Determining accurate efficiency in the presence of fuzzy data via one LP NDEA model

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Abstract

Supply chain management is crucial to successful companies because it monitors the interactions between the system components. It uses a hybrid, process-based approach to procuring, producing, and delivering goods and services to customers and aims to improve the overall system's efficiency. Within the supply chain, there may be conflict or collaboration between the different components (e.g. suppliers and manufacturers) working towards common goals. This article focuses on evaluating the performance of network structures in which segments control intermediaries. Two scenarios are examined: non-collaborative control, in which the segments manage the intermediates independently, and collaborative control, in which the partnerships manage the intermediates jointly. Fuzzy Network Data Envelopment Analysis (FNDEA) is used to evaluate the efficiency of networks with imprecise data. A two-stage system with fuzzy data for decision-making units (DMUs) is considered. Linear programming models are proposed for each scenario to calculate the efficiency of DMUs. By applying the α-cut method to convert nonlinear problems into linear ones, unique efficiency values for DMUs are obtained.

Finally, the study uses data from the particleboard industry to illustrate the model's applicability and highlight conflicts between supply chain objectives.

Key Words: Fuzzy network data envelopment analysis, a two-stage system, Triangular fuzzy data, Possibilities linear programming, definite efficiency score

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1. Introduction:

Estimating the efficiency and performance of a decision -making unit (DMU) is a very important issue for managers in the real world. Efficiency is the ratio between the actual output and the same amount of inputs. One of the most effective methods for measuring the relative efficiency of comparable DMUs is Data Envelopment Analysis (DEA). DEA is the appropriate method based on linear programming (LP) to identify efficient and inefficient DMUs. The first DEA model (CCR) was proposed by Charnes et al. Charnes, Cooper, & Rhodes [5] and is based on constant returns to scale (CRS). Then Banker et al. Banker, Charnes, & Cooper [3] proposed the BCC model with the assumption of variable returns to scale (VRS). These models generally behave like a black box and ignore the processes within the system. Therefore, such a setting determines the source of inefficiency in the system only for external inputs and final outputs. Lewis and Sexton [27]; Kao and Hwang [23, 24]. The DEA approach to evaluate the efficiency of systems (DMUs) considering their internal structures is called network DEA (NDEA). Recently, much attention has been paid to determining the efficiency of network systems. In general, three types of structures are considered for network structures: series structures, parallel structures, and a mixture of series and parallel structures. Fare & Grosskopf [13] were the first to consider the internal structure of a system as NDEA.

Many scholars have proposed articles for the development of NDEA in various methods, such as the "standard approach" which means the efficiency of the components and the system are calculated independently; for instance, see; Seiford and Zhu [38]; Sexton and Lewis [40]; Tsolas [43]; Adler, Liebert and Yazhemsky [1]. The second class "relational approach" means the overall and stage efficiency scores of the system are estimated simultaneously and the overall efficiency can be decomposed into the stage efficiencies in a specific mathematical form; for example, see; Kao and Hwang [23, 24]; Kao [20]; Chen, Cook, and Zhu [8]. The third class "aggregation approach" means that the overall and stage efficiency scores are calculated simultaneously and the overall efficiency can be aggregated from the stage efficiencies in a pre-specified mathematical form. For example: Chen and Zhu [6]; Chen, Cook, Li, and Zhu [7]; Tone and Tsutsui [41, 42]; Chiou, Lan, and Yen [9]; Liu and Lu [29].

In the real world, the different types of Production Possibility Sets (PPSs) can be presented according to various relations that can be made between shared intermediate products. In this vein, Hassanzadeh and Mostafaee [18] examined the way intermediate products affect the overall and stage efficiency scores of units in different scenarios. They identified six scenarios associated with the concept of Link control through various stages. Through their classification, intermediate products' value control via the former stage, the next stage, both first and second stages which have no cooperation, both first and second stages which have cooperated, no stages which have non-cooperative approach and no stages which have cooperative approach.

Since different authors have used different approaches without clarification, these scenarios help us compute target intermediate products in NDEA.

The information of all inputs and outputs data is crisp and precise in conventional DEA and NDEA models. However, in real-world problems, this assumption is not always true. It means the measurement of inputs and outputs is sometimes imprecise or vague. The imprecise or vague evaluation may result from unquantifiable and incomplete information. Recently, various fuzzy DEA methods have indeed been developed to address the impreciseness and ambiguity in DEA models. Fuzzy sets, initially proposed by Zadeh [39] have been utilized to determine the efficiency of Decision-Making Units (DMUs) in the presence of imprecise data. Subsequently, researchers such as Ahmadvand and Pishvaee [2]; Lozano [30]; Hatami-Marbini et al. [17], Moreno and Lozano, [34]; Esfandiari and Saati, [12]; Saati, [36] have contributed to the development of this area.

In general, fuzzy DEA methods can be classified into four categories. i) the tolerance approach (Sengupt, [39]; Kahraman and Tolga, [19] ii) $\alpha -$cut based approach Meada et al.; [33], Kao and Liu, [21], Saati et al.; [36], Hatami-Marbini and Saati, [17], iii) fuzzy ranking approach Guo and Tanaka [15] iv) possibility approach Dubois and Prade [11]; Lertworasirikul, [26]; Lertworasirikul et al.; [27].

Recently, various methods have been proposed for determining the efficiency of network systems in the presence of fuzzy data. For example, Kao and Liu, [22] have proposed a two-stage fuzzy planning approach for evaluating the efficiency of two-stage units with fuzzy data. This method takes into account the imprecise nature of the data and incorporates it into the evaluation process. Similarly, Lozano, [31] has also contributed to the field by presenting an article that focuses on the process efficiency of two-stage systems with fuzzy data. This research likely provides insights into how fuzzy sets can be utilized to assess the efficiency of two-stage systems and handle the uncertainty associated with fuzzy data. Malek Mohamadi et al., [32] have proposed a method for efficiency decomposition in two-stage networks using DEA with undesirable intermediate measures and fuzzy input and output.

In most of the existing methods for possibility LP, where the cut is used, the solution is obtained by comparing the intervals on the left and right hand side of the constraints; see; Bezdek [4]; Fuller [14]; Lai and Hwang, [25]; Sengupta, [39]. Different methodologies have been suggested for comparison of the intervals. In some of these methods, simply the end points of the interval and considered for justification that makes the model very simple and hence a lot of information might have been lost. In others, the complexity of the algorithm may cause computational inefficiency. To cope with this problem, Saati et al., [36] considered a fuzzy version of CCR model and transformed the model into a crisp LP model by applying an alternative cut approach. Then, the problem is converted to an interval programming. Then, instead of comparing the equality (or inequality) of two intervals, a variable is defined in the interval, not only satisfies the set of constraints, but also maximizes the efficiency value. Saati et al., [37] proposed a new LP model based on Saati et al., [36] method to explore congestion in Iranian hospitals.

Hatami-Marbini et al., [17] proposed a possible programming problem and through α-method converted the model to a LP problem to obtain the pure and global efficiency of a general two-stage system (without separate output and input for the first and second stage respectively). They did not pay attention to the role of intermediate products and considered them as an output of the first stage and input of the second stage. Also, they were able to accurately identify inefficient resources by decomposing performance. Malek Mohammadi et al., [32] considered intermediate products as undesirable factors in the general network and obtained the overall system performance using the Fuzzy method by decomposing the efficiency of each stage.

In contrast to the study by Hatami et al., [17] in this article, we have taken into account a two-stage system (general-network) with fuzzy inputs, outputs for each sub-process and intermediate products (see Fig.1). Then, two scenarios based on intermediate products role (link control by both stages with non-collaborative and non-stages with collaborative approach respectively) were considered that are identified by Hassanzadeh and Mostafaee [18]. The selection of these two scenarios seems to be based on the significance of intermediate products and their collaborative and non-collaborative roles in influencing overall network efficiency. Additionally, it's mentioned that the results obtained from these scenarios can provide valuable insights for real-world applications. By utilizing fuzzy logic and considering the impact of intermediate products, our proposed model can contribute to a more precise evaluation of network efficiency compared to other scenarios. Then, a possible programming model for each scenario was proposed to obtain an efficient and inefficient system. Moreover, we used the α - cut method and developed Saati et al., method [36] and transformed the non-linear model into a linear model. To the best of our knowledge, there is no research to address the exact efficiency score of network systems in the presence of fuzzy data with features of intermediate products. Eventually, in this paper, for the first time, the efficiency score of a system was determined definitely in the presence of fuzzy data and the role of the intermediate products.

