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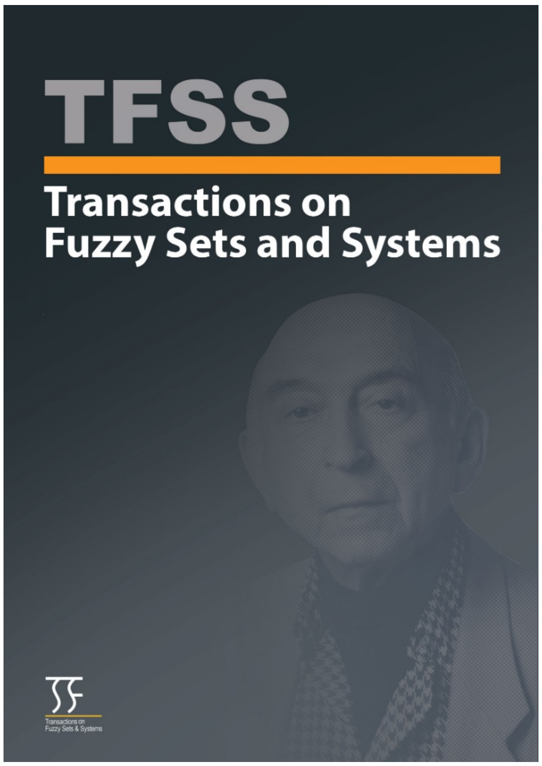

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A Modified Pythagorean Fuzzy Similarity Operator with Application in Questionnaire Analysis

Paul Augustine Ejegwa

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. This work presents a modified Pythagorean fuzzy similarity operator and utilizes its potential in the analysis of questionnaire. Similarity operator is a formidable methodology for decision-making under uncertain domains. Pythagorean fuzzy set is an extended form of intuitionistic fuzzy set with a better accuracy in complex real-world applications. Lots of discussions bordering on the uses of Pythagorean fuzzy sets have been explored based on Pythagorean fuzzy similarity operators. Among the extant Pythagorean fuzzy similarity operators, the work of Zhang et al. is significant but it contains some flaws which need to be corrected/modified to enhance reliable interpretation. To this end, this work explicates the Zhang et al.'s techniques of Pythagorean fuzzy similarity operator by pinpointing their drawbacks to develop an enhanced Pythagorean fuzzy similarity operator, which appropriately satisfies the similarity conditions and yields consistent results in comparison to the Zhang et al.'s techniques. Succinctly speaking, the aim of the work is to correct the flaws in Zhang et al.'s techniques via modifications. To theoretically validate the enhanced Pythagorean fuzzy similarity operator, we discuss it properties and find out that the similarity conditions are well satisfied. In addition, the enhanced PFSO and the Zhang et al.'s PFSOs are compared in the context of precision, and it is verified that the enhanced Pythagorean fuzzy similarity operator can successfully measure the similarity between vastly related but inconsistent PFSs and as well yields a very reasonable results. Furthermore, the enhanced Pythagorean fuzzy similarity operator is applied to the analysis of questionnaire on virtual library to ascertain the extent of awareness and effects of virtual library on students' academic performance via real data collected from fieldwork. Finally, it is certified that the enhanced Pythagorean fuzzy similarity operator can handle diverse everyday problems more precisely than the Zhang et al.'s Pythagorean fuzzy similarity operators.

AMS Subject Classification 2020: 03E72; 94D05; 03B52; 28E10
Keywords and Phrases: Decision making under uncertainty, Intuitionistic fuzzy set, Questionnaire analysis, Pythagorean fuzzy set, Similarity operator.

1 Introduction

The occurrence of vagueness and uncertainty in decision-making (DM) is a common experience witnessed by decision-makers. Due to this, fuzzy set (FS) [1] was introduced to curbed uncertainty but imprecision could not be tackled by FS. To resolve the problem of imprecision, intuitionistic fuzzy set (IFS) was developed [2], and it has been widely used to discuss practical DM problems. IFS is described by membership degree (MD) and non-membership degree (ND), where their sum cannot exceeds one. Several practical problems

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have been solved via IFS in real-life problems using distance operators [3–5], aggregation operators [6], and partial correlation coefficient operator [7]. In addition, other applications of IFSs have been discussed in medical emergency [8], selection of artificial intelligence [9], admission process [10], and decision-making [11]. The similarity metric is a vital research aspect in FS and its generalizations, and it is useful in determining the similarity index between two objects. Several techniques of similarity operator between IFS have been developed and gainfully applied in many fields, like pattern recognition [12], disaster control [13] and medical diagnostic problems [14]. From the ongoing, it is clear that similarity operators of IFSs have been effectively used in sundry fields, but there are some cases where IFSs cannot be utilized. For instance, if a decision maker has MD as 0.7 and ND as 0.5, then the IFS model cannot be applicable.

By extending the spatial scope of IFS, the term "IFS of type 2" or Pythagorean fuzzy set (PFS) was developed [15, 16]. In PFS, the sum of MD and ND may exceeds one but the square sum of MD and ND is at most one. PFS has a wider dimension of utilizations compare to IFS. In a way to discuss the usefulness of PFSs, a number of aggregation operators were discussed like Einstein operators, interactive power averaging operator, geometric aggregation operators using Einstein t-conorm and t-norm [17–19] to illustrate some DM problems. In the same vein, PFSs are pretty applicable in DM problems based on correlation coefficient operators [20–23] and distance operators [24–31]. Moreover, Hemalatha and Venkateswarlu [32] used PFSs to discuss transportation problem using mean square approach, Li et al. [33] presented an analysis of football activities using Pythagorean fuzzy approach, and various applications of PFSs have been discussed in decision-making [34–37].

In a clear term, PFS is a special case of IFS, which is fashioned to deal with some problems in which IFS is inadmissible. For that reason, the application of similarity operators on PFSs is of great important. The studies on similarity operators on PFS are carried out by modifying similar studies under IFSs. Zeng et al. [31] presented some methods of similarity operators between PFSs using some distance operators since both similarity operator and distance operator are dual in nature. Peng et al. [38] constructed some similarity operators for PFSs and used same in clustering analysis, medical diagnosis and pattern recognition. To compute the similarity between PFSs, Zhang [39] developed a similarity operator on PFSs and used it to discuss multi-criteria decision-making (MCDM) problems. Wei and Wei [40] constructed several similarity operators on PFSs through cosine function with applications in health science and pattern recognition. Recently, Zhang et al. [41] developed four methods of Pythagorean fuzzy similarity operator (PFSO), which were utilized to discuss pattern recognition problems. While the first two methods in [41] discarded the hesitation margins, the other two took into account the whole parameters of PFSs for e reliable outcomes. Nonetheless, the methods produce identical value other than one whenever the PFSs are equal, which is a violation of the similarity axioms and thus render the methods unreliable.

The interest of this work is to provide corrections to the four similarity operators between PFSs constructed in [41] by providing a new similarity operator between PFSs, which is the product of the hybridization of the four similarity operators. For emphasis, similarity operators in [41] have the following setbacks: (1) they fail to fulfill the similarity conditions if the PFSs are equal; (2) they yield similarity values that are not defined within the similarity value range, and thus lack practical interpretation. To this end, this paper proposes a hybridized similarity operator that corrects the work of Zhang et al. [41], and proves that the corrected version can successfully solve the mentioned setbacks observed in [41] via comparative examples using real collected data. This work contributes to the study of similarity operator under uncertain environments, soft computing, questionnaire analysis, and decision-making procedures.

