$L(l_{pq}^{Q'})(u) = L(l_{pq}^{Q'-\{r\}})(u) \vee ([L(l_{pr}^{Q'-\{r\}}) L((l_{rr}^{Q'-\{r\}})^*)]) L(l_{rq}^{Q'-\{r\}}))(u)$$

$$\geq \{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]: der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q,$$

$$\mathcal{M}(der_i(u)) \subseteq Q', LB(der_i(u)) = u\}.$$

We now turn to consider (b). The induction hypothesis gives

(e) $[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pq}^{Q'-\{r\}})(u)$

$\leq \vee\{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]: der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q,$

$\mathcal{M}(der_i(u)) \subseteq Q' - \{r\}, LB(der_i(u)) = u\}.$

For any $n \geq 0$ and $a_1, a_2, \ldots, a_n \in \Sigma^*$ with $u = a_1 a_2 \ldots a_n$, from (e) we can use Proposition 2.2 and Lemma 2.3 to get

$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigwedge_{i=1}^{n} L(l_{rr}^{Q'-\{r\}})(a_i)$

$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigwedge_{i=1}^{n} [[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{rr}^{Q'-\{r\}})(a_i)]$

$\leq [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigwedge_{i=1}^{n} \vee\{[path_{\tilde{F}_{\mathcal{L}}}(der(u_i))]: der(u_i) \in D_{der}(u), B(der(u_i)) = E(der(u_i)) = r,$

$\mathcal{M}(der(u_i)) \subseteq Q' - \{r\}, LB(der(u_i)) = a_i\}$

$\leq \vee\{\wedge [path_{\tilde{F}_{\mathcal{L}}}(der(u_i))]: der(u_i) \in D_{der}(u), B(der(u_i)) = E(der(u_i)) = r,$

$\mathcal{M}(der(u_i)) \subseteq Q' - \{r\}, LB(der(u_i)) = a_i$ foreach $i = 1, \ldots, n\}$

$\leq \vee\{[path_{\tilde{F}_{\mathcal{L}}}(\overline{der(u_1) der(u_2) \ldots der(u_n)})]: der(u_i) \in D_{der}(u), B(der(u_i)) =$

$E(der(u_i)) = r,$

$\mathcal{M}(\text{der}(u_i)) \subseteq Q' - \{r\}, \text{LB}(\text{der}(u_i)) = a_i$ foreach $i = 1, \ldots, n\}$

where $\overline{\text{der}(u_1)\text{der}(u_2) \ldots \text{der}(u_n)} = \text{der}(u_1)\text{der}(u'_2) \ldots \text{der}(u'_n)$, $\text{der}(u'_i)$ is the resulting string after removing the first r in $\text{der}(u_i)$ for each $i \geq 2$. Note that $\text{LB}(\overline{\text{der}(u_1)\text{der}(u_2) \ldots \text{der}(u_n)}) = a_1 a_2 \ldots a_n = u$ whenever $\text{LB}(\text{der}(u_i)) = a_i (i = 1,2, \ldots, n)$. We write

$$t = \vee\{[\text{path}_{\tilde{F}_{\mathcal{L}}}(\text{der}_i(u))] : \text{der}_i(u) \in D_{\text{der}}(u), B(\text{der}_i(u)) = E(\text{der}_i(u)) = r,$$
$$\mathcal{M}(\text{der}_i(u)) \subseteq Q', \text{LB}(\text{der}_i(u)) = u\}.$$

Then it holds that

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigwedge_{i=1}^{n} L(l_{rr}^{Q\prime - \{r\}})(a_i)] \leq t.$$

Furthermore, remind that $[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))], L(l_{rr}^{Q\prime - \{r\}})(a_i) \in [\mathcal{P}(\tilde{F}_{\mathcal{L}})]$. It follows that

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L((l_{rr}^{Q\prime - \{r\}})^*)(u)$$
$$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \vee\{\bigwedge_{i=1}^{n} L(l_{rr}^{Q\prime - \{r\}})(a_i) : n \geq 0, u = a_1 a_2 \ldots a_n\}$$
$$\leq \{[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigwedge_{i=1}^{n} L(l_{rr}^{Q\prime - \{r\}})(a_i) : n \geq 0, u = a_1 a_2 \ldots a_n\} \leq t.$$

