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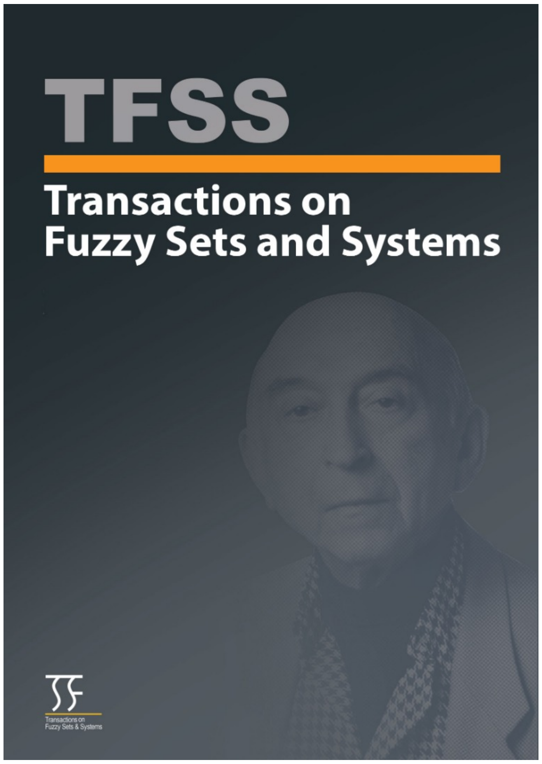


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# Nonlinear Contraction Mappings in b-metric Space and Related Fixed Point Results with Application

**Muhammed Raji, Arvind Kumar Rajpoot, Laxmi Rathour*, Lakshmi Narayan Mishra, Vishnu Narayan Mishra**

**Abstract.** The paper aims to introduce some fixed point results in the setting of sequential compact b-metric spaces to prove Eldeisten-Suzuki-type contraction for self-mappings. These contributions extend the existing literature on fixed point for ordered metric spaces and fixed point theory. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

**AMS Subject Classification 2020:** 47H10; 54H25
**Keywords and Phrases:** Fixed point, Coincidence point, Eldeisten-Suzuki-type contraction, b-metric space.

## 1 Introduction

The Banach fixed point theorem, originally proved by Stefan Banach [1] in 1922, is one of the most foundational and influential results in the field of fixed point theory. It states that a contractive mapping on a complete metric space has a unique fixed point. Since its introduction, the theorem has been generalized and extended in many ways, with applications in a variety of scientific disciplines. In particular, generalized metric spaces have been shown to be an extremely useful tool for studying fixed points in Banach spaces. The Banach contraction principle has been the subject of much research in recent years, with many different extensions and generalizations being explored. For example, Ran and Reurings [2] considered the existence of fixed points for mappings in partially ordered metric spaces, while Nieto and Lopez [3] extended this result to non-decreasing mappings. Another notable result is that of solving partial differential equations with periodic boundary conditions. Since its introduction, the Banach contraction mapping principle has been generalized and refined in numerous ways, leading to a wealth of articles dedicated to its improvement [4, 5, 6, 7, 8, 9, 10]. Czerwik's introduction of b-metric spaces [11] was a significant development in the field of generalized metric spaces. He weakened the triangle inequality in a metric space, which led to a generalized form of the Banach contraction principle. Building on this work, Boriceanu [12] provided concrete examples of b-metric spaces and investigated the fixed-point properties of set-valued operators in these spaces. Furthermore, Hussain et al. [13] introduced a new type of generalized metric space known as a dislocated b-metric space, which further extends the possibilities of the Banach contraction principle.
In 1962, mathematician Martin Edelstein [14] proved a generalization of the Banach contraction principle, a

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fundamental result in fixed point theory. Edelstein's generalization is sometimes referred to as the Edelstein fixed point theorem. It states that if a generalized metric space satisfies certain conditions, then a mapping that is contractive in the generalized metric has a unique fixed point. Motivated by the work of Banach and Edelstein, mathematician Suzuki [15] proved a refinement of their fixed point theorems, known as Suzuki's fixed point theorem. This result states that if a metric space satisfies certain additional conditions, a contractive mapping on the space has a unique fixed point. Many authors have proposed variants of Suzuki's theorem, such as those in [16, 17, 18, 19].
Based on the above insight, we present some fixed point results in the setting of sequential compact b-metric spaces to prove Eldeisten-Suzuki-type contraction for self-mappings and apply our main results to establish the existence of fixed point for ordered metric spaces. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

The common notations and terminology used in nonlinear analysis are utilized throughout this work.

## 2 Preliminaries

We begin this section by outlining a few fundamental definitions.

**Definition 2.1.** [20] Assume that $d : X \times X \to [0, +\infty)$ and $X$ are non-empty sets. $(X, d)$ is a symmetric space (also known as an $E$-space) if and only if it meets the requirements listed below:
i. $d(x, y) = 0$ if and only if $x = y$;
ii. $d(x, y) = d(y, x)$ for all $x, y \in X$.

**Remark 2.2.** [20] In the absence of triangle inequality, symmetric spaces are different from more practical metric spaces. However, a lot of concepts have definitions that are comparable to those in metric spaces.

