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Stability region of returns to scale using inverse Data Envelopment Analysis

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Abstract
The returns to scale (RTS) is an economic issue that would play a crucial role in the expansion or limitation of the decision-making unit (DMU) under-evaluation in data envelopment analysis (DEA). In this paper, we study an inverse DEA problem in which besides finding the appropriate amount of increase in output, preserving the primary classification of returns to scale for the DMU under-evaluation is considered. This research discusses two cases: when the DMU operates under constant returns to scale (CRS), and the other case considers DMUs with increasing returns to scale (IRS). Respectively for DMUs with CRS the upper bound obtained from the sensitivity analysis method is applied to determine the maximum amount of authorized output increase to preserve the primary classification of RTS. Then, we present two methods for the case of DMUs operating under the IRS. In the first one, we use an upper limit of the authorized amount of output's increase for modeling the problem in such a way that the IRS has remained unchanged. Then, the second method provides a model based on the closest most productivity scale size (MPSS) to the projection of the DMU under evaluation to solve the output estimation problem with maintaining the IRS. Finally, we give a numerical example to examine the application of the presented models.

Keywords: Inverse data envelopment analysis (IDEA); MPSS; Output/Input estimation; Returns to scale (RTS).

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric mathematical modeling method to assess the performance of several decision units (DMUs) where multiple inputs are applied to generate multiple outputs. Also, it is supposed that each DMU has at least one non-zero output and input. DEA helps managers and decision-makers make effective decisions besides improving inefficient structures to adopt a suitable policy to achieve their goals. One of the most important and widely used subfields of DEA is inverse data envelopment analysis (IDEA), which was raised with the following question: if the inputs of the DMU under evaluation (DMU_o) increase, how much will its outputs increase such that the efficiency of DMUo is maintained? This problem is called the output estimation problem. Subsequently, the same question from another point of view as the input estimation problem was proposed. In the input estimation problem, the increase in the number of inputs is estimated in the conditions that the outputs have increased and the efficiency remains constant.

Another practical topic in DEA is returns to scale (RTS). It is an economic notion that investigates the impact of changing economic variables on dependent variables. Consider a society where economic growth prevails, that is, its economic activities to be profitable. We want to know how increased investment affects profit in such a society. However, we know that an increase in investment cannot lead to an increase in profit because many factors such as fixed and variable costs can affect the efficiency of a capital increase. It is clear that if the investment leads to a higher profit, we accept the risk of this investment; otherwise, accepting the risk of investment does not seem reasonable. On the other hand, consider the opposite of the stated situation. For instance, let us assume we are dealing with a factory that is bankrupt. We want to know how downsizing the company or shutting down some factory production lines reduces the factory's losses. In both cases, we want to investigate the effect of changing the independent variables on the dependent variables. If this change is economical, the risk is reasonable and we accept it. These characteristics point to the fact that it is better to consider an optimal range (type of RTS) for economic enterprises or DMU_s and maintain the mentioned range. Because from the economic point of view, any change in the type of RTS for a DMU which intends to continue its economic activity is against its policy. Otherwise, unprincipled changes on production policy will encounter the interests of the firm or DMU with problems that can impose many costs on it or even lead it to bankruptcy. Therefore, it is crucial to consider the classification of RTS for a DMU when expansion or limitation of the mentioned DMU is taken into account. Although the problem of stability and sensitivity of classification of RTS in classic DEA has been discussed in the literature, the abovementioned issue has not been addressed in the output estimation problem.

The main contributions of this paper are as follows. In this paper, we deal with the output estimation problem on T_v where besides preserving the efficiency of the DMU, the classification of RTS remains unchanged. For this purpose, this research discusses two cases: when the DMU operates under constant returns to scale (CRS), and the other case considers DMUs with increasing returns to scale (IRS). First, we provide a method for DMUs with CRS in which the upper bound obtained from the sensitivity analysis method presented by Seiford and Joe Zh [1] u is applied to determine the maximum amount of authorized output increase to preserve the primary classification of RTS. Then, we present two methods for the case that DMUs operate under the IRS. In the first one, we use an upper limit of the authorized amount of output's increase for modeling the problem in such a way that the IRS has remained unchanged, and in the second method, a model based on the closest most productivity scale sized (MPSS) to the projection of the DMU under evaluation is presented to solve the output estimation problem with maintaining the IRS. Finally, a numerical example is given to discuss the results of the presented models.