The article is structured as follows: Section 2 recaps certain properties of PFSs; Section 3 discusses the Zhang et al.'s PFSOs and their setbacks; Section 4 provides solution to the setbacks in Zhang et al.'s PFSOs and discusses the properties of the modified PFSO; Section 5 discusses the application of the corrected Zhang et al.'s PFSOs in the analysis of questionnaire, and as well as, presents a comparative analysis to express the advantage of the corrected versions; and Section 6 concludes the paper with suggestions for future inquiries.

2 Preliminaries

This section discusses properties of PFSs and the Zhang et al's similarity functions. For clarity sake, assume A to be the universe of discourse, $\wp$ as IFS, and ℓ as PFS.

Definition 2.1. [2]*An IFS $\wp$ in A is defined by $\wp = \{(a, M_\wp(a), N_\wp(a)) : a \in A\}$, where $M_\wp \colon A \to [0,1]$ and $N_\wp \colon A \to [0,1]$ are MD and NMD of $a \in A$ in which*

$$0 \leq M_\wp(a) + N_\wp(a) \leq 1.$$

In addition, HM of $\wp$ in A is defined by $H_\wp(a) = 1 - M_\wp(a) - N_\wp(a)$.

Definition 2.2. [16] *A PFS ℓ in A is defined by $\ell = \{(a, M_\ell(a), N_\ell(a)) : a \in A\}$, where $M_\ell \colon A \to [0,1]$ and $N_\ell \colon A \to [0,1]$ are MD and NMD of $a \in A$ in which*

$$0 \leq M_\ell^2(a) + N_\ell^2(a) \leq 1.$$

In addition, HM of ℓ in A is defined by $H_\ell(a) = \sqrt{1 - M_\ell^2(a) - N_\ell^2(a)}$.

Now, we present some operations on PFSs as follows:

Definition 2.3. [16] *If ℓ, ℓ_1 and ℓ_2 are PFSs in A, then*

(i) $\ell_1 \preceq \ell_2$ *iff* $M_{\ell_1}(a) \preceq M_{\ell_2}(a)$ *and* $N_{\ell_1}(a) \preceq N_{\ell_2}(a)$ $\forall a \in A$.

(ii) $\ell_1 = \ell_2$ *iff* $M_{\ell_1}(a) = M_{\ell_2}(a)$ *and* $N_{\ell_1}(a) = N_{\ell_2}(a)$ $\forall a \in A$.

(iii) $\ell_1 \subseteq \ell_2$ *iff* $M_{\ell_1}(a) \leq M_{\ell_2}(a)$ *and* $N_{\ell_1}(a) \geq N_{\ell_2}(a)$ $\forall a \in A$.

(iv) $\overline{\ell} = \{(a, N_\ell(a), M_\ell(a)) : a \in A\}$.

(v) $\ell_1 \cap \ell_2 = \{(a, \min\{M_{\ell_1}(a), M_{\ell_2}(a)\}, \max\{N_{\ell_1}(a), N_{\ell_2}(a)\}) : a \in A\}$.

(vi) $\ell_1 \cup \ell_2 = \{(a, \max\{M_{\ell_1}(a), N_{\ell_2}(a)\}, \min\{N_{\ell_1}(a), N_{\ell_2}(a)\}) : a \in A\}$.

One of the means to estimate the similarity between PFSs is via similarity measure between them.

Definition 2.4. [1] *If ℓ, ℓ_1 and ℓ_2 are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$, then the similarity metric between ℓ_1 and ℓ_2 represented by $\Gamma(\ell_1, \ell_2)$ is a function, $\Gamma \colon PFS \times PFS \to [0,1]$ such that:*

(i) $\Gamma(\ell_1, \ell_1) = 1$, $\Gamma(\ell_2, \ell_2) = 1$,

(ii) $\Gamma(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$,

(iii) $0 \leq \Gamma(\ell_1, \ell_2) \leq 1$,

(iv) $\Gamma(\ell_1, \ell_2) = \Gamma(\ell_2, \ell_1)$,

(v) $\Gamma(\ell_1, \ell) \leq \Gamma(\ell_1, \ell_2) + \Gamma(\ell_2, \ell)$.

In short, $\Gamma(\ell_1, \ell_2) \approx 1$ implies there is high similarity between ℓ_1 and ℓ_2, and $\Gamma(\ell_1, \ell_2) \approx 0$ implies there is a negligible similarity between ℓ_1 and ℓ_2.

3 Zhang et al.'s PFSOs and Numerical Illustrations

The exponential-based techniques of similarity operators under PFSs were presented by Zhang et al. [41] because of the failures of some existing approaches of PFSOs. Zhang et al. developed four exponential based-similarity operators, enumerated as follows:

$$\Gamma_1(\ell_1, \ell_2) = \frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\max\{|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)|,|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)|\}} - 1\Big], \tag{1}$$

$$\Gamma_2(\ell_1, \ell_2) = \frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\frac{|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)|+|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)|}{2}} - 1\Big], \tag{2}$$

$$\Gamma_3(\ell_1, \ell_2) = \frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\max\{|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)|,|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)|,|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)|\}} - 1\Big], \tag{3}$$

$$\Gamma_3(\ell_1, \ell_2) = \frac{1}{Q}\Sigma_{j=1}^{Q}\Big[2^{1-\frac{|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)|+|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)|+|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)|}{2}} - 1\Big], \tag{4}$$

where ℓ_1 and ℓ_2 are the PFSs defined in A, $a_j \in A$ and $|A| = Q$. The similarity methods in (1) and (2) excluded the hesitation margins, which makes the approaches defective. Nonetheless, (1) and (2) were enhanced as (3) and (4), respectively, to yield reliable results. Howbeit, all these methods yield similar results, most especially, as the hesitation margins become small and for smaller Q. We show the defectiveness of these methods in the following examples:

Example 3.1. Suppose we have two PFSs ℓ_1 and ℓ_2 defined in $A = \{a_1, a_2, a_3\}$ as follows:

$$\ell_1 = \{(a_1, 0.5, 0.4), (a_2, 0.8, 0.1), (a_3, 0.7, 0.2)\} = \ell_2$$

This is a case of equal PFSs, and we are expected to have $\Gamma_1(\ell_1, \ell_2) = \Gamma_2(\ell_1, \ell_2) = \Gamma_3(\ell_1, \ell_2) = \Gamma_4(\ell_1, \ell_2) = 1$. Then, by applying (1)–(4) we get

$$\left.\begin{aligned}
\Gamma_1(\ell_1, \ell_2) &= \frac{2^{1-0}-1}{3} = 0.3333\\
\Gamma_2(\ell_1, \ell_2) &= \frac{2^{1-0}-1}{3} = 0.3333\\
\Gamma_3(\ell_1, \ell_2) &= \frac{2^{1-0}-1}{3} = 0.3333\\
\Gamma_4(\ell_1, \ell_2) &= \frac{2^{1-0}-1}{3} = 0.3333,
\end{aligned}\right\},$$

which violate a similarity condition, i.e., $\Gamma(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$. Hence, these approaches need to be corrected to satisfy the condition.

Again, we observe that these approaches sometimes produce results that are not within $0 \le \Gamma(\ell_1, \ell_2) \le 1$, as seen in Example 3.2.