**Definition 2.3.** [20] A sequence $\{x_n\}$ has a limit point in a symmetric space $(X, d)$ defined by $\lim_{n\to+\infty} d(x_n, x) = 0$ if and only if $\lim_{n\to+\infty} x_n = x$.

**Definition 2.4.** [20] If, for every given $\epsilon > 0$, there exists a positive integer $n(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ for all $m, n > n(\epsilon)$, then a sequence $\{x_n\} \subset X$ is a Cauchy sequence.

**Definition 2.5.** [20] If every Cauchy sequence in a symmetric space $(X, d)$ converges to a point $x$ in $X$, then the space is considered complete.

**Definition 2.6.** [21] Let $s \geq 1$ be a given real integer and let $X$ be a nonempty set. A function $d : X \times X \to [0, +\infty)$ is considered a b-metric if and only if each of the subsequent requirements holds for any $x, y, z \in X$:
i. $d(x, y) = 0$ if and only if $x = y$;
ii. $d(x, y) = d(y, x)$;
iii. $d(x, z) \leq s[d(x, y) + d(y, z)]$.
A triplet $(X, d, s)$ is called a $b$-metric space.

**Remark 2.7.** [21] The definitions of complete space, Cauchy sequence, and convergent sequence are defined as in symmetric spaces.

**Definition 2.8.** [5] If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $x$ in $X$ for each sequence $\{x_n\}$ in $X$, then a b-metric space $(X, d, s)$ is sequentially compact.

**Example 2.9.** [20] Let $d : X \times X \to [0, +\infty)$ and $X = [0, 1]$ be defined by $d(x, y) = (x - y)^2$, for all $x, y \in X$. Obviously, $(X, d, 2)$ is a b-metric space.

**Definition 2.10.** [17] Assume that $X$ is a non-empty set. $(X, \preccurlyeq)$ is referred to as an ordered b-metric space if $(X, d, s, \preccurlyeq)$ is a b-metric space and $(X, \preccurlyeq)$ is a partially ordered set. When $x \preccurlyeq y$ or $y \preccurlyeq x$ holds, then $x, y \in X$ are referred to as comparable.

**Definition 2.11.** [22, 23] If $(X, \preccurlyeq)$ is a partially ordered set, then a self-mappings $f$ is dominated if and only if $x \preccurlyeq fx$ for all $x$ in $X$ and $fx \preccurlyeq x$ for all $x$ in $X$.

**Definition 2.12.** [20] A sequential limit comparison property of an ordered b-metric space $(X, d, s, \preccurlyeq)$ exist if, for each decreasing sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$, then $x \prec x_n$.

**Definition 2.13.** [24, 25] Let $f, g : X \to X$ and $(X, d)$ be a metric space. If $fx = gx$, then there is a coincidence point at $x \in X$ for a pair of self mappings $f$ and $g$. Additionally, if $fx = gx = x$, then a point $x \in X$ is a common fixed point of $f$ and $g$.

**Definition 2.14.** [20] For every sequence $\{(x_n, y_n)\} \subset [0, +\infty) \times [0, +\infty)$, then $F : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is referred to as upper semi-continuous from the right if and only if $\lim_{n\to+\infty} x_n = x^+$ and $\lim_{n\to+\infty} y_n = y^+$, then

$$\lim_{n\to+\infty} \sup F(x_n, y_n) \leq F(x, y).$$

We represent $\Psi$ the collection of all the functions $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:
$(\phi_1)$ $\phi$ admits upper semi-continuous from the right;
$(\phi_2)$ $\phi(t, 0) \leq t$ for all $t \geq 0$.

**Definition 2.15.** [20] Consider the b-metric space $(X, d, s)$. The collection of all the functions $\alpha_L : X \times X \to [0, +\infty)$ satisfying the following assertions is also denoted by $\Psi_L$.
$(\alpha_1)$ if $\{x_n\}$ and $\{y_n\}$ are two sequences in $(X, d, s)$ such that $x_n \to x$ and $y_n \to y$, then

$$\lim_{n\to+\infty} \sup \alpha_L(x_n, y_n) \leq \alpha_L(x, y),$$

$(\alpha_2)$ $\alpha_L(x, y) = 0$ when $x = y$.

# 3 Main Results

Here is the Theorem that we use to start this section.

**Theorem 3.1.** *Let $f$ be a self mapping on $X$ and $(X,\ d, s)$ be a sequential compact b-metric space. Suppose that*

$$d(fx, fy) < a_1 d(x, y) + a_2 \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)} + a_3 \frac{d(x, fx)d(y, fy)}{d(x, y)} + \frac{a_4}{s} d(x, fy) + L d(y, fx) \tag{3.1}$$

for all $x, y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If $f$ and $d$ are continuous,
then $f$ possesses a fixed point in $X$.
Additionally,
ii. If $a_1 + \frac{a_4}{s} + L \leq 1$;
then $f$ possesses a unique fixed point.

**Proof.** Let us take any arbitrary point $x_0 \in X$ and let $\{x_n\}$ in $X$ be defined as $x_n = f^n x_0 = fx_{n-1}$. If $x_n = x_{n-1}$ for some $n \geq 1$, then $x_n$ is a fixed point of $f$ and the proof is finished.