The remainder of this paper is as follows. A literature review is provided in section 2. Then, some preliminaries on DEA and IDEA are presented in section 3. Furthermore, our proposed models are elaborated in section 4. Then, sections 5 and 6 provide a numerical example and conclusion, respectively.

2. Literature review

DEA was first presented by Charnes et al. [2] by introducing a CCR model. Then, Banker et al. [3] expanded DEA models by considering variable returns to scale (VRS). Then, regarding the technologies used in making the possible productivity set (PPS), various models were proposed to determine the efficiency of DMUs. IDEA was introduced by the question that came to Zhang and Cui [4] in 1999 when expanding an evaluation system caused the initiation of research on it. The question raised was as follows: if DMU_k continues its operation in the next period regardless of whether it is efficient or not, and the inputs increase to improve the outputs, how many additional resources should be allocated to DMU_k such that its efficiency is maintained unchanged? Wei et al. [5] developed a common form of IDEA, seeking to respond to the question as an output estimation problem and providing a multi-objective linear programming model (MOLP) to solve it. Then, they converted the MOLP to a single-objective linear programming problem and address the problem. After introducing the IDEA, many researches have been published in this field of study. Yan et al. [6] studied inputs/outputs estimation problems with preference cone constraints. They provided the properties of the IDEA by discussing the relation between weighted sum single-objective problem and MOLP. Jahanshahloo et al. [7] used IDEA models to approximate inputs for a DMU when some or all outputs and its efficiency level are increased or remain unchanged. Then, the inputs/outputs estimation problem with undesirable outputs (inputs) was investigated, and regardless of the inefficiency or efficiency of the DMU under-evaluation, a MOLP was provided to address the problem [8]. Hadi-Vencheh and Foroughi [9] studied a generalized inputs/outputs estimation problem where the increase of some outputs (inputs) and the decrease due to some of the other outputs (inputs) are considered simultaneously. Furthermore, Hadi-Vencheh et al. [10] modified the sufficient conditions provided by Wei et al. [5] for input estimation problem.

In 2014, Jahanshahloo et al. [11] investigated the IDEA problem using the modified Russell model. They presented the necessary and sufficient conditions for determining the inputs and outputs levels based on the Pareto solutions of MOLP. Also, Jahanshahloo et al. [12] proposed the concept of inter-temporal dependence with changes in the reserve capital in different periods of the production process. They introduced a new optimality concept for MOLP, the periodic weak Pareto solution. Ghobadi and Jahangiri [13] applied IDEA models to evaluate educational departments in a university and then developed some applications and properties of the problem in the presence of fuzzy data. Ghiyasi [14] provided an IDEA model where price information,

technical, cost and revenue efficiency were considered in the proposed method.
Moreover, DEA has been taken into account as one of the most applicable techniques for considering RTS issues for the last two decades. Banker [15] extended the relationship between RTS and MPSS and then applied the presented relation to expanding the CCR model for estimating the MPSS for convex PPS. Banker et al. [3] presented a new separate variable to specify if operations are executed in increasing, constant, or decreasing RTS regions. Golany and Yu [16] used the input and output-oriented models presented by Banker et al. [3] to determine precise estimates of RTS in DEA. Seiford and Joe Zhu [1] studied the determination of RTS in DEA. They provided three basic RTS methods and their modifications and addressed the equivalency between the methods mentioned above. Furthermore, they investigated the impact of multiple optimal DEA solutions on the issue of estimating RTS. Seiford and Joe Zhu [17] applied a linear programming approach to deal with the sensitivity of the RTS classification. They provided sufficient and necessary conditions for preserving the type of the primary RTS. Jahanshahloo et al. [18] critiqued the paper by Seiford and Joe Zhu [17], where the essential policies on the inputs, outputs, and DMUs are shown using the priority cone. Then, Allahyar et al. [19] modified the shortcomings of the model provided by Golany and Yu [16] and presented a new method for estimating the RTS. Benicio et al. [20] studied the efficiency of DMUs from VRS perspective. Kumar et al. [21] studied different types of efficiencies of main state industries in India by considering VRS. Mert [22] discussed positive economic growth by considering its relation with RTS. Clermont et al. [23] inspected the specification of RTS in the Business Administration research of universities in Germany. Also, many researchers applied the issue of RTS to investigate the relationship between farm size and productivity in the agriculture industry ([24], [25], [26]). Sarparast et al., [27] discussed the sensitivity of RTS classifications of the efficient DMUs in a two-stage DEA network. Gao and Reed [28] investigated effect of IRS on liquidity creation and financial fragility in the banking industry. Zhao et al., [29] dealt with the impact of RTS change on productivity of 76 China's urban commercial banks. Moreover, IRS was applied to provide methodological approach for calculating the appropriate increase in industry markups [30].
In this paper, we discuss inverse DEA problems on T_v so that the classification of RTS is also preserved. So, two cases i.e., CRS and IRS are considered to address the output estimation problem such that the primary RTS is not changed. The next section provides preliminaries which we need to use in our proposed problem.