Briefly, in this paper, taking into account the nature of the intermediate products, the efficiency of the whole system (Fig. 1) can be measured more precisely by solving solely one LP model and there is no need to calculate the efficiency of each stage separately.

The rest of this paper is organized as follows:

Section 2 covers important preliminary information. Section 3 introduces two innovative linear programming (LP) models for estimating efficiency and its quantity using fuzzy triangular numbers. In section 4, the results of the proposed models in the case study are provided. Finally, section 5 presents the conclusions drawn from the study.

2- Preliminaries

2-1 General two-stage structure

Assume we have a set of *n* DMUs including {DMU_j $j \in J = \{1, 2, . . . , n\}$}, with two-stage structure (see Fig.1). The number of the external inputs of stages *1* and *2* is denoted by m_1, m_2, respectively. Also, let s_1, s_2 be the number of final outputs to stages *1* and *2*, respectively. The vector of intermediate activities (products) from the first stage to the second stage is denoted by z. Furthermore, the observed production points are as follows:

x_{ij}^k, $i = 1, 2, ..., m_k$, $j = 1, 2, ..., n$, $k = 1, 2$. (input resource *i* of DMU_j for *k*-th stage).

y_{rj}^k, $r = 1, 2, ..., s_k$, $j = 1, 2, ..., n$, $k = 1, 2$ (Output product *r* of DMU_j for *k*-th stage).

z_{tj}^k, $t = 1, 2, ..., D$, $j = 1, 2, ..., n$ (Intermediate product *t* of DMU_j).

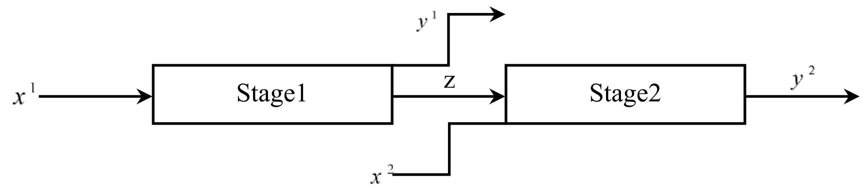

Figure 1: General two-stage process.

According to Fig.1, in the first stage, DMU$_j$ consumes the external nonnegative inputs $x_j^1 = (x_{1j}^1, ..., x_{m_1 j}^1) \neq 0$ to produce two types of nonnegative outputs, including the final outputs $y_j^1 = (y_{1j}^1, ..., y_{s_1 j}^1) \neq 0$ of the first stage and intermediate products $z = (z_1, ..., z_D) \neq 0$. Also, in the second stage DMU$_j$ consumes the external nonnegative inputs $x_j^2 = (x_{1j}^2, ..., x_{m_2 j}^2) \neq 0$ and intermediate products $z = (z_1, ..., z_D) \neq 0$ to produce the final nonnegative outputs $y_j^2 = (y_{1j}^2, ..., y_{s_2 j}^2) \neq 0$. Assume that DMU$_p$ = (x_p^k, y_p^k, z_p) for $k = 1, 2$ be the evaluated unit.

Now, among the six scenarios introduced by Hassanzadeh and Mostafaee [18], two scenarios which are more important than the rest that named *i)* intermediate product control by two stages (Non-cooperative Approach) and *ii)* intermediate product control by neither of the

two stages (cooperative approach)) and their relative PPS for the first and second scenario (PPS$^s{}_1$ and PPS$^s{}_2$ respectively) proposed as bellow:

PPS$^s{}_1$

$$= \left\{ (\mathbf{x}^k, \mathbf{z}, \mathbf{y}^k) \middle| \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_{ip}^k, \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_{rp}^k, \sum_{j=1}^{n} \lambda_j^1 z_{tj} \ge z_{tp}, \sum_{j=1}^{n} \lambda_j^2 z_{tj} \le z_{tp}, \sum_{j=1}^{n} \lambda_j^k = 1, \forall i, r, \mathrm{t}, \mathrm{k} \right\}$$

PPS $^s{}_2$

$$= \left\{ (\mathbf{x}^k, \mathbf{z}, \mathbf{y}^k) \middle| \sum_{j=1}^{n} \lambda_j^1 x_{ij}^k \le x_{io}^k, \sum_{j=1}^{n} \lambda_j^2 y_{rj}^k \ge y_{ro}^k, \sum_{j=1}^{n} \lambda_j^1 z_{lj} \ge \sum_{j=1}^{n} \lambda_j^2 z_{lj}, \sum_{j=1}^{n} \lambda_j^k = 1, \lambda_j^k \ge 0 \forall i, r, k, l \right\}$$

The input- and output- oriented radial DEA efficiency score of DMU$_p$ based on above PPS, i.e. θ^* and φ^* respectively can be obtained as follows:

$$\theta^* = \min \theta \qquad\qquad \varphi^* = \max \varphi \qquad (1)$$

$$s.t \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le \theta x_{ip}^k \quad \forall i,k \qquad\qquad s.t \sum_{j=1}^{n} \lambda_j x_{ij}^k \le x_{ip}^k \quad \forall i,k$$

$$\sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_{rp}^k \quad \forall r,k \qquad\qquad \sum_{j=1}^{n} \lambda_j y_{rj}^k \ge \varphi y_{rp}^k \quad \forall r,k$$

$$\sum_{j=1}^{n} \lambda_j^k = 1 \quad \forall k \qquad\qquad \sum_{j=1}^{n} \lambda_j^k = 1 \quad \forall k$$

$$\lambda_j^k \ge 0 \quad \forall j \qquad\qquad \lambda_j^k \ge 0 \quad \forall j$$

Plus the constraints associated with the intermediate products which can be expressed under these PPS as follows:

1. Scenario 1(PPS^s_1)

$$\sum_{j=1}^{n} \lambda_j^1 z_{tj} \le z_{tp}$$
$$\sum_{j=1}^{n} \lambda_j^2 z_{tj} \ge \theta z_{tp} \qquad (2)$$

2. Scenario 2 (PPS^s_2)

$$\sum_{j=1}^{n} \lambda_j^1 z_{tj} \ge \sum_{j=1}^{n} \lambda_j^2 z_{tj} \qquad (3)$$

2-2 Fuzzy numbers

Fuzzy sets were introduced by Zadeh [44] to represent and manipulate imprecise and accurate data associated with human cognition processes with fuzzy numbers. Now, we review some of the basic definitions of fuzzy sets, see;(Dubois and Prade [10]; Kauffman and Gupta, [35]; Zimmermann [45].

Definition 2-2-1: (Fuzzy sets) Suppose X is an un-empty set. The fuzzy set $\tilde{A}$ in X is specified by its membership function:

$$\mu_{\tilde{A}}(x) : \mathrm{X} \to [0,1]$$

where $\mu_{\tilde{A}}(x)$ is named the degree of membership of element x belongs to fuzzy set $\tilde{A}$ for each $x \in X$.

Obviously, $\tilde{A}$ is a set of ordered pairs which can be written as bellows:

$$\tilde{A}=\{(x,\mu_{\tilde{A}}(x))|x\in X\}$$

Definition 2-2-2: (α -cut) An α -cut set belongs to the fuzzy set $\tilde{A}$ of *X* is a crisp set demonstrated by A_α and is defined as follows:

$$A_\alpha=\{x\in X\ |\mu_{\tilde{A}}(x)\geq\alpha\}$$

Each member of the A_α set has a membership degree greater than or equal to the value α .

Definition 2-2-3: (Fuzzy number) A fuzzy set $\tilde{A}$ is named a fuzzy number if following conditions be hold:

i) $\mu_{\tilde{A}}(x)$ is continues.

ii) There exists at least one $x\in X$ that $\mu_{\tilde{A}}(x)=1$.

iii) $\tilde{A}$ must be normal and convex.