Example 3.2. Suppose that

$$\ell_1 = \{(a_1, 1, 0), (a_2, 0.8, 0), (a_3, 0.7, 0.1)\},$$
$$\ell_2 = \{(a_1, 0.8, 0.1), (a_2, 1, 0), (0.9, 0.1)\},$$
$$\ell_3 = \{(a_1, 0.6, 0.2), (a_2, 0.8, 0), (1, 0)\}$$

are PFSs in $A = \{a_1, a_2, a_3\}$. In case there is another PFS defined by

$$\ell = \{(a_1, 0.5, 0.3), (a_2, 0.8, 0.2), (1, 0)\}.$$

Now, we apply the approaches to find the similarities between ℓ with each of ℓ_1, ℓ_2, and ℓ_3, respectively, and get the following results:

$$\left.\begin{aligned} \Gamma_1(\ell_j,\ell) &= -0.0626, 0.0142, 0.2675 \\ \Gamma_2(\ell_j,\ell) &= 0.077, 0.1268, 0.2887 \\ \Gamma_3(\ell_j,\ell) &= -0.055, 0.0142, 0.2844 \\ \Gamma_4(\ell_j,\ell) &= -0.0626, 0.0142, 0.2675 \end{aligned}\right\},$$

for $j = 1, 2, 3$. The negative similarity values proof the failure of the PFSOs. To solve these defectiveness, the Zhang et al.'s techniques are corrected as follows:

4 Corrections to Zhang et al.'s PFSOs

Because of the problems associated with Zhang et al.'s methods, it is necessary to correct the methods to enhance reliability, precision, and the satisfication of similarity conditions.

Definition 4.1. Suppose $\ell_1 = \{(a_j, M_{\ell_1}(a_j), N_{\ell_1}(a_j)) : a_j \in A\}$ and $\ell_2 = \{(a_j, M_{\ell_2}(a_j), N_{\ell_2}(a_j)) : a_j \in A\}$ are PFSs for $A = \{a_1, a_2, \cdots, a_Q\}$, then the new similarity operator between ℓ_1 and ℓ_2, which corrects the Zhang et al.'s PFSOs is defined by:

$$\tilde{\Gamma}_*(\ell_1,\ell_2) = \sum_{j=i}^{Q} \left[2^{1-\frac{1}{3Q}\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} - 1\right]. \tag{5}$$

By incorporating the influence of weight of the elements of A, we have:

$$\tilde{\Gamma}(\ell_1,\ell_2) = \sum_{j=i}^{Q} \left[2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} - 1\right], \tag{6}$$

where $\omega_j \in [0,1]$ and $\sum_{j=1}^{Q} \omega_j = 1$.

If $\omega_j = \left(\frac{1}{Q}, \frac{1}{Q}, \cdots, \frac{1}{Q}\right)^T$, then (6) becomes (5). The first advantage of this corrected version is that, it incorporates the complete parameters of the sets. We use (5) and (6) to find the similarity between equal PFSs in Example 3.1 and get

$$\tilde{\Gamma}(\ell_1,\ell_2) = \tilde{\Gamma}_*(\ell_1,\ell_2) = 1,$$

which satisfies $\Gamma(\ell_1,\ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$. This is the second advantage of the corrected version over Zhang et al.'s methods.

In addition, the corrected approaches produce results that are within $0 \leq \Gamma(\ell_1,\ell_2) \leq 1$. To see this, we consider Example 3.2 with $\omega_j = \{0.2, 0.4, 0.4\}$, and get the following results:

$$\tilde{\Gamma}(\ell_j,\ell) = 0.6857, 0.7427, 0.8251,$$
$$\tilde{\Gamma}_*(\ell_j,\ell) = 0.6371, 0.7304, 0.7956,$$

for $j = 1, 2, 3$. Clearly, these results are better than the results from Zhang et al.'s approaches. The results from Zhang et al.'s methods and the corrected form are displayed in Table 1.

Table 1: Results for Comparison

PFSOs	Example 3.1	Example 3.2
Γ_1 [41]	0.3333	$-0.0626, 0.0142, 0.2675$
Γ_2 [41]	0.3333	$0.0770, 0.1268, 0.2887$
Γ_3 [41]	0.3333	$-0.0550, 0.0142, 0.2844$
Γ_4 [41]	0.3333	$-0.0626, 0.0142, 0.2675$
$\tilde{\Gamma}_*$	1.0000	$0.6371, 0.7304, 0.7956$

The results in Table 1 justify the faults with the methods in [41] and the superiority of the corrected form. While the results of Zhang et al.'s methods (i.e., Example 3.2) show that weak resemblance exist between the PFSs, the new method shows that the PFSs are well related in agreement to mere observation. Now, we characterize the corrected similarity operator theoretically.

Theorem 4.2. *Suppose ℓ_1 and ℓ_2 are PFSs in $A = \{a_1, a_2, \cdots, a_Q\}$, then*

(i) $\tilde{\Gamma}(\ell_1, \ell_2) = \tilde{\Gamma}(\ell_2, \ell_1)$,

(ii) $\tilde{\Gamma}(\ell_1, \ell_2) = \tilde{\Gamma}(\overline{\ell}_1, \overline{\ell}_2)$,

(iii) $0 \leq \tilde{\Gamma}(\ell_1, \ell_2) \leq 1$,

(iv) $\tilde{\Gamma}(\ell_1, \ell_2) = 1 \Leftrightarrow \ell_1 = \ell_2$.

Proof. The prove of (i) follows because

$$\begin{aligned}
\tilde{\Gamma}(\ell_1, \ell_2) &= \sum_{j=i}^{Q} \left[2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} - 1 \right] \\
&= \sum_{j=i}^{Q} \left[2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_2}(a_j)-M^2_{\ell_1}(a_j)\right|+\left|N^2_{\ell_2}(a_j)-N^2_{\ell_1}(a_j)\right|+\left|H^2_{\ell_2}(a_j)-H^2_{\ell_1}(a_j)\right|\right)} - 1 \right] \\
&= \tilde{\Gamma}(\ell_2, \ell_1).
\end{aligned}$$

Similarly, (ii) holds.

To prove $0 \leq \tilde{\Gamma}(\ell_1, \ell_2) \leq 1$, it is sufficient to show that $\tilde{\Gamma}(\ell_1, \ell_2) \leq 1$ since $\tilde{\Gamma}(\ell_1, \ell_2) \geq 0$ is straightforward.

Assume that $y = \tilde{\Gamma}(\ell_1, \ell_2)$ and $x = \dfrac{1}{3}\sum_{j=1}^{Q} \omega_j\left(\left|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\right| + \left|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\right| + \left|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\right|\right)$, where $x \in [0, 1]$. Then, we have $y = 2^{1-x} - 1$, which is a curve function with values range from 0 to 1. Thus, $0 \leq y \leq 1$ and hence, $0 \leq \tilde{\Gamma}(\ell_1, \ell_2) \leq 1$ as desired, i.e., (iii) holds.

Next, we establish (iv). Suppose $\tilde{\Gamma}(\ell_1, \ell_2) = 1$. Then, we have

$$2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} - 1 = 1 \Longrightarrow$$

$$2^{1-\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|+\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)} = 2 \Longrightarrow$$

$$1 - \frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\right| + \left|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\right| + \left|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\right|\right) = 1 \Longrightarrow$$

$$\frac{1}{3}\omega_j\left(\left|M^2_{\ell_1}(a_j) - M^2_{\ell_2}(a_j)\right| + \left|N^2_{\ell_1}(a_j) - N^2_{\ell_2}(a_j)\right| + \left|H^2_{\ell_1}(a_j) - H^2_{\ell_2}(a_j)\right|\right) = 0 \Longrightarrow$$

$$\big(\big|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\big|+\big|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\big|+\big|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\big|\big)=0 \Longrightarrow$$
$$M_{\ell_1}(a_j)=M_{\ell_2}(a_j), N_{\ell_1}(a_j)=N_{\ell_2}(a_j), H_{\ell_1}(a_j)=H_{\ell_2}(a_j).$$

Hence, $\ell_1=\ell_2$.

Conversely, if $\ell_1=\ell_2$. Then, $\big|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\big|=0$, $\big|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\big|=0$, and $\big|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\big|=0$. Thus,

$$\frac{1}{3}\omega_j\big(\big|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\big|+\big|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\big|+\big|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\big|\big)=0,$$

and hence $\tilde{\Gamma}(\ell_1,\ell_2)=1$.