3. Preliminaries

This section deals with some models and basic concepts of DEA, inverse DEA, and RTS used in the following sections.

3.1. Basic Models

Assume that the input $X \in \mathbb{R}_+^m$ is used to produce the output vector $Y \in \mathbb{R}_+^s$. All the inputs and outputs are supposed to be non-negative, and at least one component of inputs and outputs is non-zero. Then, the production possibility set (PPS) is introduced as follows:

$$PPS = \{(X,Y);\ Y\ can\ be\ produced\ by\ X\}$$

If the technology used in constructing PPS is established upon CRS, then the corresponding PPS is called T_c and can be presented as follows:

$$PPS_{T_c} = \left\{ \begin{array}{l} (X,Y);\ X \ge \sum_{j=1}^{n} \lambda_j X_j, Y \le \sum_{j=1}^{n} \lambda_j Y_j, \\ \lambda_j \ge 0;\ (j = 1,2,...,n) \end{array} \right\}$$

Applying VRS assumption in constructing the PPS is equivalent to appending the constraint $e\lambda = 1$ to T_c, where

$e = (1, 1, \ldots, 1)$, which causes to obtain T_v as follows:

$$PPS_{T_v} = \left\{ \begin{array}{l} (X,Y);\ X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0;\ (j = 1, 2, \ldots, n) \end{array} \right\}$$

In order to assess the performance of DMUs based on various aforementioned PPS, the DEA models are categorized into two types: radial and non-radial. The radial models are input- or output-oriented, depending on proper input contraction or output expansion. Suppose that $DMU_k, (k = 1, \ldots, n)$ produces the s outputs $y_{rk}, (r = 1, \ldots, s)$ by consuming m homological inputs $x_{ik}, (i = 1, \ldots, m)$. The envelopment forms of the input-oriented and output-oriented models are presented in the following, respectively [3]:

$$\begin{aligned} & Min\, \theta_o - \varepsilon \sum_{r=1}^{s} s_r^+ - \varepsilon \sum_{i=1}^{m} s_i^- && (1) \\ & s.t.\ \theta_o x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij} - s_i^- = 0, \quad i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ = y_{ro}, \qquad r = 1, 2, \ldots, s \\ & \delta_1 (\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \nu) = \delta_1 \\ & \nu \geq 0, \lambda_j \geq 0, \qquad j = 1, 2, \ldots, n \\ & s_i^- \geq 0\ ,\ s_r^+ \geq 0\ , i = 1, 2, \ldots, m\ ,\ \ r = 1, 2, \ldots, s \end{aligned}$$

and

$$\begin{aligned} & Max\, \varphi_o + \varepsilon \sum_{r=1}^{s} s_r^+ + \varepsilon \sum_{i=1}^{m} s_i^- && (2) \\ & s.t. \sum_{j=1}^{n} \lambda_j x_{ij} - s_i^- = x_{io}\ , i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ = \varphi_o y_{ro}\ , r = 1, 2, \ldots, s \\ & \delta_1 (\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \nu) = \delta_1 \\ & s_i^- \geq 0\ ,\ s_r^+ \geq 0\ , i = 1, 2, \ldots, m\ , r = 1, 2, \ldots, s \end{aligned}$$