This allows us to obtain

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge [L(l_{pr}^{Q\prime - \{r\}})L((l_{rr}^{Q\prime - \{r\}})^*)](\alpha)$$
$$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \vee\{L(l_{pr}^{Q\prime - \{r\}})(b_1) \wedge L((l_{rr}^{Q\prime - \{r\}})^*)(b_2) : \alpha = b_1 b_2\}$$
$$\leq \vee\{[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pr}^{Q\prime - \{r\}})(b_1) \wedge L((l_{rr}^{Q\prime - \{r\}})^*)(b_2) : \alpha = b_1 b_2\}$$
$$= \vee\{[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge [[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pr}^{Q\prime - \{r\}})(b_1)] \wedge\wedge [[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge$$
$$L((l_{rr}^{Q\prime - \{r\}})^*)(b_2)] : \alpha = b_1 b_2\}$$
$$\leq \vee\{[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge [\vee\{[\text{path}_{\tilde{F}_{\mathcal{L}}}(\text{der}(u_1))] : \text{der}(u_1) \in D_{\text{der}}(u), B(\text{der}(u_1)) = p,$$
$$E(\text{der}(u_1)) = r, \mathcal{M}(\text{der}(u_1)) \subseteq Q' - \{r\}, \text{LB}(\text{der}(u_1)) = b_1\}]$$
$$\wedge [\vee\{[\text{path}_{\tilde{F}_{\mathcal{L}}}(\text{der}(u_2))] : \text{der}(u_2) \in D_{\text{der}}(u), B(\text{der}(u_2)) = E(\text{der}(u_2)) = r,$$
$$\mathcal{M}(\text{der}(u_2)) \subseteq Q', \text{LB}(\text{der}(u_2)) = b_2\}] : \alpha = b_1 b_2\}$$
$$\leq \vee\{[\text{path}_{\tilde{F}_{\mathcal{L}}}(\text{der}(u_1))] \wedge [\text{path}_{\tilde{F}_{\mathcal{L}}}(\text{der}(u_2))] : \text{der}(u_1), \text{der}(u_2) \in D_{\text{der}}(u), B(\text{der}(u_1)) = p,$$
$$E(\text{der}(u_1)) = B(\text{der}(u_2)) = E(\text{der}(u_2)) = r, \mathcal{M}(\text{der}(u_1)) \subseteq Q' - \{r\},$$
$$\mathcal{M}(\text{der}(u_2)) \subseteq Q', \alpha = \text{LB}(\text{der}(u_1)) = \text{LB}(\text{der}(u_2))\}.$$

In addition, in a similar way, we can obtain

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge ([L(l_{pr}^{Q'-\{r\}})L((l_{rr}^{Q'-\{r\}})^*)]L(l_{rq}^{Q'-\{r\}}))(u)$$
$$\leq \bigvee\{[path_{\tilde{F}_{\mathcal{L}}}(der(u_1))] \wedge [path_{\tilde{F}_{\mathcal{L}}}(der(u_2))] \wedge [path_{\tilde{F}_{\mathcal{L}}}(der(u_3))]:$$
$$der(u_1), der(u_2), der(u_3) \in D_{der}(u), B(der(u_1)) = p,$$
$$E(der(u_1)) = B(der(u_2)) = E(der(u_2)) = B(der(u_3)) = r,$$
$$E(der(u_3)) = q, u = LB(der(u_1))LB(der(u_2))LB(der(u_3))\}$$
$$= \bigvee\{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]: der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q,$$
$$r \in \mathcal{M}(der_i(u)), u = LB(der_i(u))\}.$$

As a result, it holds that

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pq}^{Q'})(u) = [(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \{L(l_{pq}^{Q'-\{r\}})(u)$$
$$\vee([L(l_{pr}^{Q'-\{r\}})L((l_{rr}^{Q'-\{r\}})^*)]L(l_{rq}^{Q'-\{r\}}))(u)\}$$
$$\leq [[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pq}^{Q'-\{r\}})(u)] \vee \{[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge ([L(l_{pr}^{Q'-\{r\}})L((l_{rr}^{Q'-\{r\}})^*)]L(l_{rq}^{Q'-\{r\}}))(u)\}$$
$$\leq \bigvee\{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]: der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q,$$
$$\mathcal{M}(der_i(u)) \subseteq Q', LB(der_i(u)) = u\}.$$