□

Theorem 4.3. *Suppose ℓ_1 and ℓ_2 are PFSs in $A=\{a_1,a_2,\cdots,a_Q\}$, then*

(i) $\tilde{\Gamma}_*(\ell_1,\ell_2)=\tilde{\Gamma}_*(\ell_2,\ell_1)$,

(ii) $\tilde{\Gamma}_*(\ell_1,\ell_2)=\tilde{\Gamma}_*(\bar{\ell}_1,\bar{\ell}_2)$,

(iii) $0\le\tilde{\Gamma}_*(\ell_1,\ell_2)\le 1$,

(iv) $\tilde{\Gamma}_*(\ell_1,\ell_2)=1 \Leftrightarrow \ell_1=\ell_2$.

Proof. Follow from Theorem 4.2. □

Theorem 4.4. *Given that ℓ_1, ℓ_2, and ℓ_3 are PFSs in $A=\{a_1,a_2,\cdots,a_Q\}$ such that $\ell_1\subseteq\ell_2\subseteq\ell_3$. Then*

(i) $\tilde{\Gamma}(\ell_1,\ell_3)\le\tilde{\Gamma}(\ell_1,\ell_2)$,

(ii) $\tilde{\Gamma}(\ell_1,\ell_3)\le\tilde{\Gamma}(\ell_2,\ell_3)$,

(iii) $\tilde{\Gamma}_*(\ell_1,\ell_3)\le\tilde{\Gamma}_*(\ell_1,\ell_2)$,

(iv) $\tilde{\Gamma}_*(\ell_1,\ell_3)\le\tilde{\Gamma}_*(\ell_2,\ell_3)$.

Proof. Because $\ell_1\subseteq\ell_2\subseteq\ell_3$, we have $M_{\ell_1}(a_j)\le M_{\ell_2}(a_j)\le M_{\ell_3}(a_j)$ and $N_{\ell_1}(a_j)\le N_{\ell_2}(a_j)\le N_{\ell_3}(a_j)$ $\forall a_j\in A$. Thus,

$$\begin{aligned}\big|M^2_{\ell_1}(a_j)-M^2_{\ell_3}(a_j)\big|&\ge\big|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\big|,\\ \big|N^2_{\ell_1}(a_j)-N^2_{\ell_3}(a_j)\big|&\ge\big|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\big|,\\ \big|H^2_{\ell_1}(a_j)-H^2_{\ell_3}(a_j)\big|&\ge\big|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\big|,\end{aligned}$$

such that

$$\begin{aligned}&\big|M^2_{\ell_1}(a_j)-M^2_{\ell_3}(a_j)\big|+\big|N^2_{\ell_1}(a_j)-N^2_{\ell_3}(a_j)\big|\\ +&\big|H^2_{\ell_1}(a_j)-H^2_{\ell_3}(a_j)\big|\ge\big|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\big|\\ +&\big|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\big|+\big|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\big|.\end{aligned}$$

Clearly, $\tilde{\Gamma}(\ell_1,\ell_3)\le\tilde{\Gamma}(\ell_1,\ell_2)$ which proves (i). By using the same logic, the proofs of (ii), (iii), and (iv) hold. □

Corollary 4.5. *If ℓ_1, ℓ_2, and ℓ_3 are PFSs in $A=\{a_1,a_2,\cdots,a_Q\}$ and $\ell_1\subseteq\ell_2\subseteq\ell_3$. Then* $\tilde{\Gamma}(\ell_1,\ell_3)\le\min\big\{\tilde{\Gamma}(\ell_2,\ell_3),\tilde{\Gamma}(\ell_1,\ell_2)\big\}$ *and* $\tilde{\Gamma}_*(\ell_1,\ell_3)\le\min\big\{\tilde{\Gamma}_*(\ell_2,\ell_3),\tilde{\Gamma}_*(\ell_1,\ell_2)\big\}$.

Proof. From Theorem 4.4, $\tilde{\Gamma}(\ell_1,\ell_3)\leq\tilde{\Gamma}(\ell_1,\ell_2)$ and $\tilde{\Gamma}(\ell_1,\ell_3)\leq\tilde{\Gamma}(\ell_2,\ell_3)$. Hence, $\tilde{\Gamma}(\ell_1,\ell_3)\leq\min\left\{\tilde{\Gamma}(\ell_2,\ell_3),\tilde{\Gamma}(\ell_1,\ell_2)\right\}$. Similarly, $\tilde{\Gamma}_*(\ell_1,\ell_3)\leq\min\left\{\tilde{\Gamma}_*(\ell_2,\ell_3),\tilde{\Gamma}_*(\ell_1,\ell_2)\right\}$. □

Theorem 4.6. *Suppose $\ell_1\subseteq\ell_2\subseteq\ell_3$ are PFSs in $A=\{a_1,a_2,\cdots,a_Q\}$, then*

(i) $\tilde{\Gamma}(\ell_1,\ell_2)+\tilde{\Gamma}(\ell_2,\ell_3)\geq\tilde{\Gamma}(\ell_1,\ell_3)$,

(ii) $\tilde{\Gamma}_*(\ell_1,\ell_2)+\tilde{\Gamma}_*(\ell_2,\ell_3)\geq\tilde{\Gamma}_*(\ell_1,\ell_3)$,

(iii) $\tilde{\Gamma}(\ell_1,\ell_2)=\tilde{\Gamma}(\ell_1\cap\ell_2,\ell_1\cup\ell_2)$,

(iv) $\tilde{\Gamma}_*(\ell_1,\ell_2)=\tilde{\Gamma}_*(\ell_1\cap\ell_2,\ell_1\cup\ell_2)$.

Proof. Suppose $\ell_1\subseteq\ell_2\subseteq\ell_3$. Then, $\tilde{\Gamma}(\ell_1,\ell_3)\leq\tilde{\Gamma}(\ell_1,\ell_2)$ and $\tilde{\Gamma}(\ell_1,\ell_3)\leq\tilde{\Gamma}(\ell_2,\ell_3)$ from Theorem 4.4. Thus, $\tilde{\Gamma}(\ell_1,\ell_3)\leq\tilde{\Gamma}(\ell_1,\ell_2)+\tilde{\Gamma}(\ell_2,\ell_3)$, which proves (i). Similarly, (ii) follows from (i).

The proof of (iii) follows by using intersection and union of PFSs in terms of $\tilde{\Gamma}$. Thus,

$$\begin{aligned}\tilde{\Gamma}(\ell_1\cap\ell_2,\ell_1\cup\ell_2)&=\sum_{j=i}^{Q}\Big[2\times 2^{\frac{2}{3}\omega_j\left|\left(\min\{M_{\ell_1}(a_j),M_{\ell_2}(a_j)\}\right)^2-\left(\max\{M_{\ell_1}(a_j),M_{\ell_2}(a_j)\}\right)^2\right|}\\&\quad\times 2^{-\omega_j\left(\left|\left(\max\{N_{\ell_1}(a_j),N_{\ell_2}(a_j)\}\right)^2-\left(\min\{N_{\ell_1}(a_j),N_{\ell_2}(a_j)\}\right)^2\right|+\left|H^2_{\ell_1\cap\ell_2}(a_j)-H^2_{\ell_1\cup\ell_2}(a_j)\right|\right)}-1\Big]\\&=\sum_{j=i}^{Q}\Big[2\times 2^{\frac{2}{3}\omega_j\left|M^2_{\ell_1}(a_j)-M^2_{\ell_2}(a_j)\right|}\times 2^{-\omega_j\left(\left|N^2_{\ell_1}(a_j)-N^2_{\ell_2}(a_j)\right|+\left|H^2_{\ell_1}(a_j)-H^2_{\ell_2}(a_j)\right|\right)}-1\Big]\\&=\tilde{\Gamma}(\ell_1,\ell_2),\end{aligned}$$

which proves (iii). The proof of (iv) is similar to (iii). □

5 Application in Questionnaire Analysis

This section deliberates on the use of the new PFSO in the analysis of questionnaire due to the fuzziness in filling questionnaire. The questionnaire is constructed to measure the extents of awareness and use of virtual library esources (VLR) by undergraduate medical students. Virtual library (VL) is the incorporation of ICT into library services, and this has brought remarkable progress in the academic performance of students in universities [42]. The majority of works done on virtual library made used of questionnaire to decide their aim and objectives. The process of filling questionnaire is characterized with hesitation on the part of the respondents and equally, some of the questions in the questionnaire could be ambiguous. This is the reason why PFS is necessary for questionnaire analysis. This work is governed by the following questions, namely: (i) what is the level of awareness of the VLR in the department by the students? (ii) what are the effects of VL on the medical students' academic wellbeing in the department?