In which ν is nonnegative number that turns the inequality into an equality. It It can be easily shown that for $(\delta_1, \delta_2, \delta_3) = (0,*,*)$ in the models (1) and (2) the production technology is CRS, and the obtained models are called CCR. Moreover, for $(\delta_1, \delta_2, \delta_3) = (1,0,*)$ the production technology is VRS, and the corresponding models are named BCC.

Note: DMU_o is Pareto efficient if and only if (A) or (B) are held [17]:

(A) $s_i^{-*} = s_r^{+*} = 0;\ (i = 1, \ldots, m\ \&\ r = 1, \ldots, s)$ and $\theta^* = 1$, in the model (1).

(B) $s_i^{-*} = s_r^{+*} = 0;\ (i = 1, \ldots, m\ \&\ r = 1, \ldots, s)$ and $\varphi^* = 1$, in the model (2).

3.2. Stability of RTS

As mentioned, the type of RTS for the DMU under-evaluation can be constant, increasing, or decreasing. First, it should be mentioned that the type of RTS for DMUo is determined using proposed methods (see Banker [15] and Seiford and Zhu [17]).

Seiford and Zhu [1] have presented some definitions and models regarding the RTS in DEA. Assume that:

$$E_o = \left\{ \begin{array}{l} j;\ \lambda_j > 0, \\ for\ some\ optimal\ solutions\ for\ DMU_o \end{array} \right\}$$

Then, consider the following model:

$$\begin{aligned} & Min\, \theta && (3) \\ & s.t. \sum_{j \in E_o} \lambda_j x_{ij} \leq \theta x_{io}\ , \quad i = 1, 2, \ldots, m \\ & \sum_{j \in E_o} \lambda_j y_{rj} \geq y_{ro}\ , \quad r = 1, 2, \ldots, s \\ & \lambda_j \geq 0 \qquad j \in E_o \end{aligned}$$

Model (3) is named the input-oriented CCR model.

The output-oriented CCR model is expressed below:

$$
\begin{aligned}
& Max\,\varphi \qquad (4) \\
& s.t. \sum_{j \in E_o} \lambda_j x_{ij} \le x_{io} \;, \quad i = 1,2,...,m \\
& \sum_{j \in E_o} \lambda_j y_{rj} \ge \varphi y_{ro} \quad , \; r = 1,2,...,s \\
& \lambda_j \ge 0 \qquad j \in E_o
\end{aligned}
$$

Seiford and Zhu [1] called the output proportional changes as χ and used the following two models to calculate the amount of χ.

$$
\begin{aligned}
& (\tau_o^*)^{-1} = Min \sum_{j \in E_o} \hat{\lambda}_j \qquad (5) \\
& s.t. \sum_{j \in E_o} \hat{\lambda}_j x_{ij} \le \theta^* x_{io} \;, \; i = 1,2,...,m \\
& \sum_{j \in E_o} \hat{\lambda}_j y_{rj} \ge y_{ro} \quad , \; r = 1,2,...,s \\
& \hat{\lambda}_j \ge 0 \qquad j \in E_o
\end{aligned}
$$

and

$$
\begin{aligned}
& (\tau_o^*)^{-1} = Max \sum_{j \in E_o} \hat{\lambda}_j \qquad (6) \\
& s.t. \sum_{j \in E_o} \hat{\lambda}_j x_{ij} \le \theta^* x_{io} \;, \; i = 1,2,...,m \\
& \sum_{j \in E_o} \hat{\lambda}_j y_{rj} \ge y_{ro} \quad , \; r = 1,2,...,s \\
& \hat{\lambda}_j \ge 0 \qquad j \in E_o
\end{aligned}
$$

in which θ^* is the optimal solution of model (3) in evaluating efficiency of DMU_o.