After proving (a), we can declare that

$$[u \in L(\mathcal{K}(\tilde{F}_{\mathcal{L}}))] = \bigvee_{p,q\in Q}[\tilde{R}_{\mathcal{L}}(p) \wedge \tilde{T}_{\mathcal{L}}(q) \wedge L(l_{pq}^{Q})(u)]$$
$$\geq \bigvee[(\tilde{R}_{\mathcal{L}}(p) \wedge \tilde{T}_{\mathcal{L}}(q)) \wedge \bigvee\{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]:$$
$$der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q, LB(der_i(u)) = u\}]$$
$$\geq \bigvee\bigvee\{\tilde{R}_{\mathcal{L}}(p) \wedge \tilde{T}_{\mathcal{L}}(q) \wedge path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]:$$
$$der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q, LB(der_i(u)) = u\}$$
$$= [rec_{\tilde{F}_{\mathcal{L}}}(u)].$$

By utilizing (b) and Proposition 2.2 and Lemma 2.3, we have

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge [u \in L(\mathcal{K}(\tilde{F}_{\mathcal{L}}))]$$
$$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigvee_{p,q\in Q}[\tilde{R}_{\mathcal{L}}(p) \wedge \tilde{T}_{\mathcal{L}}(q) \wedge L(l_{pq}^{Q})(u)]$$
$$\leq \bigvee_{p,q\in Q}[\tilde{R}_{\mathcal{L}}(p) \wedge \tilde{T}_{\mathcal{L}}(q) \wedge [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge L(l_{pq}^{Q})(u)]$$
$$\leq \bigvee_{p,q\in Q}[(\tilde{R}_{\mathcal{L}}(p)) \wedge \tilde{T}_{\mathcal{L}}(q)) \wedge [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}))] \wedge \bigvee\{[path_{\tilde{F}_{\mathcal{L}}}(der_i(u))]:$$
$$der_i(u) \in D_{der}(u), B(der_i(u)) = p, E(der_i(u)) = q, LB(der_i(u)) = u\}]$$

$$\leq \bigvee_{p,q\in Q} \vee\{\widetilde{R}_{\mathcal{L}}(p) \wedge \widetilde{T}_{\mathcal{L}}(q) \wedge [\mathrm{path}_{\widetilde{F}_{\mathcal{L}}}(\mathrm{der_i}(u))]:$$

$$\mathrm{der_i}(u) \in D_{\mathrm{der}}(u), B(\mathrm{der_i}(u)) = p, E(\mathrm{der_i}(u)) = q, LB(\mathrm{der_i}(u)) = u\}$$

$$= [\mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}(u)].$$

Therefore, (i) and (ii) are proved, and the part (1) implies (2) of (iii) is a simple corollary of (ii), and (2) to (1) is clear.

Theorem 3.3 (The pumping lemma) Let $\mathcal{L} = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice, and let $\rightarrow = \rightarrow_3$. For any $B \in L^{\Sigma^*}$, if the Range(B) is finite, then

$$\overset{\mathcal{L}}{\vDash} \mathrm{CReg}_{\Sigma}(B) \rightarrow (\exists s \geq 0)(\forall u \in \Sigma^*)[u \in B \wedge |u| \geq s \rightarrow (\exists x, y, z \in \Sigma^*)$$

$$(u = xyz \wedge |xy| \leq s \wedge |y| \geq 1 \wedge (\forall i \geq 0)(xy^i z) \in B))]$$

where for any word $\alpha = x_1 x_2 \dots x_k \in \Sigma^*$, $|\alpha|$ stands for the length k of α.

Proof. For simplicity, we employ $T(u, s)$ to mean the statement that $x, y, z \in \Sigma^*$, $u = xyz$, $|xy| \leq s$ and $|y| \geq 1$ for each $u \in \Sigma^*$ and $s \geq 0$. Then we only need to show that

$$[\mathrm{CReg}_{\Sigma}(B)] \leq \bigvee_{s\geq 0, u\in\Sigma^*} \bigwedge_{|u|\geq s} (B(u) \rightarrow \bigvee_{T(u,s)} \bigwedge_{i\geq 0} B(xy^i z)).$$

From Definition 2.8 we hold that

$$[\mathrm{CReg}_{\Sigma}(B)] = \bigvee_{\widetilde{F}_{\mathcal{L}}\in\mathcal{A}(Q,\Sigma,\mathcal{L})} ([\gamma(\mathcal{P}(\widetilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [B \equiv \mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}]).$$