5.1 Data description and presentation

The data for the analysis is drawn from 198 students in the Department of Medicine and Surgery, Benue State University, Makurdi, Nigeria. 198 students out of the 392 students in the department are gotten by using the Yamane's sampling technique [43]. The collected data are presented in Tables 2 and 3, where strongly agree is represented by ℓ_1, agree is ℓ_2, disagree is ℓ_3, strongly disagree is ℓ_4, and the questions are Q_1, Q_2, Q_3, Q_4 and Q_5, respectively.

Table 2: Level of Awareness on the Availability of VL

Questions/Scales	ℓ_1	%	ℓ_2	%	ℓ_3	%	ℓ_4	%
Q_1	95	48	63	31.8	22	11.1	18	9.1
Q_2	68	34.3	51	25.8	49	24.7	30	15.2
Q_3	39	19.7	50	25.3	78	39.4	31	15.7
Q_4	34	17.2	78	39.4	65	32.8	21	10.6
Q_5	105	53	73	36.9	16	8.1	4	2

Table 3: Effects of VL on Academic Performance

Questions/Scales	ℓ_1	%	ℓ_2	%	ℓ_3	%	ℓ_4	%
Q_1	47	23.7	81	40.9	45	22.7	25	12.6
Q_2	51	25.8	75	37.9	46	23.2	26	13.1
Q_3	34	17.2	83	41.9	50	25.3	31	15.7
Q_4	42	21.2	72	36.4	57	28.8	27	13.6
Q_5	52	26.3	62	31.3	56	28.3	28	14.1

Due to the fuzziness in filling the questionnaire, we transform the data in Tables 2 and 3 into PFD as displayed in Tables 4 and 5, by taking the percentages of each of the scales as the MGs while $1-$MGs are the NMGs.

Table 4: Data on Level of Awareness of VL

Scales	Q_1	Q_2	Q_3	Q_4	Q_5
ℓ_1	$(0.480, 0.520)$	$(0.343, 0.657)$	$(0.197, 0.803)$	$(0.172, 0.828)$	$(0.53, 0.47)$
ℓ_2	$(0.318, 0.682)$	$(0.258, 0.742)$	$(0.253, 0.747)$	$(0.394, 0.606)$	$(0.369, 0.631)$
ℓ_3	$(0.111, 0.889)$	$(0.247, 0.753)$	$(0.394, 0.606)$	$(0.328, 0.672)$	$(0.081, 0.919)$
ℓ_4	$(0.091, 0.909)$	$(0.152, 0.848)$	$(0.157, 0.843)$	$(0.106, 0.894)$	$(0.02, 0.98)$

Table 5: Data on Effects of Virtual Library

Scales	Q_1	Q_2	Q_3	Q_4	Q_5
ℓ_1	$(0.237, 0.763)$	$(0.258, 0.742)$	$(0.172, 0.828)$	$(0.212, 0.788)$	$(0.263, 0.737)$
ℓ_2	$(0.409, 0.591)$	$(0.379, 0.621)$	$(0.419, 0.581)$	$(0.364, 0.636)$	$(0.313, 0.687)$
ℓ_3	$(0.227, 0.773)$	$(0.232, 0.768)$	$(0.253, 0.747)$	$(0.288, 0.712)$	$(0.283, 0.717)$
ℓ_4	$(0.126, 0.874)$	$(0.131, 0.869)$	$(0.157, 0.843)$	$(0.136, 0.864)$	$(0.141, 0.859)$

Now, we find the similarity between the scales in Tables 4 and 5 using the new similarity operator (5) and get the outcomes in Table 6, which are presented in Figure 1.

Table 6: Results for Analysis

Awareness/Effects	$\tilde{\Gamma}_*(\ell_1, \ell_2)$	$\tilde{\Gamma}_*(\ell_1, \ell_3)$	$\tilde{\Gamma}_*(\ell_1, \ell_4)$	$\tilde{\Gamma}_*(\ell_2, \ell_3)$	$\tilde{\Gamma}_*(\ell_2, \ell_4)$	$\tilde{\Gamma}_*(\ell_3, \ell_4)$
Awareness	0.8410	0.6950	0.6994	0.8128	0.7129	0.8243
Effects	0.8178	0.9408	0.8692	0.8545	0.6990	0.8323

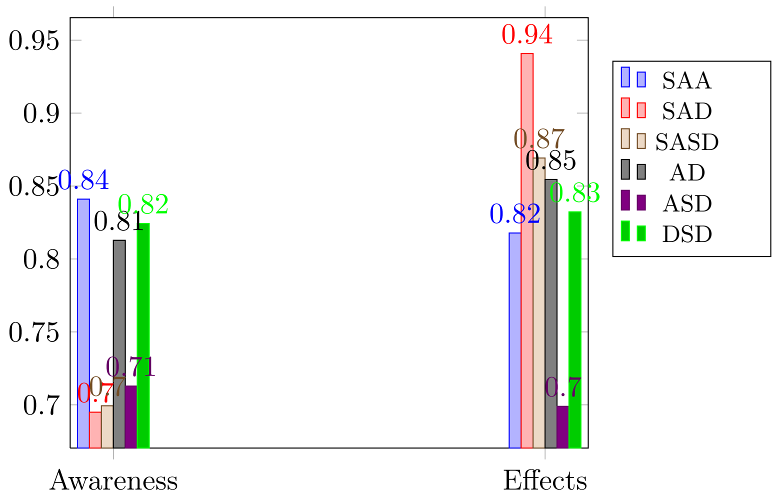

Figure 1: Plot of Results

From these results, for the case of level of awareness of VL, we see that the similarity between SA and A (i.e., SAA) is the greatest, which implies that the undergraduate medical students are aware of the virtual library resources in their department. Similarly, for the case of effects of virtual library, it is observed that the similarity between SA and D (i.e., SAD) is the closest, which implies that the effect of virtual library on the academic performance of the students is not satisfactory.

5.2 Comparison I

To determine the effectiveness of the corrected similarity method, we show its results side by side with the results from the methods in [41]. The comparative results are expressed in Tables 7 and 8.

Table 7: Level of Awareness of VL

Methods	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)	(ℓ_1, ℓ_4)	(ℓ_2, ℓ_3)	(ℓ_2, ℓ_4)	(ℓ_3, ℓ_4)
$\tilde{\Gamma}_*$	0.8410	0.6950	0.6994	0.8128	0.7129	0.8243
Γ_1	0.0486	−0.0338	−0.0254	0.0440	−0.0089	0.0576
Γ_2	0.0149	−0.0844	−0.0821	−0.0086	−0.0749	0.0007
Γ_3	0.0149	−0.0844	−0.0821	−0.0086	−0.0749	0.0007
Γ_4	0.0149	−0.0844	−0.0821	−0.0086	−0.0749	0.0007

From Table 7, we see that the similarity between SA and A, and D and SD are very close using the corrected similarity operator. Among the relations, the similarity between scales SA and A is the greatest. This implies that the medical students are aware of the existent of VL on their campus. It is observed that the methods in [41] fail a similarity condition by giving negative results. Therefore, the methods are not appropriate PFSOs, which justifies the effected correction.