Now, let $d_n = d(x_n, x_{n+1})$ and $d_{n-1} = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Assume that $x_n \neq x_{n+1}$, for all $n \geq 1$. From condition (3.1) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d_n =& d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \\ &< a_1 d(x_{n-1}, x_n) + a_2 \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})d(x_n, fx_n)}{d(x_n, fx_{n-1}) + d(x_{n-1}, fx_n)} \\ &\quad + a_3 \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{d(x_{n-1}, x_n)} + \frac{a_4}{s} d(x_{n-1}, fx_n) + L d(x_n, fx_{n-1}) \\ &= a_1 d(x_{n-1}, x_n) + a_2 \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_n)d(x_n, x_{n+1})}{d(x_n, x_n) + d(x_{n-1}, x_{n+1})} \\ &\quad + a_3 \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \frac{a_4}{s} d(x_{n-1}, x_{n+1}) + L d(x_n, x_n) \\ &= a_1 d_{n-1} + a_2 d_{n-1} + a_3 d_n + \frac{a_4}{s} d(x_{n-1}, x_{n+1}) \\ &\leq a_1 d_{n-1} + a_2 d_{n-1} + a_3 d_n + a_4 [d_{n-1} + d_n]. \end{aligned} \tag{3.2}$$

From (3.2), we get $[1 - (a_3 + a_4)]d_n < (a_1 + a_2 + a_4)d_{n-1}$. Since $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, we have $[1 - (a_3 + a_4)] > 0$ and so

$$d_n < \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)} d_{n-1} = d_{n-1}.$$

Consequently, $\{d_n\}$ is a decreasing sequence of positive real numbers and hence there exists $d^* \geq 0$ such that $\lim_{n \to \infty} d_n = d^*$. By using the sequentially compactness of $X$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in X$ as $i \to +\infty$. Again, by using the continuity of $d$ and $f$, we have

$$d_{n_i} = d(x_{n_i}, x_{n_i+1}) = d(x_{n_i}, fx_{n_i}) \to d(x^*, fx^*) \text{ as } i \to +\infty$$

Similarly,

$$d_{n_i+1} = d(x_{n_i+1}, x_{n_i+2}) = d(fx_{n_i}, ffx_{n_i}) \to d(fx^*, ffx^*) \text{ as } i \to +\infty.$$

If $x^* = fx^*$, then $f$ has a fixed point. Assume that $x^* \neq fx^*, d^* = d(x^*, fx^*) > 0$, with $x = x^*$ and $y = fx^*$ in (3.2), we have

$$\begin{aligned} d^* &= d(fx^*, ffx^*) \\ &< a_1 d(x^*, fx^*) + a_2 \frac{d(x^*, fx^*)d(x^*, ffx^*) + d(fx^*, fx^*)d(fx^*, ffx^*)}{d(fx^*, fx^*) + d(x^*, ffx^*)} \\ &\quad + a_3 \frac{d(x^*, fx^*)d(fx^*, ffx^*)}{d(x^*, fx^*)} + \frac{a_4}{s} d(x^*, ffx^*) + L d(fx^*, fx^*) \\ &\leq (a_1 + a_2 + a_3)d^* + \frac{a_4}{s} d(x^*, ffx^*) \\ &\leq (a_1 + a_2 + a_3)d^* + a_4 [d(x^*, fx^*) + d(fx^*, ffx^*)] \\ &= (a_1 + a_2 + a_3 + 2a_4)d^* = d^*, \end{aligned}$$

a contradiction. Hence, $d^* = d(x^*, fx^*) = 0$, that is $x^* = fx^*$. Thus, $x^*$ represent a fixed point of $f$.

To prove the uniqueness of the fixed point, suppose $z$ is another fixed point of $f$ different from $x^*$, so that $d(z, x^*) > 0$. Using $x = z$ and $y = x^*$ in (3.1), we have

$$\begin{aligned} d(z, x^*) &= d(fz, fx^*) \\ &< a_1 d(z, x^*) + a_2 \frac{d(z, fz)d(z, fx^*) + d(x^*, fz)d(x^*, fx^*)}{d(x^*, fz) + d(z, fx^*)} \\ &\quad + a_3 \frac{d(z, fz)d(x^*, fx^*)}{d(z, x^*} + \frac{a_4}{s} d(z, fx^*) + Ld(x^*, fz) \\ &= \Big(a_1 + \frac{a_4}{s} + L\Big) d(z, x^*) \\ &\leq d(z, x^*), \end{aligned}$$

a contradiction and hence $z = x^*$. □

**Example 3.2.** Consider $X = [0, 1]$ and assume $d : X \times X \to [0, +\infty)$. endowed with $d(x, y) = (x - y)^2$, for all $x, y \in X$. Then, let $f : X \to X$ be defined as

$$f(x) = \frac{1}{4(x^2 + 1)}.$$

Clearly, $(X, d, 2)$ represent a sequentially compact b-metric space. Since

$$d(fx, fy) = \left|\frac{x + y}{4(x^2 + 1)(y^2 + 1)}\right|^2 |x - y|^2 < |x - y|^2 = d(x, y) \text{ for all } x, y \in X, x \neq y.$$

Thus, all the hypotheses of Theorem 3.1 are verified, with $a_1 = 1, a_2 = a_3 = a_4 = L = 0$ and hence $f$ has a unique fixed point.