Definition 2-2-4: (Triangular number) A fuzzy number $\tilde{A}=(a^l,\mathrm{a}^m,\mathrm{a}^u)$ is named a triangular fuzzy number whose membership function $\mu_{\tilde{A}}$ satisfies the following properties; see Fig 2.

i) $\mu_{\tilde{A}}(x)$ is a continues mapping from R to the closed interval [0,1].

ii) $\mu_{\tilde{A}}(x)=0$ for all $x\in(-\infty,a^l]$

iii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $a^l\leq x\leq a^m$

iv) $\mu_{\tilde{A}}(x)=1$ for $x=a^m$

v) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $a^m\leq x\leq a^u$

vi) $\mu_{\tilde{A}}(x)=0$ for all $x\in[\mathrm{a}^u,\infty)$

The membership function $\mu_{\tilde{A}}(x)$ of $\tilde{A}$ is given as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases}0 & x<a^l \\ \dfrac{x-a^l}{a^m-a^l} & a^l\le x\le a^m \\ \dfrac{a^u-x}{a^u-a^m} & a^m\le x\le a^u \\ 0 & x>a^u\end{cases}$$

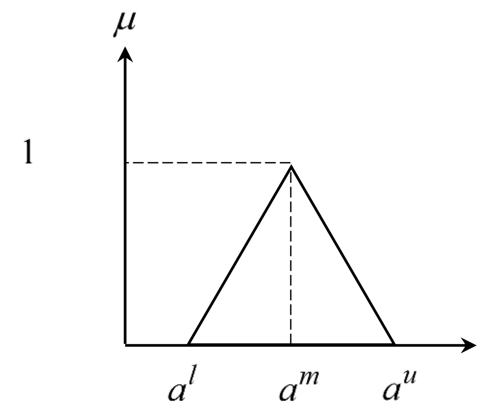

Fig 2. Triangular fuzzy number

Without loss of generality, we assume that all fuzzy numbers used throughout the ape are triangular fuzzy numbers.

Definition A fuzzy number $\tilde{A}$ is called positive if its membership function is $\mu_{\tilde{A}}(x)=0$ for each $x<0$.

Definition 2-2-5: Let two positive triangular fuzzy number $\tilde{A}=(a^L,a^m,a^u)$ and, $\tilde{B}=(\mathrm{b}^L,\mathrm{b}^m,\mathrm{b}^u)$, the arithmetic operations of these two triangular fuzzy numbers are defined as follows:

Addition:

$$\tilde{A}(+)\tilde{B}=(a^L+b^L,a^m+b^m,a^u+b^u)$$

Subtraction:

$$\tilde{A}(-)\tilde{B}=(a^L-b^u,a^m-b^m,a^u-b^L)$$

Multiplication:

$$\tilde{A}(\times)\tilde{B}\approx(a^L\times b^L,a^m\times b^m,a^u\times b^u)\quad \tilde{a},\tilde{b}>0$$

$$k\tilde{A} = (\mathrm{k}a^L, \mathrm{k}a^m, \mathrm{k}a^u), \quad \forall \mathrm{k} \in \mathrm{R}^+$$

Inverse:

$$(\tilde{A})^{-1} = (\frac{1}{a^u}, \frac{1}{a^m}, \frac{1}{a^L}) \quad \tilde{a} > 0$$

Division:

$$\tilde{A}(\div)\tilde{B} = \tilde{A}(\times)(\tilde{B})^{-1} = (\frac{a^L}{b^u}, \frac{a^m}{b^m}, \frac{a^u}{b^L}) \qquad \tilde{a}, \tilde{b} > 0$$

2-3 Fuzzy efficiency measurement

Suppose that there are n DMUs, DMU$_j$=$(\tilde{\mathbf{x}}_j, \tilde{\mathbf{y}}_j)$; $j \in \{1, \ldots, n\}$ with imprecise data can be represented by $\tilde{\mathbf{x}}_j = (\mathbf{x}_j^l, \mathbf{x}_j^m, \mathbf{x}_j^u)$ and $\tilde{\mathbf{y}}_j = (\mathbf{y}_j^l, \mathbf{y}_j^m, \mathbf{y}_j^u)$.

The multiplier form of CCR model with fuzzy data for evaluating DMU$_p$ is as follows:

$$\begin{aligned} \max w_p &= \sum_{r=1}^{s} u_r \tilde{y}_{rp} \\ s.t \quad & \sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1 \\ & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \quad \forall j \\ & u_r, v_i \geq 0 \quad \forall r, i \end{aligned} \tag{4}$$

By placing fuzzy data, Model (4) is rewritten as Model (5):

$$\begin{aligned} \max w_p &= \sum_{r=1}^{s} u_r (y_p^l, y_p^m, y_p^u) \\ s.t \quad & \sum_{i=1}^{m} v_i (x_p^l, x_p^m, \mathrm{x}_p^u) = (1^l, 1^m, 1^u) \\ & \sum_{r=1}^{s} u_r (y_j^l, y_j^m, y_j^u) - \sum_{i=1}^{m} v_i (x_p^l, x_p^m, \mathrm{x}_p^u) \leq 0 \quad \forall j \\ & u_r, v_i \geq 0 \quad \forall r, i \end{aligned} \tag{5}$$

Model (4) is a possible programming problem that many methods have been proposed to solve. In most of these methods applying the cut method, the intervals in both sides of the constraints are compared. There are many methods for comparing the intervals. In this vein, Saati et al., [36] proposed a new approach in which instead of comparing the intervals, defined variables in the intervals such that they satisfied the set of constraints and at the same time the objective function was maximized.

Now, in this study, we consider a two-stage system with separate input and output for each stage as shown in Fig.1. To be close to the real world, the data is considered triangular fuzzy numbers. Also, we consider the two important roles of intermediate products in finding efficient and inefficient systems. Toward this gain, two scenarios are considered, and propose a fuzzy non-linear problem for each one. Then, we developed the method proposed by Saati et al., [36] to convert models into an LP problem to determine a definite efficiency score.

3- Crisp efficiency score via fuzzy data

In this section, we will determine the efficiency of a two-stage system (Fig.1) by considering the intermediate products role (link control by neither of the two stages (cooperative approach) and two stages (Non-cooperative Approach)) in the presence of triangular fuzzy data. In this regard, a non-linear FNDEA model is proposed for each scenario, and then by developing the Saati et al., method [36], each model is changed to the novel LP model. Finally, for the first time, the crisp efficiency score is determined for each scenario, and efficient and inefficient DMUs can be identified. Now, suppose that there are n two-stage DMUs, DMU$_j = (x_j^k, y_j^k, z_j); k = 1, 2,\ j \in \{1, \ldots, n\}$ with imprecise data can be represented by

$$\tilde{\mathbf{x}}_j = (\mathbf{x}_j^l, \mathbf{x}_j^m, \mathbf{x}_j^u), \tilde{\mathbf{z}}_j = (\mathbf{z}_j^l, \mathbf{z}_j^m, \mathbf{z}_j^u) \text{ and } \tilde{\mathbf{y}}_j = (\mathbf{y}_j^l, \mathbf{y}_j^m, \mathbf{y}_j^u).$$

Scenario 1: Link control by two stages (Non-cooperative Approach)

The intermediate products are simultaneously and independently specified by both the former and the latter stages. Hence, the intermediate products play the input role in the latter stage and the output role in the former stage.
It is worth mentioning that in this article, we aim to evaluate the maximum profit from production, so we consider the upper limit of efficiency. In this vein, according to the PPSs1, the upper-efficiency measurement can be calculated through the following mathematical model:

$$\begin{aligned} \theta^{u-s_1} = \max &\{\min \quad \theta \quad \text{s.t} \;\; constra\text{int}s \;\; (1) \; and \; (2)\} \\ & X \in X \\ & Y \in Y \\ & Z \in Z \end{aligned}$$

The inner program, the second level program, calculates the efficiency score for each set of $(x_{ij}^k, y_{rj}^k, z_{tj}); i = 1, \ldots, \mathrm{m}^k; r = 1, \ldots, s^k; t = 1, \ldots, D; k = 1, 2$ where the outer program determines the set of

$(x_j^k, y_j^k, z_j); k=1,2$ that produces the highest efficiency score. Therefore, the following NLP model based on θ^u can be expressed follows:

$$
\begin{aligned}
&\theta_p = Max\ \ Min\ \theta && (6)\\
&s.t \quad \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le \theta x_{ip}^k \quad \forall i,k\\
&\qquad \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_{rp}^k \quad \forall r,k\\
&\qquad \sum_{j=1}^{n} \lambda_j^1 z_{tj} \ge z_{tp} \quad \forall t\\
&\qquad \sum_{j=1}^{n} \lambda_j^2 z_{tj} \le \theta z_{tp} \quad \forall t\\
&\qquad \sum_{j=1}^{n} \lambda_j^k = 1 \quad \forall k\\
&\qquad \lambda_j^k \ge 0 \quad \forall j,k\\
&\theta : URS
\end{aligned}
$$