Table 8: Effects of VL

Methods	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)	(ℓ_1, ℓ_4)	(ℓ_2, ℓ_3)	(ℓ_2, ℓ_4)	(ℓ_3, ℓ_4)
$\tilde{\Gamma}_*$	0.8178	0.9408	0.8692	0.8545	0.6990	0.8323
$\mathit{\Gamma}_1$	0.0392	0.1451	0.0926	0.0637	−0.0250	0.0654
$\mathit{\Gamma}_2$	−0.0046	0.1193	0.0409	0.0270	−0.0823	0.0074
$\mathit{\Gamma}_3$	−0.0046	0.1193	0.0409	0.0270	−0.0823	0.0074
$\mathit{\Gamma}_4$	−0.0046	0.1193	0.0409	0.0270	−0.0823	0.0074

The information in Table 8 shows that the similarity between the scales SA and D is the closest based on the corrected PFSO. The implication of this is that, VL has not effect on the academic wellbeing of the medical students because awareness does not translates into effectiveness if the VLR are not put into use. We observe that the defective methods in [41] produce outcomes that are undefined in the range of the similarity values. Throughout the study, we see that $\mathit{\Gamma}_2$, $\mathit{\Gamma}_3$, and $\mathit{\Gamma}_4$ in [41] yield the same results.

5.3 PFSO based-MCDM Approach of Analyzing Questionnaire

MCDM is a process of choice making in social sciences, medicine, engineering, etc. MCDM determines the best option by assessing more than one criteria for the purpose of selection. Due to the present of imprecision in choice making, MCDM has been studied under PFSs using various information measures. Here, we present the MCDM approach of analyzing questionnaire of VL based on the corrected similarity operator because it has been proven to be effective, consistent and reliable with the most precise results compare to the methods in [41].

5.3.1 Algorithm for the MCDM

The algorithm are as follows:
Step 1. Obtain the Pythagorean fuzzy decision matrix (PFDM) denoted by $\tilde{\ell}_j = \{Q_i(\tilde{\ell}_j)\}_{(m\times n)}$ for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, where Q_i are the questions.
Step 2. Formulate the normalized PFDM $\tilde{\ell} = \langle M_{\tilde{\ell}^*_j}(Q_i), N_{\tilde{\ell}^*_j}(Q_i)\rangle_{m\times n}$, where $\langle M_{\tilde{\ell}^*_j}(Q_i), N_{\tilde{\ell}^*_j}(Q_i)\rangle$ are the PFD, and $\tilde{\ell}$ is defined as:

$$\langle M_{\tilde{\ell}^*_j}(Q_i), N_{\tilde{\ell}^*_j}(Q_i)\rangle = \begin{cases} \langle M_{\tilde{\ell}_j}(Q_i), N_{\tilde{\ell}_j}(Q_i)\rangle, & \text{for benefit criterion of } \tilde{\ell} \\ \langle N_{\tilde{\ell}_j}(Q_i), M_{\tilde{\ell}_j}(Q_i)\rangle, & \text{for cost criterion of } \tilde{\ell} \end{cases} \tag{7}$$

Step 3. Compute PIS (positive ideal solution) and NIS (negative ideal solution) given by

$$\begin{aligned} \tilde{\ell}^+ &= \{\tilde{\ell}^+_1, \cdots, \tilde{\ell}^+_n\} \\ \tilde{\ell}^- &= \{\tilde{\ell}^-_1, \cdots, \tilde{\ell}^-_n\} \end{aligned} \tag{8}$$

where

$$\tilde{\ell}^+ = \begin{cases} \langle \max\{M_{\tilde{\ell}_j}(Q_i)\}, \min\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the BC} \\ \langle \min\{M_{\tilde{\ell}_j}(Q_i)\}, \max\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the CC,} \end{cases} \tag{9}$$

$$\tilde{\ell}^- = \begin{cases} \langle \min\{M_{\tilde{\ell}_j}(Q_i)\}, \max\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the BC} \\ \langle \max\{M_{\tilde{\ell}_j}(Q_i)\}, \min\{N_{\tilde{\ell}_j}(Q_i)\}\rangle, & \text{if } Q_i \text{ is the CC,} \end{cases} \tag{10}$$

where BC is benefit criterion and CC is cost criterion.
Step 4. Compute the similarities $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$ and $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$.
Step 5. Find the closeness coefficients $\nabla_*(\tilde{\ell}_j)$ by

$$\nabla_*(\tilde{\ell}_j) = \frac{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)}{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+) + \tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)}, \tag{11}$$

for $j = 1, 2, \cdots, n$.
Step 6. Determine the greatest closeness coefficient for the interpretation.

In case either $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$ or $\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$ is negative (which ought not to happen except the similarity operator is defective), we find $\nabla^+(\tilde{\ell}_j)$ and $\nabla^-(\tilde{\ell}_j)$ thus:

$$\nabla^+(\tilde{\ell}_j) = \frac{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^+)}{\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^+) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^+)}, \tag{12}$$

$$\nabla^-(\tilde{\ell}_j) = \frac{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^-)_{\min}}{\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^-) - \tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^-)} \tag{13}$$

before Step 5. Then (11) becomes:

$$\nabla_*(\tilde{\ell}_j) = \frac{\nabla^+(\tilde{\ell}_j)}{\nabla^+(\tilde{\ell}_j) + \nabla^-(\tilde{\ell}_j)}, \tag{14}$$

for $j = 1, 2, \cdots, n$.
Note that

$$\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^+) = \max_{1\le j\le n}\{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)\},$$
$$\tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^+) = \min_{1\le j\le n}\{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)\},$$

$$\tilde{\Gamma}_{\max}(\tilde{\ell}_j, \tilde{\ell}^-) = \max_{1\le j\le n}\{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)\},$$
$$\tilde{\Gamma}_{\min}(\tilde{\ell}_j, \tilde{\ell}^-) = \min_{1\le j\le n}\{\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)\}.$$

The algorithm is captured in Figure 2.