Consequently, it suffices to prove that for any $\widetilde{F}_{\mathcal{L}} \in \mathcal{A}(Q, \Sigma, \mathcal{L})$,

$$[\gamma(\mathcal{P}(\widetilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [B \equiv \mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}] \leq \bigvee_{s\geq 0, u\in\Sigma^*} \bigwedge_{|u|\geq s} (B(u) \rightarrow \bigvee_{T(u,s)} \bigwedge_{i\geq 0} B(xy^i z)).$$

Let Q be the set of states of $\widetilde{F}_{\mathcal{L}}$. First, it holds that for any $u \in \Sigma^*$ with $|u| \geq |Q|$,

$$\mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}(u) \leq \bigvee_{T(u,s)} \bigwedge_{i\geq 0} \mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}(xy^i z). \qquad (1)$$

Indeed, assume that $u = a_1 a_2 \dots a_k$. Then

$$\mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}(u) = \vee[\widetilde{R}_{\mathcal{L}}(p_0) \wedge \widetilde{T}_{\mathcal{L}}(p_k) \wedge \bigwedge_{i=0}^{k-1} \delta_{\mathcal{L}}(p_i, a_{i+1}, p_{i+1})]. \qquad (2)$$

Thus, it suffices to demonstrate that for any $p_0, p_1, \dots, p_k \in Q$,

$$\widetilde{R}_{\mathcal{L}}(p_0) \wedge \widetilde{T}_{\mathcal{L}}(p_k) \wedge \bigwedge_{i=0}^{k-1} \delta_{\mathcal{L}}(p_i, a_{i+1}, p_{i+1}) \leq \bigvee_{T(u,s)} \bigwedge_{i\geq 0} \mathrm{rec}_{\widetilde{F}_{\mathcal{L}}}(xy^i z). \quad (3)$$

Given that $k = |u| \geq |Q|$, There are two identical states present among $p_0, p_1, \dots, p_{|Q|}$; in other words, there are $t \geq 0$ and $s > 0$ such that $t + s \leq |Q|$ and $p_t = p_{t+s}$. We set $x_0 = a_1 \dots a_t, y_0 = a_{t+1} \dots a_s$, and $z_0 = a_{t+s+1} \dots a_k$. Then $u = x_0 y_0 z_0$, $|z_0 y_0| = t + s \leq |Q|, |y| = s \geq 1$, and

$$\bigvee_{T(u,s)} \bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(xy^i z) \geq \bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(x_0 y_0^i z_0). \qquad (4)$$

Based on the definition of $rec_{\tilde{F}_{\mathcal{L}}}$, it is clear that for all $i \geq 0$,

$$rec_{\tilde{F}_{\mathcal{L}}}(x_0 y_0^i z_0) \geq [path_{\tilde{F}_{\mathcal{L}}}(p_0 a_1 p_1 \dots a_t p_t \times (a_{t+1} p_{t+1} \dots a_{t+s} p_{t+s})^i a_{t+s+1} q_{t+s+1} \dots a_k p_k)]$$
$$= \tilde{R}_{\mathcal{L}}(p_0) \wedge \tilde{T}_{\mathcal{L}}(p_k) \wedge \bigwedge_{j=0}^{t+s-1} \delta_{\mathcal{L}}(p_j, a_{j+1}, p_{j+1}) \wedge \bigwedge_{t=1}^{i-1} [\delta_{\mathcal{L}}(q_{t+s}, a_{t+1}, q_{t+1})$$
$$\wedge \bigwedge_{j=t+1}^{t+s-1} \delta_{\mathcal{L}}(p_j, a_{j+1}, p_{j+1})] \wedge \bigwedge_{j=t+s}^{k-1} \delta_{\mathcal{L}}(p_j, a_{j+1}, p_{j+1})$$
$$= \tilde{R}_{\mathcal{L}}(p_0) \wedge \tilde{T}_{\mathcal{L}}(p_k) \wedge \bigwedge_{j=0}^{k-1} \delta_{\mathcal{L}}(p_j, a_{j+1}, p_{j+1}) \qquad (5)$$

Since $p_{t+s} = p_t$ and $\delta_{\mathcal{L}}(p_{t+s}, a_{t+1}, p_{t+1}) = \delta_{\mathcal{L}}(p_t, a_{t+1}, p_{t+1})$. Consequently, by combining (4) and (5), we attain (3) which, together (2) yields (1).