Now, we see that the inner and outer programs have different directions for optimization. To tackle this problem, the dual of the inner problem should be considered. So, the above model is changed as follows:

$$
\begin{aligned}
&Max\ \ Max\ \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k y_{rp}^k + \sum_{t=1}^{D} w_t^1 z_{tp} + \sum_{k=1}^{2} w_o^k && (7)\\
&s.t \quad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k x_{ip}^k + \sum_{t=1}^{D} w_t^2 z_{tp} = 1\\
&\qquad \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k y_{rj}^k - \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k x_{ij}^k + \sum_{t=1}^{D} w_t^1 z_{tj} - \sum_{t=1}^{D} w_t^2 z_{tj} + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j\\
&\qquad v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i,r,t,k \qquad w_o^k : free \quad k=1,2
\end{aligned}
$$

Model (7) with triangular fuzzy data $\tilde{x}_{ij}^k$, $\tilde{y}_{rj}^k$ and $\tilde{z}_{tj}^k$ is changed as Model (8):

(8)

$$
\begin{aligned}
&Max\ \ Max\ \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k\, \tilde{y}_{rp}^{k} + \sum_{t=1}^{D} w_t^1 \tilde{z}_{tp} + \sum_{k=1}^{2} w_o^k \\
&s.t \qquad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k\, \tilde{x}_{ip}^{k} + \sum_{t=1}^{D} w_t^2 \tilde{z}_{tp} = 1 \\
&\qquad \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k\, \tilde{y}_{rj}^{k} - \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k\, \tilde{x}_{ij}^{k} + \sum_{t=1}^{D} w_t^1 \tilde{z}_{tj} - \sum_{t=1}^{D} w_t^2 \tilde{z}_{tj} + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j \\
&\qquad v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1,2
\end{aligned}
$$

Let $\tilde{x}_{ij}^{k} = (x_{ij}^{k,m}, x_{ij}^{k,L}, x_{ij}^{k,u})$, $\tilde{y}_{rj}^{k} = (y_{rj}^{k,m}, y_{rj}^{k,L}, y_{rj}^{k,u})$ and $\tilde{z}_{tj}^{k} = (z_{tj}^{k,m}, z_{tj}^{k,L}, z_{tj}^{k,u})$ for $k = 1,2$.
According to fuzzy data, the above model can be written as bellow:

(9)

$$
\begin{aligned}
&Max\ \ Max\ \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k\, (y_{rp}^{k,m}, y_{rp}^{k,L}, y_{rp}^{k,u}) + \sum_{t=1}^{D} w_t^1 (z_{tp}^{m}, z_{tp}^{L}, z_{tp}^{u}) + \sum_{k=1}^{2} w_o^k \\
&s.t \qquad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k\, (x_{ip}^{k,m}, x_{ip}^{k,L}, x_{ip}^{k,u}) + \sum_{t=1}^{D} w_t^2 (z_{tj}^{m}, z_{tj}^{L}, z_{tj}^{u}) = (1^L, 1^u) \\
&\qquad \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k\, (y_{rj}^{k,m}, y_{rj}^{k,L}, y_{rj}^{k,u}) - \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k\, (x_{ij}^{k,m}, x_{ij}^{k,L}, x_{ij}^{k,u}) + \sum_{t=1}^{D} w_t^1 (z_{tj}^{m}, z_{tj}^{L}, z_{tj}^{u}) \\
&\qquad - \sum_{t=1}^{D} w_t^2 (z_{tj}^{m}, z_{tj}^{L}, z_{tj}^{u}) + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j \\
&\qquad v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1,2
\end{aligned}
$$

Where, $1^L \le 1$ and $1^u \ge 1$ are real number. Above model is a possibilistic linear programming. There are various methods to solve this model. Now, we apply the concept of the $\alpha -$ cut method so, above model with $\alpha -$cuts of objective function and constraints is obtained as follows:

(10)

$$\begin{aligned}
&Max\ \ Max\ \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k\,[\alpha\, y_{rp}^{k,m}+(1-\alpha)\, y_{rp}^{k,L},\alpha\, y_{rp}^{k,m}+(1-\alpha)\, y_{rp}^{k,u}]+\\
&\qquad \sum_{t=1}^{D} w_t^1[\alpha\, z_{tp}^{m}+(1-\alpha)\, z_{tp}^{L},\alpha\, z_{tp}^{m}+(1-\alpha)\, z_{tp}^{u}]+\sum_{k=1}^{2} w_o^k\\
&s.t\qquad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k\,[\alpha x_{ip}^{k,m}+(1-\alpha)x_{ip}^{k,L},\alpha x_{ip}^{k,m}+(1-\alpha)x_{ip}^{k,u}]+\\
&\qquad \sum_{t=1}^{D} w_t^2[\alpha\, z_{tp}^{m}+(1-\alpha)\, z_{tp}^{L},\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tp}^{u}]=(\alpha+(1+\alpha)1^L,\alpha+(1+\alpha)1^u)\\
&\qquad \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k\,[\alpha\, y_{rj}^{k,m}+(1-\alpha)\, y_{rj}^{k,L},\alpha\, y_{rj}^{k,m}+(1-\alpha)\, y_{rj}^{k,u}]-\\
&\qquad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k\,[\alpha x_{ij}^{k,m}+(1-\alpha)x_{ij}^{k,L},\alpha x_{ij}^{k,m}+(1-\alpha)x_{ij}^{k,u}]+\sum_{t=1}^{D} w_t^1[\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tj}^{L},\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tj}^{u}]\\
&\qquad -\sum_{t=1}^{D} w_t^2[\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tj}^{L},\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tj}^{u}]+\sum_{k=1}^{2} w_o^k\le 0\quad \forall j\\
&\qquad v_i^k,u_r^k,w_t^k\ge 0\quad \forall i,r,t,k\qquad w_o^k:free
\end{aligned}$$

By developing Saati et al., method [36], the new variables are introduced as follows:

$$\hat{x}_{ij}^{k}\in[\alpha x_{ij}^{k,m}+(1-\alpha)x_{ij}^{k,L},\alpha x_{ij}^{k,m}+(1-\alpha)x_{ij}^{k,u}]$$

$$\hat{y}_{rj}^{k}\in[\alpha\, y_{rj}^{k,m}+(1-\alpha)\, y_{rj}^{k,L},\alpha\, y_{rj}^{k,m}+(1-\alpha)\, y_{rj}^{k,u}]$$

$$\hat{z}_{tj}\in[\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tj}^{L},\alpha\, z_{tj}^{m}+(1-\alpha)\, z_{tj}^{u}]$$

$$\mathrm{L}\in[\alpha+(1+\alpha)1^L,\ \alpha+(1+\alpha)1^u]$$

By substituting the new variables, above model can be written as follows:

$$\max \; \max \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k \, \hat{y}_{rp}^k + \sum_{t=1}^{D} w_t^1 \hat{z}_{tp} + \sum_{k=1}^{2} w_o^k \tag{11}$$

$$s.t \quad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k \hat{x}_{ip}^k + \sum_{t=1}^{D} w_t^2 \hat{z}_{tp} = L$$

$$\sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k \, \hat{y}_{rj}^k - \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k \hat{x}_{ij}^k + \sum_{t=1}^{D} w_t^1 \hat{z}_{tj} - \sum_{t=1}^{D} w_t^2 \hat{z}_{tj} + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j$$

$$\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,L} \le \hat{x}_{ij}^k \le \alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,u} \quad \forall j, i, k$$

$$\alpha \, y_{rj}^{k,m} + (1-\alpha) \, y_{rj}^{k,L} \le \hat{y}_{rj}^k \le \alpha \, y_{rj}^{k,m} + (1-\alpha) \, y_{rj}^{k,u} \quad \forall j, r, k$$

$$\alpha \, z_{tj}^m + (1-\alpha) \, z_{tj}^L \le \hat{z}_{tj} \le \alpha \, z_{tj}^m + (1-\alpha) \, z_{tj}^u \quad \forall j, t, k$$

$$\alpha + (1+\alpha) 1^L \le \mathrm{L} \le \alpha + (1+\alpha) 1^u$$

$$v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free$$

Above model is a non-linear model. In order to linearize this model, following substitutions are performed.