**Corollary 3.3.** *Let $f$ be a self mapping on $X$ and $(X,\ d, s)$ be a sequential compact b-metric space. Suppose that*

$$d(fx, fy) < a_1 d(x, y) + a_2 \frac{d(x, fx)d(y, fy)}{d(x, y)} + \frac{a_3}{s} d(x, fy) + Ld(y, fx) \tag{3.3}$$

for all $x, y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_2 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If $f$ and $d$ are continuous,
then $f$ possesses a fixed point in $X$.
Additionally,
ii. If $a_1 + \frac{a_4}{s} + L \leq 1$;
then $f$ possesses a unique fixed point.
**Proof.** Theorem 3.1 provides the basis for the Corollary's proof. □

**Corollary 3.4.** *Let $f$ be a self mapping on $X$ and $(X,\ d, s)$ be a sequential compact b-metric space. Suppose that*

$$d(fx, fy) < a_1 \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)} + a_2 \frac{d(x, fx)d(y, fy)}{d(x, y)} + \frac{a_3}{s} d(x, fy) + Ld(y, fx) \tag{3.4}$$

for all $x, y \in X$, $x \neq y$, where $a_1 + a_2 + 2a_3 = 1$, $a_2 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If $f$ and $d$ are continuous,
then $f$ possesses a fixed point in $X$.
Additionally,
ii. If $a_1 + \frac{a_3}{s} + L \leq 1$;
then $f$ possesses a unique fixed point.
**Proof.** It is evident that the proof of the Corollary follows from Theorem 3.1. □

**Corollary 3.5.** *Let $f$ be a self mapping on $X$ and $(X,\ d,s)$ be a sequential compact b-metric space. Suppose that*

$$d(fx,fy) < a_1 d(x,y) + a_2 \frac{d(x,fx)d(x,fy)+d(y,fx)d(y,fy)}{d(y,fx)+d(x,fy)} + a_3 \frac{d(x,fx)d(y,fy)}{d(x,y)} + \frac{a_4}{s} d(x,fy) \tag{3.5}$$

for all $x, y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:
i. If $f$ and $d$ are continuous,
then $f$ possesses a fixed point in $X$.
Additionally,
ii. If $a_1 + \frac{a_4}{s} + L \leq 1$;
then $f$ possesses a unique fixed point.
**Proof.** Theorem 3.1 provides the proof of the Corollary in the case where $L = 0$. □

The next theorem is the Suzuki type fixed point result.

**Theorem 3.6.** *Let $f$ be a self mapping on $X$ and $(X,d,s)$ be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s} d(x,fx) < d(x,y)$$

*implies*

$$d(fx,fy) < \varphi\left(d(y,fx), \frac{d(x,fx)d(x,fy)+d(y,fx)d(y,fy)}{d(y,fx)+d(x,fy)}\right) + \alpha_L d(y,fx) \tag{3.6}$$

*for all $x, y \in X$ and $d$ is continuous, then $f$ has a fixed point.*

**Proof.** Let $r = \inf d(x,fx) : x \in X$. We define a sequence $\{x_n\}$ in $X$ be

$$\lim_{n\to\infty} d(x_n, fx_n) = r. \tag{3.7}$$

Since $X$ is sequentially compact, we assume that $x_n \to u$ and $fx_n \to v$ with $u, v \in X$. Now we prove that $r = 0$. Assume on the contrary that $r > 0$. Using the continuity of $d$, we have

$$\lim_{n\to+\infty} d(x_n, v) = (u,v) = \lim_{n\to+\infty} d(x_n, fx_n) = r \tag{3.8}$$

and

$$\lim_{n\to+\infty} d(u, fx_n) = (u,v) = \lim_{n\to+\infty} d(x_n, fx_n) = r. \tag{3.9}$$

Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{2}{3s} < d(x_n, v) \text{ and } d(x_n, fx_n) < \frac{4}{3} f, \text{ for all } n \geq n_1.$$

For all $n \geq n_1$, we have

$$\frac{1}{2s} d(x_n, fx_n) < \frac{1}{2s}\frac{4}{3} r = \frac{1}{s}\frac{2}{3} r < \frac{1}{s} d(x_n, v) \leq d(x_n, v),$$

and by (3.6), we get

$$d(fx_n, fv) < \varphi\left(d(v, fx_n), \frac{d(x_n, fx_n)d(x_n, fv)+d(v, fx_n)d(v, fv)}{d(v, fx_n)+d(x_n, fv)}\right) + \alpha_L d(v, fx_n). \tag{3.10}$$

Taking the *limsup* as $n \to +\infty$ in (3.10), we get

$$\begin{aligned} d(v, fv) &= \limsup_{n\to+\infty} d(fx_n, fv) \\ &\leq \limsup_{n\to+\infty} \varphi\left(d(v, fx_n), \frac{d(x_n, fx_n)d(x_n, fv) + d(v, fx_n)d(v, fv)}{d(v, fx_n) + d(x_n, fv)}\right) \\ &\quad + \limsup_{n\to+\infty} \alpha_L d(v, fx_n) \\ &\leq \varphi(0, d(u, v)) + \alpha_L d(v, v) \leq d(u, v) = r. \end{aligned} \tag{3.11}$$