Now, we apply Lemma 2.4 (1) and (3) and obtain

$$\bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(xy^i z) \to \bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} (xy^i z)$$
$$\geq [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge \bigwedge_{T(u,|Q|)} (\bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(xy^i z) \to \bigwedge_{i\geq 0} B(xy^i z))$$
$$\geq [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge \bigwedge_{T(u,|Q|)} \bigwedge_{i\geq 0} (rec_{\tilde{F}_{\mathcal{L}}}(xy^i z) \to \bigwedge_{i\geq 0} B(zy^i z))$$
$$\geq [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge \bigwedge_{u\in\Sigma^*)} (rec_{\tilde{F}_{\mathcal{L}}}(u) \to B(\beta))$$
$$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [rec_{\tilde{F}_{\mathcal{L}}} \subseteq B].$$

Additionally, from the above inequality, we have

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [rec_{\tilde{F}_{\mathcal{L}}} \equiv B]$$
$$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [A \subseteq rec_{\tilde{F}_{\mathcal{L}}}] \wedge [rec_{\tilde{F}_{\mathcal{L}}} \subseteq B]$$
$$= [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge \bigwedge_{u\in\Sigma^*} (B(u) \to rec_{\tilde{F}_{\mathcal{L}}}(u)) \wedge [rec_{\tilde{F}_{\mathcal{L}}} \subseteq B]$$
$$\leq [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge \bigwedge_{u\in\Sigma^*, |u|\geq|Q|} (B(u) \to rec_{\tilde{F}_{\mathcal{L}}}(u)) \wedge [rec_{\tilde{F}_{\mathcal{L}}} \subseteq B]$$
$$= \bigwedge_{u\in\Sigma^*, |u|\geq|Q|} ([\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge (B(u) \to rec_{\tilde{F}_{\mathcal{L}}}(u)) \wedge [\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [rec_{\tilde{F}_{\mathcal{L}}} \subseteq B]$$
$$\leq \bigwedge_{u\in\Sigma^*, |u|\geq|Q|} ([\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge (B(u) \to rec_{\tilde{F}_{\mathcal{L}}}(u))$$
$$\wedge (\bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(xy^i z) \to \bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} B(xy^i z)).$$

From (1), it can be concluded that

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [rec_{\tilde{F}_{\mathcal{L}}} \equiv B]$$
$$\leq \bigwedge_{u\in\Sigma^*,|u|\geq|Q|} ([\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge (B(u) \rightarrow \bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(xy^i z))$$
$$\wedge (\bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} rec_{\tilde{F}_{\mathcal{L}}}(xy^i z) \rightarrow \bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} A(xy^i z))).$$

By applying Lemma 2.4 (1) and (3) we hold that

$$[\gamma(\mathcal{P}(\tilde{F}_{\mathcal{L}}) \cup r(B))] \wedge [rec_{\tilde{F}_{\mathcal{L}}} \equiv B]$$
$$\leq \bigwedge_{u\in\Sigma^*,|u|\geq|Q|} (B(u) \rightarrow \bigvee_{T(u,|Q|)} \bigwedge_{i\geq 0} B(xy^i z))$$
$$\leq \bigvee_{u\in\Sigma^*,s\geq 0} \bigwedge_{|u|\geq s} (B(u) \rightarrow \bigvee_{T(u,s)} \bigwedge_{i\geq 0} B(xy^i z)),$$

and therefore this completes the proof.

4. Conclusions

In this research, we have generalized the Kleene theorem regarding the equivalence of regular expressions and general fuzzy automata within the context of quantum logic. Additionally, we have proposed a pumping lemma specifically for orthomodular lattice-valued regular languages. To advance the theory of computation based on quantum logic, our focus has also included examining the behavior of various computational models, such as pushdown automata and Turing machines, through the lens of quantum logic. One of the most compelling aspects of this work is its connection to different mathematical models of quantum computation. Consequently, our theory of computation grounded in quantum logic has been developed using an algebraic semantic approach. As a suggestion for future research, establishing a theory of computation that employs Kripke semantics for quantum logic could be beneficial, allowing for comparisons with the current study's framework.

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