$v_i^k \hat{x}_{ij}^k = \bar{x}_{ij}^k$, $u_r^k \hat{y}_{rj}^k = \bar{y}_{rj}^k$, $\mathrm{w}_t^1 \hat{z}_{tj} = \bar{z}_{tj}^1$, $\mathrm{w}_t^2 \hat{z}_{tj} = \bar{z}_{tj}^2$

By these substitutions, Model (11) becomes a LP model as follows:

$$\max \sum_{k=1}^{2}\sum_{r=1}^{s^k} \bar{y}_{rp}^k + \sum_{t=1}^{D} \bar{z}_{tp}^1 + \sum_{k=1}^{2} w_o^k \tag{12}$$

$$s.t \quad \sum_{k=1}^{2}\sum_{i=1}^{m^k} \bar{x}_{ip}^k + \sum_{t=1}^{D} \bar{z}_{tp}^2 = L$$

$$\sum_{k=1}^{2}\sum_{r=1}^{s^k} \bar{y}_{rj}^k - \sum_{k=1}^{2}\sum_{i=1}^{m^k} \bar{x}_{ij}^k + \sum_{t=1}^{D} \bar{z}_{tj}^1 - \sum_{t=1}^{D} \bar{z}_{tj}^2 + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j$$

$$v_i^k (\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,L}) \le \bar{x}_{ij}^k \le v_i^k (\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,u}) \quad \forall j, i, k$$

$$u_r^k (\alpha \, y_{rj}^{k,m} + (1-\alpha) \, y_{rj}^{k,L} \le \bar{y}_{rj}^k \le \alpha \, y_{rj}^{k,m} + (1-\alpha) \, y_{rj}^{k,u}) \quad \forall j, r, k$$

$$w_t^1 (\alpha \, z_{tj}^m + (1-\alpha) \, z_{tj}^L) \le \bar{z}_{tj}^1 \le w_t^1 (\alpha \, z_{tj}^m + (1-\alpha) \, z_{tj}^u) \quad \forall j, t$$

$$w_t^2 (\alpha \, z_{tj}^m + (1-\alpha) \, z_{tj}^L) \le \bar{z}_{tj}^2 \le w_t^2 (\alpha \, z_{tj}^m + (1-\alpha) \, z_{tj}^u) \quad \forall j, t$$

$$\alpha + (1+\alpha) 1^L \le \mathrm{L} \le \alpha + (1+\alpha) 1^u$$

$$v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1, 2$$

The objective function of Model (12) reflects the value of efficiency of the under evaluation DMU. Here, the $\bar{y}_{rp}^{k}, \forall r, k$ and $\bar{z}_{tp}^{1}, \forall t$ are variables in the intervals

$$u_r^k\,(\alpha\, \mathrm{y}_{rj}^{k,m} + (1-\alpha)\, \mathrm{y}_{rj}^{k,L} \le \bar{\mathrm{y}}_{rj}^{k} \le \alpha\, \mathrm{y}_{rj}^{k,m} + (1-\alpha)\, \mathrm{y}_{rj}^{k,u}) \quad \forall j, r, k \;\text{ and}$$

$$w_t^{1}(\alpha\, \mathrm{z}_{tj}^{m} + (1-\alpha)\, \mathrm{z}_{tj}^{L}) \le \bar{\mathrm{z}}_{tj}^{1} \le w_t^{1}(\alpha\, \mathrm{z}_{tj}^{m} + (1-\alpha)\, \mathrm{z}_{tj}^{u}) \qquad \forall j, t \;\text{ respectively.}$$

Hence, the efficiency value is represented in interval. The optimal value of these variables is the optimizing point that satisfies the set of constraints and, at the same time, the summation of these variables evaluates the efficiency of DMUp. Consider the first and second constraints of Model (12).

If $1^{u} > 1$ then, some of the efficiency score of DMUs be greater than 1. So, it must be equal to 1. Therefore, the last constraint of Model (12) can be written as follows:

$\alpha + (1+\alpha)1^{L} \le \mathrm{L} \le 1$ then above model can be equivalently written as:

$$
\begin{aligned}
e^{s_1^*} = Max\ & \sum_{k=1}^{2}\sum_{r=1}^{s^k} \bar{\mathrm{y}}_{rp}^{k} + \sum_{t=1}^{D} \bar{\mathrm{z}}_{tp}^{1} + \sum_{k=1}^{2} w_o^{k} \\
s.t\quad & \sum_{k=1}^{2}\sum_{i=1}^{m^k} \bar{x}_{ip}^{k} + \sum_{t=1}^{D} \bar{\mathrm{z}}_{tp}^{2} = 1 \\
& \sum_{k=1}^{2}\sum_{r=1}^{s^k} \bar{\mathrm{y}}_{rj}^{k} - \sum_{k=1}^{2}\sum_{i=1}^{m^k} \bar{x}_{ij}^{k} + \sum_{t=1}^{D} \bar{\mathrm{z}}_{tj}^{1} - \sum_{t=1}^{D} \bar{\mathrm{z}}_{tj}^{2} + \sum_{k=1}^{2} w_o^{k} \le 0 \quad \forall j \\
& v_i^k\,(\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,L}) \le \bar{\mathrm{x}}_{ij}^{k} \le v_i^k\,(\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,u}) \quad \forall j, i, k \\
& u_r^k\,(\alpha\, \mathrm{y}_{rj}^{k,m} + (1-\alpha)\, \mathrm{y}_{rj}^{k,L} \le \bar{\mathrm{y}}_{rj}^{k} \le \alpha\, \mathrm{y}_{rj}^{k,m} + (1-\alpha)\, \mathrm{y}_{rj}^{k,u}) \quad \forall j, r, k \\
& w_t^{1}(\alpha\, \mathrm{z}_{tj}^{m} + (1-\alpha)\, \mathrm{z}_{tj}^{L}) \le \bar{\mathrm{z}}_{tj}^{1} \le w_t^{1}(\alpha\, \mathrm{z}_{tj}^{m} + (1-\alpha)\, \mathrm{z}_{tj}^{u}) \qquad \forall j, t \\
& w_t^{2}(\alpha\, \mathrm{z}_{tj}^{m} + (1-\alpha)\, \mathrm{z}_{tj}^{L}) \le \bar{\mathrm{z}}_{tj}^{2} \le w_t^{2}(\alpha\, \mathrm{z}_{tj}^{m} + (1-\alpha)\, \mathrm{z}_{tj}^{u}) \qquad \forall j, t \\
& \qquad v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1, 2
\end{aligned}
\tag{13}
$$

This model is equivalent to a parametric programming while $\alpha \in [0,1]$ is a parameter. Thus, the fuzzy LP problem can be equivalent to a crisp parametric LP problem. It is noted that for each α , we have an optimal solution.

Theorem 3-1: Model (13) is feasible and bounded.

Proof: The dual form of Model (13) is written as follows:

$$
\begin{aligned}
\theta^* = & Min \quad \theta \qquad (14)\\
s.t \quad & \theta - \mu_p + \eta_p'^k - \eta_p^k \geq 0 \quad k = 1,2\\
& -\mu_j + \eta_j'^k - \eta_j^k \geq 0 \qquad \forall j \neq p, k = 1,2\\
& \mu_p + s_p'^k - s_p^k \geq 1 \quad k = 1,2\\
& \mu_j + s_j'^k - s_j^k \geq 1 \quad \forall j \neq p \quad k = 1,2\\
& \mu_p + n_p'^1 - n_p^1 \geq 1\\
& \theta - \mu_p + n_p'^2 - n_p^2 \geq 0\\
& -\mu_j + n_j'^k - n_j^k \geq 0 \quad j \neq p \quad k = 1,2\\
& -\eta_j'^k(\alpha x_{ij}^{k1,m} + (1-\alpha)x_{ij}^{k,u}) + \eta_j^k(\alpha x_{ij}^{k,m} + (1-\alpha)x_{ij}^{k,L}) - Q^k \leq 0\\
& -s_j'^k(\alpha y_{rj}^{k,m} + (1-\alpha)y_{rj}^{k,u}) + s_j^k(\alpha y_{rj}^{k,m} + (1-\alpha)y_{rj}^{k,L}) - \tau^k \leq 0\\
& -n_j'^k(\alpha z_{tj}^{k,m} + (1-\alpha)z_{tj}^{k,u}) + n_j^k(\alpha z_{tj}^{k,m} + (1-\alpha)z_{tj}^{k,L}) - P^k \leq 0\\
& \tau^k + P^k + Q^k \leq 0\\
& \mu_j = 1 \quad \forall j\\
& \alpha \in [0,1]
\end{aligned}
$$

Let $(\theta, \mu, \eta^k, \eta'^k, s^k, s'^k, n^k, n'^k, \tau^k, p^k, Q^k)$ is a feasible solution such that $\theta = 1, \mu_p = 1, \mu_j = \eta^k = \eta'^k = s^k = s'^k = n^k = n'^k = \tau^k = p^k = Q^k = 0$.