Thus, from (3.11), we have $d(v, fv) = r$. Since $r > 0, v \neq fv$. So

$$\frac{1}{2s} d(v, fv) < d(v, fv).$$

And by condition (3.6), we get

$$d(fv, ffv) < \varphi\left(d(fv, fv), \frac{d(v, fv)d(v, ffv) + d(fv, fv)d(fv, ffv)}{d(fv, fv) + d(v, ffv)}\right) + \alpha_L d(fv, fv)$$

implies

$$d(fv, ffv) < d(v, fv) = r, \tag{3.12}$$

a contradiction with the given definition of $r$. Thus, $r = 0$ and hence $u = v$. Now, we prove by contradiction. Assume on the contrary that $f$ does not have fixed points. Since

$$\frac{1}{2s} d(x_n, fx_n) < d(x_n, fx_n), \text{ for all } n \geq 1,$$

by condition (3.6), we have

$$\begin{aligned} d(fx_n, ffx_n) < \varphi&\left(d(fx_n, fx_n), \frac{d(x_n, fx_n)d(x_n, ffx_n) + d(fx_n, fx_n)d(fx_n, ffx_n)}{d(fx_n, fx_n) + d(x_n, ffx_n)}\right) \\ &+ \alpha_L d(fx_n, fx) \end{aligned}$$

implies

$$d(fx_n, ffx_n) < d(x_n, fx_n), \text{ for all } n \geq 1. \tag{3.13}$$

From

$$d(u, ffx_n) \leq s[d(u, fx_n) + d(fx_n, ffx_n)] \leq s[d(u, fx_n) + d(x_n, fx_n)],$$

as $n \to +\infty$, we have $f^2 x_n \to u$ and $fx_n \to u$. Suppose that there exists $n \geq 1$ such that

$$\frac{1}{2s} d(x_n, fx_n) \geq d(x_n, u) \text{ and } \frac{1}{2s} d(fx_n, ffx_n) \geq d(fx_n, u),$$

then by (3.13), we get

$$\begin{aligned} d(x_n, fx_n) &\leq s[d(x_n, u) + d(u, fx_n)] \\ &\leq s\frac{1}{2s} d(x_n, fx_n) + s\frac{1}{2s} d(fx_n, ffx_n) \\ &\leq \frac{1}{2} d(x_n, fx_n) + \frac{1}{2} d(x_n, fx_n) \\ &= d(x_n, fx_n), \end{aligned}$$

a contradiction. Hence, for every $n \geq 1$, we have

$$\frac{1}{2s}d(x_n, fx_n) < d(x_n, u), \text{ or } \frac{1}{2s}d(fx_n, ffx_n) < d(fx_n, u).$$

By (3.6) for each $n \geq 1$,

$$d(fx_n, fu) < \varphi\left(d(u, fx_n), \frac{d(x_n, fx_n)d(x_n, fu) + d(u, fx_n)d(u, fu)}{d(u, fx_n) + d(x_n, fu)}\right) + \alpha_L d(u, fx_n) \tag{3.14}$$

or

$$d(ffx_n, fu) < \varphi\left(d(u, ffx_n), \frac{d(fx_n, ffx_n)d(fx_n, fu) + d(u, ffx_n)d(u, fu)}{d(u, ffx_n) + d(fx_n, fu)}\right) + \alpha_L d(u, ffx_n) \tag{3.15}$$

(3.14) and(3.15) hold.
Assume that (3.14) holds for every $n \in J \subset \mathbb{N}$. If $J$ is infinite set, then

$$\begin{aligned} d(u, fu) &= \limsup_{n\to+\infty, n\in J} d(fx_n, fu) \\ &\leq \limsup_{n\to+\infty, n\in J} \varphi\left(d(u, fx_n), \frac{d(x_n, fx_n)d(x_n, fu) + d(u, fx_n)d(u, fu)}{d(u, fx_n) + d(x_n, fu)}\right) \\ &\quad + \limsup_{n\to+\infty, n\in J} \alpha_L d(u, fx_n) \\ &\leq 0, \end{aligned}$$

Thus, $u = fu$. The same conclusion satisfies if $\mathbb{N} \setminus J$ represents an infinite set, in this case we use condition (3.15). In (3.14) and (3.15), we have shown that $u$ is a fixed point of $f$. $\square$

**Corollary 3.7.** *Let $f$ be a self mapping on $X$ and $(X, d, s)$ be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s}d(x, fx) < d(x, y)$$

*implies*

$$d(fx, fy) < \varphi\left(d(x, y), d(y, fx)\right) + \alpha_L d(y, fx) \tag{3.16}$$

*for all $x, y \in X$ and $d$ is continuous, then $f$ has a fixed point.*

**Proof.** Clearly, the proof of the corollary follows from Theorem 3.2. $\square$

If we take $\alpha_L d(y, fx)$ in Theorem 3.2 to be $L \min\{d(y, fx), d(x, fx), d(y, fy)\}$ with $L \geq 0$, we have

**Corollary 3.8.** *Let $f$ be a self mapping on $X$ and $(X, d, s)$ be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s}d(x, fx) < d(x, y),$$

*implies*

$$d(fx, fy) < \varphi\left(d(y, fx), \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)}\right) + L\min\{d(y, fx), d(x, fx), d(y, fy)\} \tag{3.17}$$