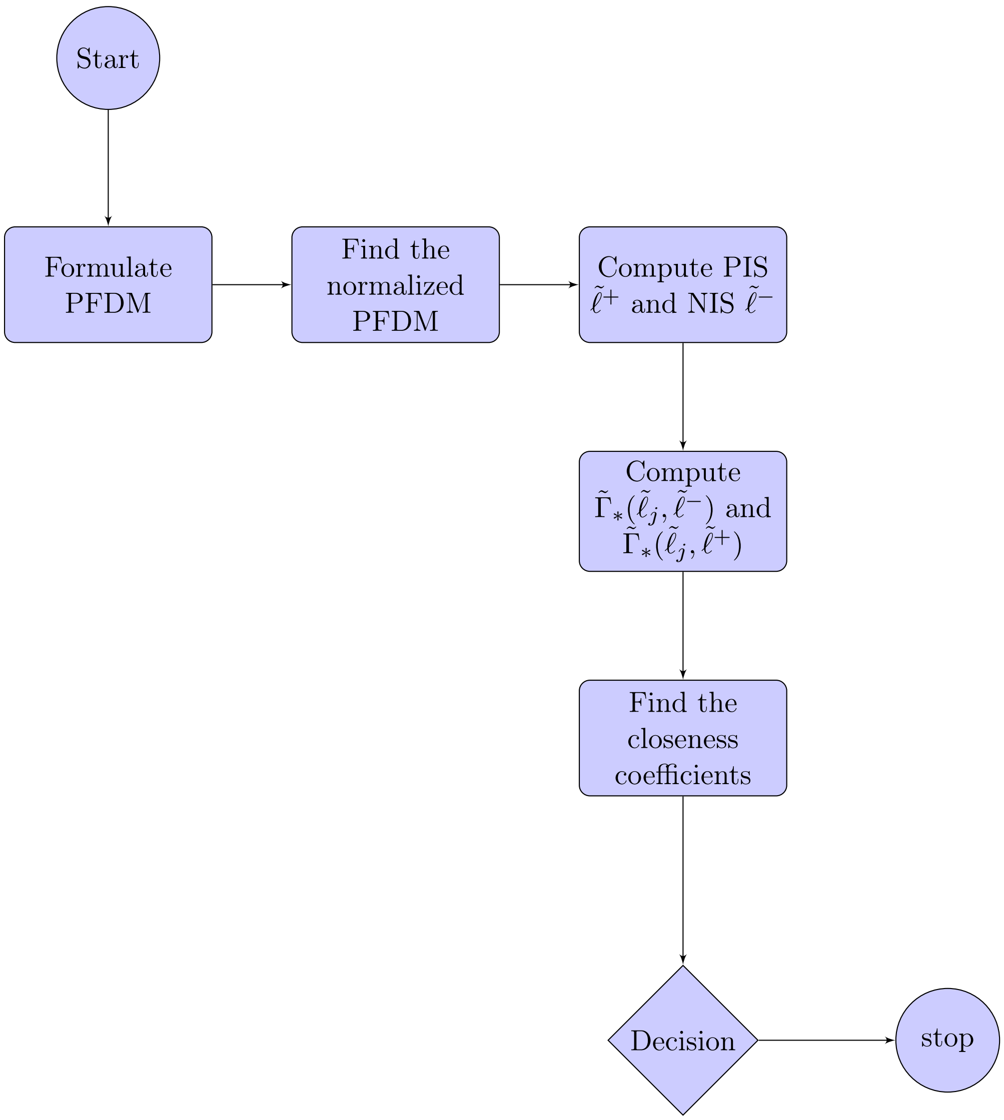

Figure 2: Flowchart for Implementation

5.3.2 Case I

Here, we discuss the questionnaire on the level of awareness of VL as presented in Table 4 via MCDM technique, where Q_5 is cost criterion since the question gives the least MDs. By Step 2, we get Table 9.

Table 9: Normalized PFDM for Level of Awareness

Scales	$\tilde{\ell}_1$	$\tilde{\ell}_2$	$\tilde{\ell}_3$	$\tilde{\ell}_4$
Q_1	$(0.48, 0.52)$	$(0.318, 0.682)$	$(0.111, 0.889)$	$(0.091, 0.909)$
Q_2	$(0.343, 0.657)$	$(0.258, 0.742)$	$(0.247, 0.753)$	$(0.152, 0.848)$
Q_3	$(0.197, 0.803)$	$(0.253, 0.747)$	$(0.394, 0.606)$	$(0.157, 0.843)$
Q_4	$(0.172, 0.828)$	$(0.394, 0.606)$	$(0.328, 0.672)$	$(0.106, 0.894)$
Q_5	$(0.47, 0.53)$	$(0.631, 0.369)$	$(0.919, 0.081)$	$(0.98, 0.02)$

Using Step 3, we obtain the PIS and NIS in Table 10.

Table 10: PIS and NIS for Level of Awareness

Scales	$\tilde{\ell}^-$	$\tilde{\ell}^+$
Q_1	$(0.091, 0.909)$	$(0.48, 0.52)$
Q_2	$(0.152, 0.848)$	$(0.343, 0.657)$
Q_3	$(0.157, 0.843)$	$(0.394, 0.606)$
Q_4	$(0.106, 0.894)$	$(0.394, 0.606)$
Q_5	$(0.98, 0.02)$	$(0.47, 0.53)$

By Step 4, we compute the similarities between $\tilde{\ell}_j$ and $\tilde{\ell}^-$, and $\tilde{\ell}_j$ and $\tilde{\ell}^+$ using (5) to obtain the results in Table 11.

Table 11: Similarities of $(\tilde{\ell}_j, \tilde{\ell}^-)$ and $(\tilde{\ell}_j, \tilde{\ell}^+)$

Scales	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$
$\tilde{\ell}_1$	0.7088	0.8824
$\tilde{\ell}_2$	0.6719	0.8883
$\tilde{\ell}_3$	0.6883	0.7730
$\tilde{\ell}_4$	0.8302	0.6173

Next, by using (11) in Step 6, we obtain the closeness coefficients in Table 12.

Table 12: Closeness Coefficients for Level of Awareness

Scales	$\nabla_*(\tilde{\ell}_j)$	Ranking
$\tilde{\ell}_1$	0.5546	2^{nd}
$\tilde{\ell}_2$	0.5694	1^{st}
$\tilde{\ell}_3$	0.5290	3^{rd}
$\tilde{\ell}_4$	0.4265	4^{th}

From Table 12, we see that the medical students are aware of the existent of VLR because the scale $\tilde{\ell}_2$ (i.e., A) is ranked first, which tallies with the finding in Table 7.

5.3.3 Case II

Now, we consider the questionnaire on the effects of VLR on the academic wellbeing via MCDM method using the PFDM in Table 5, where Q_3 is taken as the cost criterion. By Step 2, we get Table 13.

Table 13: Normalized PFDM for Effects of VL

Scales	$\tilde{\ell}_1$	$\tilde{\ell}_2$	$\tilde{\ell}_3$	$\tilde{\ell}_4$
Q_1	$(0.237, 0.763)$	$(0.409, 0.591)$	$(0.227, 0.773)$	$(0.126, 0.874)$
Q_2	$(0.258, 0.742)$	$(0.379, 0.621)$	$(0.232, 0.768)$	$(0.131, 0.869)$
Q_3	$(0.828, 0.172)$	$(0.581, 0.419)$	$(0.747, 0.253)$	$(0.843, 0.157)$
Q_4	$(0.212, 0.788)$	$(0.364, 0.636)$	$(0.288, 0.712)$	$(0.136, 0.864)$
Q_5	$(0.263, 0.737)$	$(0.313, 0.687)$	$(0.283, 0.717)$	$(0.141, 0.859)$

Using Step 3, we obtain the PIS and NIS in Table 14.

Table 14: PIS and NIS for Effects of VL

Scales	$\tilde{\ell}^-$	$\tilde{\ell}^+$
Q_1	$(0.126, 0.874)$	$(0.409, 0.591)$
Q_2	$(0.131, 0.869)$	$(0.379, 0.621)$
Q_3	$(0.843, 0.157)$	$(0.581, 0.419)$
Q_4	$(0.136, 0.864)$	$(0.364, 0.636)$
Q_5	$(0.141, 0.859)$	$(0.313, 0.687)$

By Step 4 via (5), we get Table 15.

Table 15: Similarities between $(\tilde{\ell}_j, \tilde{\ell}^-)$ and $(\tilde{\ell}_j, \tilde{\ell}^+)$

Scales	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^-)$	$\tilde{\Gamma}_*(\tilde{\ell}_j, \tilde{\ell}^+)$
$\tilde{\ell}_1$	0.7593	0.7908
$\tilde{\ell}_2$	0.6737	0.9703
$\tilde{\ell}_3$	0.7505	0.8269
$\tilde{\ell}_4$	0.8771	0.6737

Next, we find the closeness coefficients for the similarity values using (11) and get Table 16.