The optimal objective function value of Model (14) is equal or less than one ($\theta^* \leq 1$). According to the duality theorem, the value of the objective function of a dual and primal problem is equal to optimality. Therefore, we have $e^{s1^*} = \theta^* \leq 1$. So, Model (13) is bounded.■

Scenario 2: Control of intermediate product's value by neither of the two stages (cooperative approach)

In this scenario, there is no continuity in the quantity of sending and receiving intermediate products between two stages. Like the reason given in the previous scenario, we consider the upper bound of efficiency because our goal is to maximize the amount of objective function (profit). It should be mentioned that if the goal is to reduce the cost or to evaluate the lowest amount of production, the objective function of the probabilistic model will be as follows:

$$\theta^{L-S_2} = \min\{\min\ \theta \quad \text{s.t}\ \ constraints\ \ (1)\ and\ \ (3)\}$$
$$X \in X$$
$$Y \in Y$$
$$Z \in Z$$

Therefore, according to the PPS$^{s}{}_{2}$ (3) the efficiency measurement of DMU$_p$ can be obtained as follow:

$$\theta^{u-S_2} = \max\{\min\ \theta\ \ \text{s.t}\ constraints\ \ (1)\ and\ \ (3)\}$$
$$X \in X$$
$$Y \in Y$$
$$Z \in Z$$

Based on above, the efficiency score can be calculated through solving Model [15]:

$$\max \min \theta \tag{15}$$
$$
\begin{aligned}
s.t \quad & \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le \theta x_{ip}^k \quad \forall i, k \\
& \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_{rp}^k \quad \forall r, k \\
& \sum_{j=1}^{n} \lambda_j^1 z_{tj} \ge \sum_{j=1}^{n} \lambda_j^2 z_{tj} \quad \forall t \\
& \sum_{j=1}^{n} \lambda_j^k = 1 \quad \forall k \\
& \lambda_j^k \ge 0 \quad \forall j, k
\end{aligned}
$$

As in the first scenario, the two objective functions are different and to determine the efficiency, we should write the dual form of inner problem. Thus, Model (15) is written as follows:

$$Max\ Max\ \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k y_{rp}^k + \sum_{k=1}^{2} w_o^k \tag{16}$$

$$s.t \quad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k x_{ip}^k = 1$$

$$\sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k y_{rj}^k - \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k x_{ij}^k + \sum_{t=1}^{D} w_t^1 z_{tj} - \sum_{t=1}^{D} w_t^2 z_{tj} + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j$$

$$v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1, 2$$

Similar to scenario 1, Model (16) in presence of fuzzy triangular data and utilizing $\alpha-$ cut method transformed to the Model (17)

$$Max\ Max\ \sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k [\alpha\, y_{rp}^{k,m} + (1-\alpha)\, y_{rp}^{k,L}, \alpha\, y_{rp}^{k,m} + (1-\alpha)\, y_{rp}^{k,u}] + \sum_{k=1}^{2} w_o^k \tag{17}$$

$$s.t \quad \sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k [\alpha x_{ip}^{k,m} + (1-\alpha) x_{ip}^{k,L}, \alpha x_{ip}^{k,m} + (1-\alpha) x_{ip}^{k,u}] = (\alpha + (1+\alpha)1^L, \alpha + (1+\alpha)1^u)$$

$$\sum_{k=1}^{2}\sum_{r=1}^{s^k} u_r^k [\alpha\, y_{rj}^{k,m} + (1-\alpha)\, y_{rj}^{k,L}, \alpha\, y_{rj}^{k,m} + (1-\alpha)\, y_{rj}^{k,u}] -$$

$$\sum_{k=1}^{2}\sum_{i=1}^{m^k} v_i^k [\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,L}, \alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,u}] + \sum_{t=1}^{D} w_t^1 [\alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^L, \alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^u]$$

$$-\sum_{t=1}^{D} w_t^2 [\alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^L, \alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^u] + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j$$

$$v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free$$

Obviously, Model (17) is a non-linear model. In order to linearize this model, following substitutions are performed:

$$v_i^k \hat{x}_{ij}^k = \bar{x}_{ij}^k,\ u_r^k \hat{y}_{rj}^k = \bar{y}_{rj}^k,\ \mathrm{w}_t^1 \hat{z}_{tj} = \bar{z}_{tj}^1,\ \mathrm{w}_t^2 \hat{z}_{tj} = \bar{z}_{tj}^2$$

By these substitutions, above model become a LP as follows:

$$Max \sum_{k=1}^{2}\sum_{r=1}^{s^k}\overline{y}_{rp}^{k} + \sum_{k=1}^{2} w_o^k \tag{18}$$

$$s.t \quad \sum_{k=1}^{2}\sum_{i=1}^{m^k}\overline{x}_{ip}^{k} = L$$

$$\sum_{k=1}^{2}\sum_{r=1}^{s^k}\overline{y}_{rj}^{k} - \sum_{k=1}^{2}\sum_{i=1}^{m^k}\overline{x}_{ij}^{k} + \sum_{t=1}^{D}\overline{z}_{tj}^{1} - \sum_{t=1}^{D}\overline{z}_{tj}^{2} + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j$$

$$v_i^k(\alpha x_{ij}^{k,m} + (1-\alpha)x_{ij}^{k,L}) \le \overline{x}_{ij}^{k} \le v_i^k(\alpha x_{ij}^{k,m} + (1-\alpha)x_{ij}^{k,u}) \quad \forall j, i, k$$

$$u_r^k(\alpha\, y_{rj}^{k,m} + (1-\alpha)\, y_{rj}^{k,L} \le \overline{y}_{rj}^{k} \le \alpha\, y_{rj}^{k,m} + (1-\alpha)\, y_{rj}^{k,u}) \quad \forall j, r, k$$

$$w_t^1(\alpha\, z_{tj}^{m} + (1-\alpha)\, z_{tj}^{L}) \le \overline{z}_{tj}^{1} \le w_t^1(\alpha\, z_{tj}^{m} + (1-\alpha)\, z_{tj}^{u}) \qquad \forall j, t$$

$$w_t^2(\alpha\, z_{tj}^{m} + (1-\alpha)\, z_{tj}^{L}) \le \overline{z}_{tj}^{2} \le w_t^2(\alpha\, z_{tj}^{m} + (1-\alpha)\, z_{tj}^{u}) \qquad \forall j, t$$

$$\alpha + (1+\alpha)1^L \le \mathrm{L} \le \alpha + (1+\alpha)1^u$$

$$v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1, 2$$

Hence, the efficiency value is represented in terms of interval. The optimal value of these variables is the optimizing point that satisfies the set of constraints and at the same time, the summation of these variables evaluates the efficiency of DMU*p*.

Consider first and second constraints of Model (18). If $1^u > 1$ then, some of the efficiency score of DMUs be greater than 1. So, it must be equal to 1. Therefore, the last constraint of Model (18) can be written as follows:

$\alpha + (1+\alpha)1^L \le \mathrm{L} \le 1$ then above model can be equivalently written as follows:

$$
e^{s_2^*} = Max \sum_{k=1}^{2}\sum_{r=1}^{s^k} \bar{y}_{rp}^k + \sum_{k=1}^{2} w_o^k \tag{19}
$$

$$
\begin{aligned}
s.t \quad & \sum_{k=1}^{2}\sum_{i=1}^{m^k} \bar{x}_{ip}^k = 1 \\
& \sum_{k=1}^{2}\sum_{r=1}^{s^k} \bar{y}_{rj}^k - \sum_{k=1}^{2}\sum_{i=1}^{m^k} \bar{x}_{ij}^k + \sum_{t=1}^{D} \bar{z}_{tj}^1 - \sum_{t=1}^{D} \bar{z}_{tj}^2 + \sum_{k=1}^{2} w_o^k \le 0 \quad \forall j \\
& v_i^k (\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,L}) \le \bar{x}_{ij}^k \le v_i^k (\alpha x_{ij}^{k,m} + (1-\alpha) x_{ij}^{k,u}) \quad \forall j, i, k \\
& u_r^k (\alpha\, y_{rj}^{k,m} + (1-\alpha)\, y_{rj}^{k,L} \le \bar{y}_{rj}^k \le \alpha\, y_{rj}^{k,m} + (1-\alpha)\, y_{rj}^{k,u}) \quad \forall j, r, k \\
& w_t^1 (\alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^L) \le \bar{z}_{tj}^1 \le w_t^1 (\alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^u) \quad \forall j, t \\
& w_t^2 (\alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^L) \le \bar{z}_{tj}^2 \le w_t^2 (\alpha\, z_{tj}^m + (1-\alpha)\, z_{tj}^u) \quad \forall j, t \\
& v_i^k, u_r^k, w_t^k \ge 0 \quad \forall i, r, t, k \qquad w_o^k : free \quad k = 1, 2
\end{aligned}
$$

This model is equivalent to a parametric programming while $\alpha \in [0,1]$ is a parameter. Thus, the fuzzy LP problem can be equivalent to a crisp parametric LP problem. It is noted that for each α , we have an optimal solution.