*for all $x, y \in X$ and $d$ is continuous, then $f$ has a fixed point.*

**Proof.** The proof of the Corollary follows from Theorem 3.2 if $\alpha_L d(y,fx) = L\min\{d(y,fx), d(x,fx), d(y,fy)\}$ with $L \geq 0$. $\square$

**Corollary 3.9.** *Let $f$ be a self mapping on $X$ and $(X,d,s)$ be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s}d(x,fx) < d(x,y),$$

*implies*

$$d(fx,fy) < \varphi\left(d(x,y), d(y,fx)\right) + L\min\{d(y,fx), d(x,fx), d(y,fy)\} \tag{3.18}$$

*for all $x,y \in X$ and $d$ is continuous with $L \geq 0$, then $f$ has a fixed point.*

**Proof.** Clearly the proof of the Corollary follows from Theorem 3.2. $\square$

**Corollary 3.10.** *Let $f$ be a self mapping on $X$ and $(X,d,s)$ be a sequential compact b-metric space. Suppose that*

$$\frac{1}{2s}d(x,fx) < d(x,y),$$

*implies*

$$d(fx,fy) < d(x,y) + L\min\{d(y,fx), d(x,fx), d(y,fy)\} \tag{3.19}$$

*for all $x,y \in X$ and $d$ is continuous with $L \geq 0$, then $f$ has a fixed point.*

# 4 Application

Ran and Reurings pioneered the study of fixed point results on partially ordered sets in their paper [22], where they explored the applications of these results to the solution of matrix equations. Nieto and Rodrıguez-Lopez continued this research direction in their paper [26], in which they provided several applications to differential equations.

We obtain the subsequent theorems in partially ordered metric spaces through the application of our previously demonstrated results.

**Theorem 4.1.** *Assume that $d$ is continuous in the ordered b-metric space $(X,d,s,\preccurlyeq)$ and let $f,g : X \to X$ be such that $f(X) \subset g(X)$, $g(X)$ represents a sequentially compact subspace of $X$, $f$ a dominated mapping and $g$ a dominating mapping. Suppose that*

$$d(fx,fy) < a_1 d(gx,gy) + a_2\frac{d(gx,fx)d(gx,fy) + d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} + a_3\frac{d(gx,fx)d(gy,fy)}{d(gx,gy)} + \frac{a_4}{s}d(gx,fy) + Ld(gy,fx) \tag{4.1}$$

*for every comparable elements $x,y \in X$, $gx \neq gy$, where $a_1+a_2+a_3+2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:*
*(i) $X$ possesses a sequential limit comparison property, then $g$ and $f$ possesses a coincidence point in $X$. Additionally,*
*(ii) If $a_1 + \frac{a_4}{s} + L \leq 1$,*
*then the points of coincidence of $g$ and $f$ is well ordered if and only if $g$ and $f$ possesses one and only one point of coincidence.*

**Proof.** Let us take any arbitrary point $x_0 \in X$ and let $\{x_n\}$ in $X$ be defined as

$$g_{n+1} = fx_n, \text{ for all } n \geq 0.$$

Since the range of $g$ contains the range of $f$. If $d(gx_n, gx_{n+1}) = 0$ for some $n \geq 0$, then $gx_n = gx_{n+1} = fx_n$ and so $x_n$ is a coincidence point of $f$ and $g$. Assume that $d(gx_n, gx_{n+1} > 0$ for all $n \geq 0$. On using the property of the mappings $f$ and $g$, we have

$$x_{n+1} \preccurlyeq gx_{n+1} = fx_n \preccurlyeq gx_n \text{ for all } n \geq 0.$$

Then $x_n$ and $x_{n+1}$ are comparable for all $n \geq 0$. Since $d(gx_n, gx_{n+1} > 0$, we get that $gx_{n+1} \prec gx_n$, for all $n \geq 0$. Thus $\{gx_n\}$ is a decreasing sequence. Using the hypothesis that $g(X)$ is a sequentially compact subspace of $X$, we can assume that $gx_n \to gu$ for some $u \in X$. Now, condition $(i)$ guarantees that $gu \prec gx_n$, for all $n \geq 0$. Now, we prove that $fu = gu$. We have

$$\begin{aligned}
d(gu, fu) &= \lim_{n\to+\infty} d(gx_{n+1}, fu) = \lim_{n\to+\infty} d(fx_n, fu) \\
&\leq \lim_{n\to+\infty} \Bigg[a_1 d(gx_n, gu) + a_2 \frac{d(gx_n, fx_n)d(gx_n, fu) + d(gu, fx_n)d(gu, fu)}{d(gu, fx_n) + d(gx_n, fu)} \\
&\quad + a_3 \frac{d(gx_n, fx_n)d(gu, fu)}{d(gx_n, gu)} + \frac{a_4}{s} d(gx_n, fu) + L d(gu, fx_n)\Bigg] \\
&= a_3 d(gy, fu) + \frac{a_4}{s} d(gu, fu) \\
&= \left(a_3 + \frac{a_4}{s}\right) d(gu, fu) \\
&< d(gu, fu)
\end{aligned}$$