Suppose that DMU_o operates under CRS, then the optimal solutions of the models (5) and (6) are $\tau_o^* = (\sum_{j \in E_o} \hat{\lambda}_j^*)^{-1} \ge 1$ and $\delta_o^* = (\sum_{j \in E_o} \hat{\lambda}_j^*)^{-1} \le 1$, respectively.

Seiford and Zhu [1] shown stated that $\hat{\lambda}_j^* \; (j \in E_o)$ such that $\sum_{j \in E_o} \hat{\lambda}_j^* \le 1$ and $\sum_{j \in E_o} \hat{\lambda}_j^* \ge 1$ are also the optimal solutions of (3) and (4), respectively.

Proposition 1: Suppose DMU_o exhibits CRS, then its classification of RTS is preserved if $\chi \in R^{CRS} = \left\{ \chi;\; \min\left\{1,\, \delta_o^*\right\} \le \chi \le \max\left\{1, \tau_o^*\right\} \right\}$ which χ shows the output proportional changes, namely $\hat{y}_{ro} = \chi y_{ro};\; (r = 1,...,s)$ and τ_o^* and δ_o^* are as defined in models (5), (6) [18].

Proposition 2: Let DMU_o exhibits IRS, then its classification of RTS is preserved if $\chi \in R^{IRS} = \left\{ \chi;\, 1 \le \chi \le \delta_o^* \right\}$ that shows output proportional changes defined in the model (6) [18].

3.3. Most Productive Scale Size

Definition 1: A possible production $(X_o, Y_o) \in T_v$ is called MPSS, if and only if for each $\alpha > 0$ and $\beta > 0$:

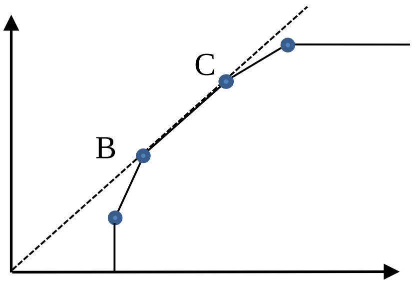
$$
(\beta X_o, \alpha Y_o) \in T_v \Rightarrow \frac{\alpha}{\beta} \le 1
$$

If $(X_o, Y_o) \in T_c$ and $(\lambda^*,\, \theta_o^*,\, s^{-^*},\, s^{+^*})$ is the optimal solution of model (1), then the projection of MPSS corresponding to DMU_o is obtained by using the following formula [2]:

$$
(X_o^{MPSS}, Y_o^{MPSS}) = \left(\frac{\theta_o^* X_o - s^{-^*}}{\sum_{j=1}^{n} \lambda_j^*} \;,\; \frac{Y_o + s^{+^*}}{\sum_{j=1}^{n} \lambda_j^*} \right)
$$

Note that the methods based on MPSS scale the projection of DMU_o on the frontier T_c again by extending it towards the MPSS. In this way, the resulting projection has a better scale than the efficient DMUs in the CRS region. However, the MPSS may not be unique if there are alternative optimal solutions for the model.

The following figures illustrate the mentioned above cases:



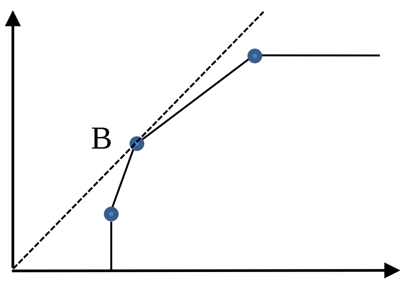


Fig. 1 Multiple MPSS and **Fig. 2** Unique MPSS

As shown in Figure 1, the facet of BC is MPSS, whereas, in Figure 2, the MPSS is unique (point B). Banker et al. [3] proposed a method to choose the projections of alternative MPSS corresponding to each DMU. They stated that the projections of MPSS must be close to the technically efficient projection of the DMUs under-evaluation. Thus, for DMUs that exhibit IRS, the smallest projection of feasible MPSS is considered. Note that DMUs exhibiting CRS are still in the MPSS, so moving on the frontier is unnecessary.