Theorem 3-2: Model (18) is a feasible and bounded model.

Proof: The dual form of Model (18) is written as follows:

$$\theta^* = Min \quad \theta \tag{19}$$

$$\begin{aligned}
s.t \quad & \theta - \mu_p + \eta'^k_p - \eta^k_p \geq 0 \quad k = 1,2 \\
& -\mu_j + \eta'^k_j - \eta^k_j \geq 0 \qquad \forall j \neq p, k = 1,2 \\
& \mu_p + s'^k_p - s^k_p \geq 1 \quad k = 1,2 \\
& \mu_j + s'^k_j - s^k_j \geq 1 \quad \forall j \neq p \quad k = 1,2 \\
& \mu_j + n'^1_j - n^1_j \geq 0 \\
& -\mu_j + n'^k_j - n^k_j \geq 0 \quad j \neq p \quad k = 1,2 \\
& -\eta'^k_j (\alpha x^{k\,1,m}_{ij} + (1-\alpha) x^{\mathrm{k},u}_{ij}) + \eta^k_j (\alpha x^{\mathrm{k},m}_{ij} + (1-\alpha) x^{k,L}_{ij}) - Q^k \leq 0 \\
& -s'^k_j (\alpha y^{\mathrm{k},m}_{rj} + (1-\alpha) y^{\mathrm{k},u}_{rj}) + s^k_j (\alpha y^{\mathrm{k},m}_{rj} + (1-\alpha) y^{\mathrm{k},L}_{rj}) - \tau^k \leq 0 \\
& -n'^k_j (\alpha z^{\mathrm{k},m}_{tj} + (1-\alpha) z^{\mathrm{k},u}_{tj}) + n^k_j (\alpha z^{\mathrm{k},m}_{tj} + (1-\alpha) z^{\mathrm{k},L}_{tj}) - P^k \leq 0 \\
& \tau^k + P^k + Q^k \leq 0 \\
& \mu_j = 1 \quad \forall j \\
& \alpha \in [0,1]
\end{aligned}$$

Let $(\theta, \mu, \eta^k, \eta'^k, s^k, s'^k, n^k, n'^k, \tau^k, p^k, Q^k)$ is a feasible solution such that $\theta = 1, \mu_p = 1, \mu_j = \eta^k = \eta'^k = s^k = s'^k = n^k = n'^k = \tau^k = p^k = Q^k = 0$.

The optimal value of Model (19) is equal or less than one ($\theta^* \leq 1$). According to the duality theorem, we have $e^{s_2^*} \leq \theta^* \leq 1$. Therefore, Model (18) is bonded. ■

4- Case Study

In this section, we consider the practical example of Malek Mohammadi et al., [32]. In this example, the chipboard industry of wood lumber was considered as a two-stage process. The first stage is considered as the assembly, and the second stage is considered as the most pressing part.

The lumber wood, a number of nails, and glue are considered first-stage inputs to produce some tables and chairs. In the meantime, the remaining wood chips from the first stage products enter the second stage as inputs which is named intermediate product. In the second stage, wood chips are combined with materials like Special glue, Urea formaldehyde glue, Solid paraffin, and chemicals as second-stage inputs which lead to producing chipboard as final output (see Fig 3).

In this study, by considering the whole system (supply chain) and characterizing the intermediate product, unlike Malek Mohammadi et al., [32] method, we can consider wood chips as the desirable outputs of stage one and input of stage 2. In our system, the value of the intermediate product (wood chip) is determined simultaneously by both stages and independently. In this case study, it is quite clear that the goals of both stages conflict with each other. The reason is that in the first stage, the tables and chairs are made of pure wood and the lower amount of impurity of wood leads to the higher output quality and efficiency. On the other hand, chipboard production for the second stage from wood chips is more profitable and less expensive. The quality of tables and chairs made of pure wood is much higher than chipboard products. Therefore, the goals of Stage 1 and Stage 2 conflict with each other.

Eventually, to explore the efficiency of the systems, scenario 1 is considered and an efficiency score can be obtained via solving Model (13).

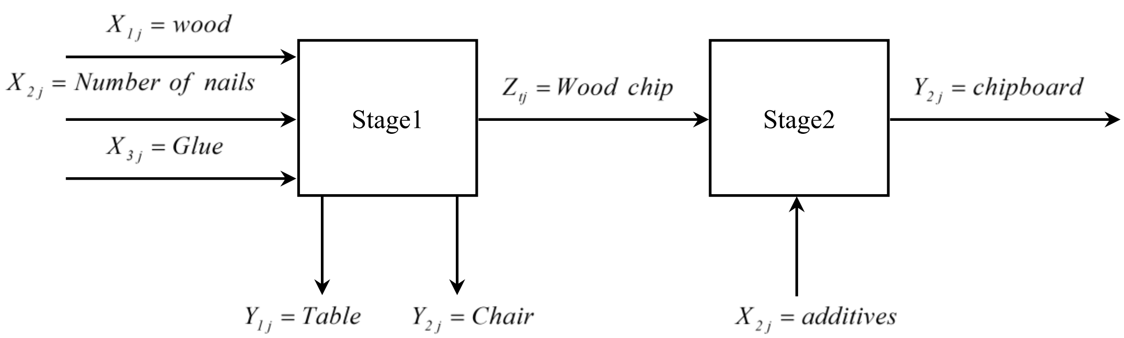

Figure 3: Two- stage process of workshop

It is anticipated that 15 workshops will be conducted in January and July, with each workshop serving as a decision-making unit (DMU) denoted by *j* (*j*=1,…,15) representing the number of workshops. The inputs and outputs are expressed as a fuzzy triangular number. The data used in this paper are the average figures for these items from June to July 2019 and are shown in Table 1. Averages may not adequately capture the variability and nuances present in the data, leading to potentially misleading results. By using fuzzy numbers, which allow for a range of

values rather than a single-point estimate, researchers can better account for uncertainty and variability in the data.

The input of stage 1 are presented as a $\tilde{x}_{1j}$ (lumber wood), $\tilde{x}_{2j}$ (nails) and $\tilde{x}_{3j}$ (glue) while wood chips ($\tilde{z}_{1j}$)are used as intermediate measures and outputs in the first stage are tables and chairs ($\tilde{y}_{1j}$, $\tilde{y}_{2j}$). In the second stage, additives ($\tilde{x}'_{2j}$) are inputs, and chipboard ($\tilde{y}'_{2j}$) is the output. Using the data from June to July to represent the domain of fuzzy numbers and their averages to show the vertex, triangular fuzzy numbers are constructed by $\tilde{x}_{1j}, \tilde{x}_{2j}, \tilde{x}_{3j}, \tilde{y}_{1j}, \tilde{y}_{2j}, \tilde{z}_{1j}, \tilde{x}'_{2j}, \tilde{y}'_{2j} \quad j = 1,\dots,15$, as shown in Table 1.