a contradiction. That is, $d(gu, fu) = 0$ and hence $fu = gu$. Therefore, $u$ is a coincidence point of $f$ and $g$. Now, suppose that the set of points of coincidence of $f$ and $g$ is well ordered. We claim that the point of coincidence of $f$ and $g$ is unique. Assume on the contrary that there exists another point $v$ in $X$ such that $fv = gv$ with $gu \neq gv$. Assume that $gu \prec gv$, then $u \preccurlyeq gu \prec gv = fv \preccurlyeq v$ and $u, v$ are comparable. Now, using the condition (4.1), we get

$$\begin{aligned}
d(fu, fv) &< a_1 d(gu, gv) + a_2 \frac{d(gu, fu)d(gu, fv) + d(gv, fu)d(gv, fv)}{d(gv, fu) + d(gu, fv)} \\
&\quad + a_3 \frac{d(gu, fu)d(gv, fv)}{d(gu, gv)} + \frac{a_4}{s} d(gu, fv) + L d(gv, fu) \\
&= \left(a_1 + \frac{a_4}{s} + L\right) d(fu, fv) \\
&\leq d(fu, fv),
\end{aligned}$$

a contradiction and hence $gu = gv$. The same holds if $gv \prec gu$. Therefore $fu = gu = z$ is the unique point of coincidence of $f$ and $g$ in $X$. Conversely, if $f$ and $g$ have one and only one point of coincidence, then the set of points of coincidence of $f$ and $g$ being singleton is well ordered. □

**Theorem 4.2.** *Consider all the hypotheses of Theorem 4.1 with the following assertions:*
*(ii) If $\{gx_n\}$ possess a decreasing sequence that converges to $gu$ for some $u \in X$, then $ggu \preccurlyeq gu$;*
*(iii) $g$ and $f$ possess a weakly compatible;*
*then $g$ and $f$ possesses a common fixed point in $X$.*
*Additionally, $g$ and $f$ possesses a unique common fixed point in $X$ if coincidence of $g$ and $f$ is well ordered.*

**Proof.** Let us take any arbitrary point $x_0 \in X$ and let $\{x_n\}$ in $X$ be defined as

$$gx_{n+1} = fx_n \text{ for all } n \geq 0.$$

Continuing as in the proof of Theorem 4.1, we deduce that $\{gx_n\}$ is a decreasing sequence that converges to $gu$ for some $u \in X$ and $gu = fu = z$. Using condition $(ii)$, we have $gz \preccurlyeq gu$. Since, the mappings $f$ and $g$ are weakly compatible we obtain that $fz = fgu = gfu = gz$. If $gz = gu = z$, then $z$ is a common fixed point of $f$ and $g$. If $gz \prec gu$, then $u, z$ are comparable and using the condition (4.1), we get $gz = gu$. So $z$ is a common fixed point of $f$ and $g$. If the set of points of coincidence of $f$ and $g$ is well ordered, then $f$ and $g$ have a unique point of coincidence and so $z$ is a unique common fixed point of $f$ and $g$. □

**Corollary 4.3.** *Assume that $d$ is continuous in the ordered b-metric space $(X, d, s, \preccurlyeq)$ and let $f : X \to X$ be such that $f(X) \subset X$ possess a sequentially compact subspace of $X$, $f$ a dominated mapping. Suppose that*

$$d(fx, fy) < a_1 d(x, y) + a_2 \frac{d(x, fx)d(x, fy) + d(y, fx)d(y, fy)}{d(y, fx) + d(x, fy)} + a_3 \frac{d(x, fx)d(y, fy)}{d(x, y)} + \frac{a_4}{s} d(x, fy) + Ld(y, fx) \tag{4.2}$$

*for every comparable elements $x, y \in X$, $x \neq y$, where $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$, $L \geq 0$ and satisfies the following conditions:*
*(i) $X$ possess a sequential limit comparison property,*
*then $f$ possesses a fixed point in $X$.*
*Additionally,*
*(ii) If $a_1 + \frac{a_4}{s} + L \leq 1$,*
*then $f$ possesses a unique fixed point.*

## 5 Conclusion

The main findings of this study demonstrate applicability of sequential compact b-metric spaces in establishing fixed point theorems for Eldeisten-Suzuki-type contraction mappings. This study provides significant advancements in the understanding of sequential compact b-metric spaces, with potential applications in differential equations and nonlinear integral equation.

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## References

[1] Banach S. Sur les op´erations dans les ensembles abstraits et leur application aux equations itegrales. *Fundamenta mathematicae.* 1922; 3(1): 133-181. DOI: https://doi.org/10.4064/fm-3-1-133-181

[2] Ran ACM, Reurings MCB. A fixed point theorem in partially ordered sets and some applications to matrix equations. *proceedings of the American Mathematical Society.* 2004; 132(5): 1435-1443. https://www.ams.org/journals/proc/2004-132-05/S0002-9939-03-07220-4/S0002-9939-03-07220-4.pdf

[3] Nieto JJ, Rodrıguez-Lopez R. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Mathematica Sinica, English Series.* 2007; 23(12): 2205-2212. DOI: https://doi.org/10.1007/s10114-005-0769-0

[4] Rus I A, Petru¸sel A, Petru¸sel G. *Fixed Point Theory.* Cluj-Napoca: Cluj University Press; 2(2), 2008.