In order to find the closest MPSS to DMU_o, according to what stated in Banker et al. [3], it is enough to find the optimal solution of the model (1) and then consider one of the situations below:

- If $\sum_{j=1}^{n} \lambda^{*}_{j} \le 1$, then, the DMU under-evaluation exhibits constant/ increasing RTS. In this case, the following model should be solved:

$$
\begin{aligned}
Max \;& \sum_{j=1}^{n} \lambda_j + \varepsilon \sum_{r=1}^{s} s_r^{+} + \varepsilon \sum_{i=1}^{m} s_i^{-} && (7)\\
s.t. \;& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^{-} = \theta_o x_{io}, && i = 1,2,\dots,m\\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^{+} = y_{ro}, && r = 1,2,\dots,s\\
& \sum_{j=1}^{n} \lambda_j \le 1 &&\\
& s_i^{-} \ge 0, s_r^{+} \ge 0, \; i = 1,2,\dots,m, \; r = 1,2,\dots,s &&\\
& \lambda_j \ge 0, && j = 1,2,\dots,n
\end{aligned}
$$

Now, if $\sum_{j=1}^{n} \lambda_j^{*} < 1$, then IRS prevails for the DMU under-evaluation; otherwise, when $\sum_{j=1}^{n} \lambda_j^{*} = 1$, then CRS prevails for the DMU. Then, by using the below formula the closest MPSS to DMU_o would be obtained:

$$
(X_o^{MPSS}, Y_o^{MPSS}) = \left(\frac{\theta_o^{*} X_o - s^{-^{*}}}{\sum_{j=1}^{n} \lambda_j^{*}} , \frac{Y_o + s^{+^{*}}}{\sum_{j=1}^{n} \lambda_j^{*}} \right) \quad (8)
$$

- If $\sum_{j=1}^{n} \lambda_j^{*} \ge 1$ then the DMU under-evaluation exhibits decreasing or constant RTS which is not discussed in this section.

3.4. Output estimation problem

As stated, before Wei et al. [5] came up with a common form of inverse DEA, seeking to answer the question as an output estimation problem as follows:

If specified inputs of DMU_o increase by a fixed amount, how much should we increase the outputs of this DMU so that the efficiency of the DMU_o is not changed?

Let the inputs of DMU_o increase from X_o to $\alpha_o = X_o + \Delta X_o$; ($\Delta X_o \ge 0$ and

$\Delta X_o \neq 0$). We want to estimate the vector of outputs β_o in which $\beta_o = Y_o + \Delta Y_o$, $\Delta Y_o \geq 0$, such that the efficiency remains at its previous level (φ_o^*) and DMU$_{n+1}$ is the new DMU which represents DMU$_o$ after changes in its inputs and outputs. The model (9) was provided by Wei et al. [5] to assess the new DMU i.e., DMU$_o$:

$$
\begin{aligned}
& Max\,(\beta_{1o}, \beta_{2o}, \ldots, \beta_{so}) && (9)\\
& s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \leq \alpha_{io}, && i = 1,2,\ldots,m\\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi_o^* \beta_{ro}, && r = 1,2,\ldots,s\\
& y_{ro} \leq \beta_{ro}, && r = 1,2,\ldots,s\\
& \sum_{j=1}^{n} \lambda_j = 1 &&\\
& \lambda_j \geq 0, && j = 1,2,\ldots,n
\end{aligned}
$$

where $\alpha_o = X_o + \Delta X_o$; ($\Delta X_o \geq 0$ and $\Delta X_o \neq 0$) and φ_o^* is given as the optimal solution of the model (10).

$$
\begin{aligned}
& \varphi^* = Max\,\varphi && (10)\\
& s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, && i = 1,2,\ldots,m\\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \varphi y_{ro}, && r = 1,2,\ldots,s\\
& \sum_{j=1}^{n} \lambda_j = 1 &&\\
& \lambda_j \geq 0, && j = 1,2,\ldots,n
\end{aligned}
$$

It shall be reminded that the following Vector Maximum theorem can be used as an appropriate method to solve MOLP models.