Table 1: Geometric mean of fuzzy inputs and outputs and intermediate product

Workshop	**Wood (lumber) (500 kg/m3)**	**Number of nails**	**Glue (cc)**	**Average table**	**Average chair**	**Wood chip (kg)**	**Additives**	**Chipboard**
DMU1	(835,838,840)	(32732,32741,32750)	(136609,136613,136621)	(501,512,522)	(497,505,512)	(1197,1199,1201)	(15,20,34)	(520,528,541)
DMU2	(833,839,844)	(34422,34429,34436)	(136610,136615,136620)	(500,532,538)	(498,504,515)	(1196,1200,1204)	(58,63,68)	(1127,1167,1198)
DMU3	(800,802,804)	(32402,32409,32415)	(128423,128449,128465)	(500,501,504)	(490,495,501)	(2000,2005,2010)	(110,112.5,115)	(2100,2135,2160)
DMU4	(730,756,775)	(30539,30545,30560)	(118345,118350,118355)	(440,446,452)	(525,530,535)	(810,810,824)	(37,44,59)	(715,802,890)
DMU5	(1009,1012,1014)	(36274,36280,36288)	(153970,153975,153980)	(520,530,545)	(460,462,467)	(1000,1013,1021)	(47,59,65)	(1021,1058,1099)
DMU6	(908,912,916)	(36281,36285,36291)	(153960,153970,153980)	(525,532,544)	(400,405,406)	(860,862,864)	(36,44,56)	(800,849,861)
DMU7	(909,914,918)	(36276,36285,36293)	(153960,153972, 153983)	(523,530,548)	(402,405,409)	(931,938,946)	(40,48,60)	(860,920,966)
DMU8	(870,874,877)	(31835,31839,31848)	(108331,108340,108349)	(430,437,440)	(490,494,500)	(995,1001,1005)	(47,54,63)	(1019,1056,1073)
DMU9	(970,972,974)	(37451,37460,37464)	(155170,155176,155183)	(640,642,645)	(505,507,511)	(939,941,943)	(43, 50,58)	(955,995,1015)
DMU10	(875,880,910)	(32300,32308,32319)	(137835,137842,137849)	(370,373,376)	(474,478,481)	(757,760,763)	(34,40,46)	(878,890,914)
DMU11	(1020,1032,1040)	(39742,39748,39755)	(151600,151607,151613)	(579,580,582)	(634,639,644)	(1028.1033,1038)	(50,55,65)	(1045,1089,1106)
DMU12	(968,977,985)	(38975,38981,38987)	(154430,154438,154444)	(428,453,466)	(358,365,372)	(910,916,924)	(43,47,53)	(950,995,1016)
DMU13	(868,870,872)	(35012,35020,35025)	(137000,137004,137009)	(431,449,469)	(528,541,555)	(931,937,940)	(46,48,52)	(1070,1112,1134)
DMU14	(881,895,913)	(35159,35164,35170)	(133020,133030,133039)	(536,540,546)	(587,590,594)	(830.835,840)	(35,40,55)	(735,779.5,806)
DMU15	(1000,1003,1006)	(38731,38741,38748)	(176460,176462, 176464)	(440,452,464)	(490,499,501)	(870.876,883)	(42,50,64)	(700,850,926)

Now, to assess and explore the efficient and inefficient 15 DMUs units and identify areas for improvement and optimization, we solve Model (13) with various $\alpha \in [0,1]$ amounts. The results are reported in Table 2 for various value of α .

Table 2: The efficiency scores of DMUs via solving Model (12)

workshop	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
DMU **1**	1.000	1.000	1.000	1.000	1.000
DMU 2	0.823	0.695	0.681	0.542	0.360
DMU 3	1.000	1.000	0.894	0.867	0.859
DMU 4	0.759	0.360	0.298	0.249	0.241
DMU 5	1.000	0.983	0.954	0.913	0.910
DMU 6	0.275	0.247	0. 229	0.187	0.103
DMU 7	1.000	1.000	1.000	1.000	1.000
DMU 8	0.578	0.469	0.434	0.378	0.298
DMU 9	1.000	1.000	0.823	0.786	0.362
DMU 10	1.000	1.000	1.000	1.000	0.973
DMU 11	0.258	0.241	0.212	0.148	0.135
DMU 12	0.978	0.975	0.971	0.895	0.824
DMU 13	1.000	1.000	1.000	0.639	0.592
DMU 14	1.000	1.000	1.000	1.000	1.000
DMU 15	0.998	0.874	0.733	0.697	0.512

In Table 2, as seen, the efficiencies are decreased by increasing α . But DMU_1 , DMU_{07} and DMU_{14} are efficient for all $\alpha \in (0,1]$. Briefly, out of 15 workshops, only three of them and 7 of them are efficient and inefficient for each α respectively. In other words, 20% of units are efficient and 46.6% of units are inefficient. Therefore, only 26.6% of the efficient units should increase their demand to the $\alpha \in [0.25,1]$ to reach higher efficiency.

For $\alpha = 0$, 8 units are identified as an efficient DMUs and the rest are inefficient units. Among the inefficient DMUs, DMU_6 and DMU_{11} had the worst performance with scores of 0.275 and 0.258, respectively. Also, DMU_6 and DMU_{11} have lower- efficiency scores than the rest of the units in all α values. According to Table 2, the number of determined efficient units decreases as the value of α increases.

As mentioned before, Malek Mohammadi et al., [32] ignored the nature of the intermediate product (wood chips) and considered it as an undesirable output of stage 1. They calculated the efficiency of both stages and the overall efficiency as fuzzy numbers which are shown in Table 3.

Table 3: The results of Malek Mohammadi's et al. (2022) method

Workshop (DMU)	$\tilde{e}_0$	$\tilde{e}_1$	$\tilde{e}_2$
1	(1,1,1)	(1,1,1)	(1,1,1)
2	(0.966,0.985,0.988)	(0.956,1,1)	(0.823,0.823,0.829)
3	(1,1,1)	(1,1,1)	(1,1,1)
4	(0.943,0.947,0.969)	(1,1,1)	(0.765,0.846,0.904)
5	(0,837,0.873,0.906)	(0.873,0.879,0.890)	(0.826,0.832,0.921)
6	(0.897, 0.899, 0.941)	(0.912,0.913,0.921)	(0.810,0.860,1)
7	(0.898,0.902,0.934)	(0.907,0.909,0.925)	(0.778,0.846,0.889)
8	(0.972, 0.974, 0.983)	(1,1,1)	(0.844,0.871,0.898)
9	(0.971, 0.973,0.984)	(0.999,1,1)	(0.849,0.886,0.905)
10	(0.971,0.976,0.980)	(0.913,0.938,0.936)	(1,1,1)
11	(0.820,0.821,0.824)	(1,1,1)	(0.836,0.873,0.887)
12	(0.820,0.821,0.824)	(0.779,0.785,0.793)	(0.887,0.926,0.920)
13	(0.820,0.932,0.946)	(0.881,0.896,0.915)	(1,1,1)
14	(0.961,0.975,0.983)	(1,1,1)	(0.793,0.889,0.909)
15	(0.750,0.751,0.774)	(0.743,0.747,0.752)	(0.702,0.802,0.828)

regarding Tables 2 and 3, it can be noted that the efficiency value of our proposed model is greater than Malek Mohammadi's et al., Model [32]. Because the obtained optimizing point provides the best situation for each DMU in contrast to Malek Mohammadi's model, which is based on the comparison of the intervals.

The proposed method to deal with uncertainty in the real world makes data analysis easier and clearly shows the efficiency of the system.

5- Conclusion

Computing the efficiency and assessing the performance of a system as a Decision-Making Unit (DMU) is so noticeable topic for managers in the real world.

One of the appropriate methods to measure the relative efficiency of a system with network structure is Network Data Envelopment Analysis (NDEA). The efficiency score of a system with NDEA models is calculated with accurate data. However, in the real world, data is sometimes inaccurate and ambiguous. To deal with this problem, Fuzzy Network DEA (FNDEA) method was introduced. Many studies have been carried out to determine the overall efficiency of the system and its sub-processes based on FNDEA models.

In this study, we considered two roles for intermediate products (control linked by two stages with non-cooperative and control linked by neither of stages with cooperative) and proposed a linear model for each scenario by developing Saati's et al. method.

In this study, for the first time, according to the nature of the intermediate products, development of Saati's et al., [36] and α -cut method, we introduced an LP model and calculated the definitive efficiency score for a two-stage system where each stage has separate inputs and outputs.

Then, we utilized the data of Malek Mohammadi et al., [32] example to demonstrate the application of our proposed models. Unlike the result of Malek Mohammadi et al., [32] method, our results are completely definite and more reliable. The results show that the efficiency score of each DMU decreases with the increase of the α value. DMU6 and DMU 11 have the lowest efficiency score i.e. the worst performance in these two months. According to the efficiency scores, we find that less than 50 percent that is, less than half of the DMUs are inefficient.

In this study, for the first time, considering the role of intermediate products, we considered two scenarios and presented a linear model for each one and calculated the accurate efficiency score. By using α-method and developing Saati et al. method, and role of intermediate products we determined the definite efficiency score in the presence of fuzzy data with less and easier calculation. In future studies, we can develop models for multi-stage systems. Also, a method can be developed for trapezoidal fuzzy data.

Data Availability

The data are shown in Table 1.

Conflict of interest

We have no conflict of interest to disclose.

Authors' Contributions

All authors have main contributions in writing the original draft preparation and in also writing a review and editing the paper. All authors read and approved the final manuscript.

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