[5] Raji M, Ibrahim MA. Fixed point theorems for fuzzy contractions mappings in a dislocated b-metric spaces with applications. *Annals of Mathematics and Computer Science.* 2024; 21: 1-13. DOI: https://doi.org/10.56947/amcs.v21.233

[6] Shah MH, Hussain N. Nonlinear contractions in partially ordered quasi b-metric spaces. *Communications of the Korean Mathematical Society.* 2012; 27(1): 117-128. DOI: http://dx.doi.org/10.4134/CKMS.2012.27.1.117

[7] Kannan R. Some results on fixed points. *Bull Calcutta Math Soc.* 1968; 60: 71-76. DOI: https://doi.org/10.2307/2316437

[8] Raji M, Ibrahim MA, Rauf K, Kehinde R. Common fixed point results for fuzzy $F$-contractive mappings in a dislocated metric spaces with application. *Qeios.* 2024; 7; 1-21. DOI: https://doi.org/10.32388/SV98CN

[9] Dewangan K, Rathour L, Mishra VN, Raiz M. On multi-valued nonexpansive mappings in UCBS, *Journal of Computational Analysis and Applications.* 2024; 33(1): 396-406. https://eudoxuspress.com/index.php/pub/article/view/31

[10] Mishra LN, Raiz M, Rathour L, Mishra VN. Tauberian Theorems for Weighted Means of Double Sequences in Intuitionistic Fuzzy Normed Spaces. *Yugoslav Journal of Operations Research.* 2022; 377-388. DOI: https://doi.org/10.2298/YJOR210915005M

[11] Czerwik S. Contraction mappings in b-metric spaces. *Acta mathematica et informatica universitatis ostraviensis.* 1993; 1(1): 5-11.

[12] Boriceanu M. Strict fixed point theorems for multivalued operators in b-metric Spaces. *International Journal of Modern Mathematical Sciences.* 2009; 4(3): 285-301.

[13] Hussain N, Doric D, Kadelburg Z, Radenovic S. Suzuki-type fixed point results in metric type spaces. *Fixed point theory and applications.* 2012; 2012: 1-12. DOI: https://doi.org/10.1186/1687-1812-2012-126

[14] Edelstein M. On fixed and periodic points under contractive mappings. *Journal of the London Mathematical Society.* 1962; 1(1): 74-79. DOI: https://doi.org/10.1112/jlms/s1-37.1.74

[15] Suzuki T. A new type of fixed point theorem in metric spaces. *Nonlinear Analysis: Theory, Methods & Applications.* 2009; 71(11): 5313-5317. DOI: https://doi.org/10.1016/j.na.2009.04.017

[16] Doric D, Kadelburg Z, Radenovic S. Edelstein-Suzuki-type fixed point results in metric and abstract metric spaces.*Nonlinear Analysis: Theory, Methods & Applications.* 2012; 75(4): 1927-1932. DOI: https://doi.org/10.1016/j.na.2011.09.046

[17] Paesano D, Vetro P. Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces. *Topology and its Applications.* 2012; 159(3): 911-920. DOI: https://doi.org/10.1016/j.topol.2011.12.008

[18] Salimi P, Karapinar E. Suzuki-Edelstein type contractions via auxiliary functions. *Mathematical Problems in Engineering.* 2013; 2013(1): 648528. DOI: https://doi.org/10.1155/2013/648528

[19] Radenovic S, Salimi P, Vetro C, Dosenovic T. Edelstein-Suzuki-type results for self-mappings in various abstract spaces with application to functional equations. *Acta Mathematica Scientia.* 36(1): 94-110. DOI: https://doi.org/10.1016/S0252-9602(15)30081-3

[20] Cosentino M, Salimi P, Vetro P. Fixed point results on metric-type spaces. *Acta Mathematica Scientia.* 2014; 34(4):1237-1253. DOI: https://doi.org/10.1016/S0252-9602(14)60082-5

[21] Czerwik S. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Sem. Mat. Univ. Modena.* 1998; 46: 263-276.

[22] Nieto JJ, Rodrıguez-Lopez R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order*. 2005; 22: 223-239. DOI: https://doi.org/10.1007/s11083-005-9018-5

[23] Raji M, Rathour L, Mishra LN, Mishra VN. Generalized twisted $(\alpha, \beta)$-$\psi$ contractive type mappings and related fixed point results with applications. *Int. J. Adv. Sci. Eng.* 2024; 10(4): 3639-3654. DOI: https://doi.org/10.29294/IJASE.10.4.2024.3639-3654

[24] Raji M. Generalized $\alpha$-$\psi$ contractive type mappings and related coincidence fixed point theorems with applications. *The Journal of Analysis*. 2023; 31(2): 1241-1256. DOI: https://doi.org/10.1007/s41478-022-00498-8

[25] Raji M, Ibrahim MA. Fixed point theorems for modified $F$-weak contractions via $\alpha$- admissible mapping with application to periodic points. *Annals of Mathematics and Computer Science*. 2024; 20: 82-97. DOI: https://doi.org/10.56947/amcs.v20.232

[26] Nieto JJ, Pouso RL, Rodrıguez-Lopez R. Fixed point theorems in ordered abstract spaces. *Proceedings of the American Mathematical Society*. 2007; 135(8): 2505-2517. DOI: https://doi.org/10.1090/S0002-9939-07-08729-1

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