Table 16: Closeness Coefficients for Effects of Virtual Library

Scales	$\nabla_*(\tilde{\ell}_j)$	Ranking
$\tilde{\ell}_1$	0.5102	3rd
$\tilde{\ell}_2$	0.5902	1st
$\tilde{\ell}_3$	0.5242	2nd
$\tilde{\ell}_4$	0.4344	4th

The values of the closeness coefficient indicate that $\tilde{\ell}_2 \succeq \tilde{\ell}_3 \succeq \tilde{\ell}_1 \succeq \tilde{\ell}_4$. The interpretation of the ranking is somehow confusing because it oscillates between agree and disagree, which infers that the medical students agree to a minimal effect of VLR on their academic wellbeing possibly due to a very poor use of the VLR, which may be caused by technological barriers, user interface issues, and competing academic commitments.

5.4 Comparison II

Again, we show the effectiveness of the corrected similarity method via MCDM in comparison with the defective methods in [41]. The comparative results are shown in Tables 17 and 18, and Figures 3 and 4.

Table 17: MCDM Comparative Results for Case 1

Methods	$\nabla_*(\tilde{\ell}_1)$	$\nabla_*(\tilde{\ell}_2)$	$\nabla_*(\tilde{\ell}_3)$	$\nabla_*(\tilde{\ell}_4)$	Ordering	Verdict
$\tilde{\Gamma}_*$	0.5546	0.5694	0.5290	0.4265	$\tilde{\ell}_2 \succ \tilde{\ell}_1 \succ \tilde{\ell}_3 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_1 [41]	0.8403	1	0.8513	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_2 [41]	0.6187	0.7997	1	0	$\tilde{\ell}_3 \succ \tilde{\ell}_2 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_3$
Γ_3 [41]	0.8491	0.9359	0.4522	0	$\tilde{\ell}_2 \succ \tilde{\ell}_1 \succ \tilde{\ell}_3 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_4 [41]	0.8403	1	0.8513	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$

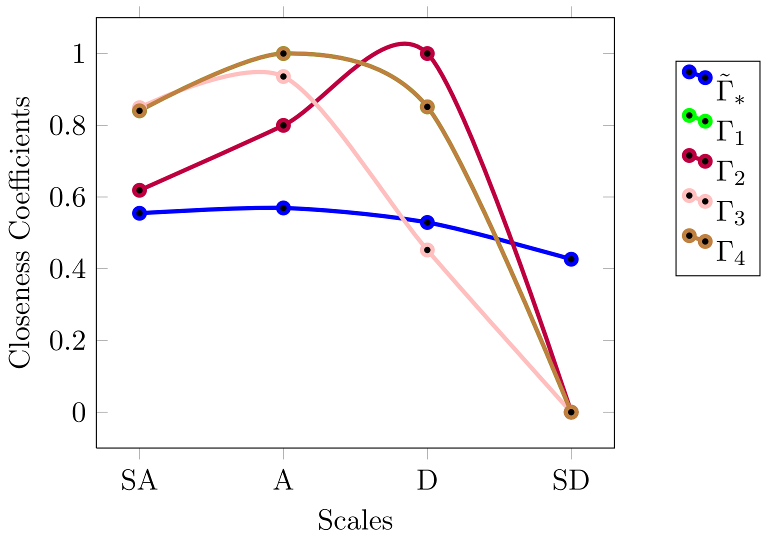

Figure 3: Plot for Comparison

From Table 17, we see that the medical students agree that they are aware of the existent of VLR in their department. While all the methods give the same interpretation, Γ_2 gives different interpretation. The existing methods give zero and one closeness coefficients due to their defectiveness. From Figure 3, it is only the corrected similarity operator that shows consistency.

Table 18: MCDM Comparative Results for Case 2

Methods	$\nabla_*(\tilde{\ell}_1)$	$\nabla_*(\tilde{\ell}_2)$	$\nabla_*(\tilde{\ell}_3)$	$\nabla_*(\tilde{\ell}_4)$	Ordering	Verdict
$\tilde{\Gamma}_*$	0.5102	0.5902	0.5242	0.4344	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_1 [41]	0.4531	1	0.5687	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_2 [41]	0.4731	1	0.5318	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_3 [41]	0.5881	1	0.7151	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$
Γ_4 [41]	0.4531	1	0.5687	0	$\tilde{\ell}_2 \succ \tilde{\ell}_3 \succ \tilde{\ell}_1 \succ \tilde{\ell}_4$	$\tilde{\ell}_2$

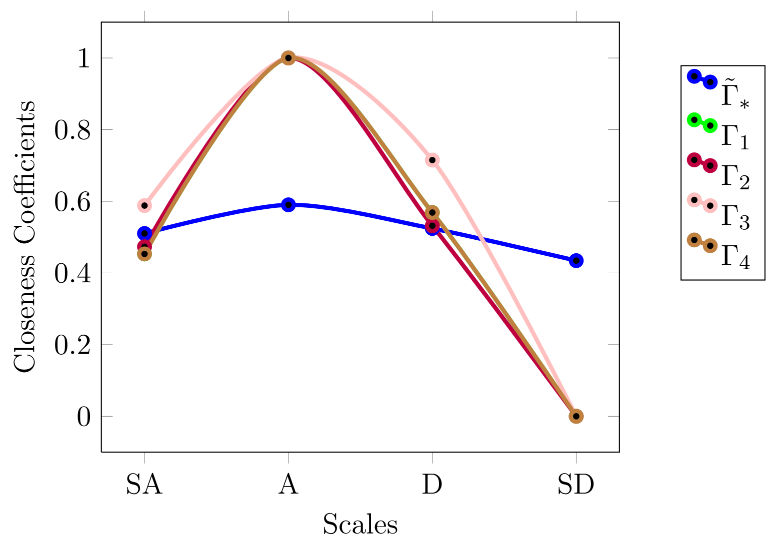

Figure 4: Plot for Comparison

From Table 18, it follows that the medical students agree that the use of VLR has effect on their academic wellbeing. We observe that the closeness coefficients based on the existing PFSOs [41] for $\tilde{\ell}_2$ and $\tilde{\ell}_4$ give zero and one, respectively due to their defectiveness unlike the corrected PFSO. From Figure 4, we see the consistency of the corrected PFSO.

6 Conclusion

In this paper, a new PFSO was developed to ease decision-making in imprecise environments. The new PFSO is the corrected form of the PFSOs in [41], where four PFSOs were constructed which we have demonstrated to be defective. The new PFSO can be used with or without weight vector. Some numerical illustrations were used to showcase the defectiveness of the PFSOs in [41] and to demonstrate the overriding significant of the new PFSO. While the PFSOs in [41] violated the axioms of similarity function, the new PFSO yields reliable and precise results which are consistent with the axioms of similarity function. In addition, some theoretic results of the new PFSO were considered and proved. Furthermore, the new PFSO was used to analyze questionnaire on VL where the collected data were transformed to Pythagorean fuzzy data (PFD). The questionnaire was designed and distributed to 198 undergraduate medical students for the purpose of data collection, after which the data were converted to PFD. It is observed that the corrected version of PFSO could be helpful in decision-making under indeterminate domains since the PFSO is well equipped to control hesitations that may constitute bottleneck for decision-makers. Exploring the potential real-world applications of the new PFSO in different imprecise domains is an interesting research direction for future endeavor. The construction of the modified PFSO limits its application to only Pythagorean fuzzy environment. Thus, the modified PFSO cannot be used to model decision-making problems under picture fuzzy sets [44], q-rung orthopair fuzzy sets [45], Fermatean fuzzy sets [46], etc. because the distinct properties of these sets are not represented in the modified PFSO. However, with some alterations, the modified PFSO could be stretched to the aforementioned domains and use to solve real-world applications.

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