Theorem 1 [31]**:** let $S = \left\{ X \in \mathbb{R}^n;\ AX = b,\ \ X \geq 0,\ \ b \in \mathbb{R}^m \right\}$ is the feasible region of the vector-maximum problem, then $\bar{X} \in S$ is efficient if and only if there exists a $\lambda \in \Lambda$ when

$$
\Lambda = \left\{ \lambda \in \mathbb{R}^l;\ \lambda_i > 0,\ \sum_{i=1}^{l} \lambda_i = 1 \right\}
$$

such that $\bar{X}$ maximizes the weighted-sums (composite) LP:

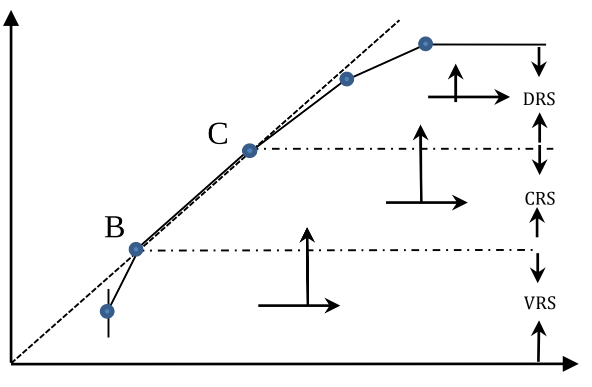
$$
Max\ \left\{ \lambda^T;\ CX;\ X \in S \right\}.
$$

4. Proposed methods

In this paper, we consider the estimation of the output in the inverse DEA problems on T_v under the condition that the classification of RTS is preserved. Namely, if the DMU$_o$ under-evaluation exhibits increasing, decreasing, or constant RTS, and its inputs increase, how much do the outputs of DMU$_o$ change so that the new DMU$_o$ exhibits the previous classification of RTS and the efficiency remains unchanged? As seen in figure 3, it is clear that when DRS prevails for DMU$_o$, an increase in outputs cannot change the primary classification of RTS for DMU$_o$. Hence, this paper considers only DMUs that exhibit constant or increasing RTS. Hence, the two cases mentioned above are provided below.

Fig. 3 Classification of RTS

4.1. Constant Returns to Scale (CRS)



Now, suppose that DMU$_o$ exhibits CRS. This section considers the output estimation problem and the condition of preserving the type of RTS. To that end, the following model is proposed which δ_o^*

and τ_o^* given in proposition (1) are applied to determine the boundary of β_o:

$$Max(\beta_{1o},\beta_{2o},...,\beta_{so}) \qquad (11)$$
$$s.t.\sum_{j=1}^{n}\lambda_j x_{ij} \le \alpha_{io}, \quad i=1,2,...,m$$
$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \varphi^*\beta_{ro}, \; r=1,2,...,s$$
$$\underset{\sim}{y}_{ro} \le \beta_{ro} \le \tilde{y}_{ro}, \quad r=1, 2, ..., s$$
$$\sum_{j=1}^{n}\lambda_j = 1$$
$$\lambda_j \ge 0, \qquad j=1,2,...,n$$

By using model (11), the permissible amount of output increase is obtained, while the type of RTS is maintained in which $\underset{\sim}{Y}_o = \min\{1, \delta_o^*\}Y_o$ & $\tilde{Y}_o = \max\{1,\tau_o^*\}Y_o$. Moreover, α_o, φ_o^*, δ_o^*,τ_o^*, are as defined in section 3.4.

Since the model (11) is a MOLP, the Vector Maxima theorem [32] with weights equal to 1 is applied to tackle the presented model.

Theorem 2: Let (λ^*,β_o^*) is the Pareto optimal solution of model (11). If (X_o,Y_o) exhibits CRS then, $(\alpha_o = X_o+\Delta X_o,\beta_o^*)$ exhibits CRS.

Proof: First consider the following remarks:

Remark 1: If (X_o,Y_o) exhibits CRS, then input decrease or increase does not change the type of RTS. Therefore, for all $\bar{X} \ge X_o$, $\bar{X} \ge X_o$ also exhibits CRS (as shown in Fig.3.).

Remark 2: In the case of DMU$_o$ exhibits CRS, suppose that τ_o^* and δ_o^* are the optimal solutions of models (5) and (6), respectively. Let $\tau_o^*=\delta_o^*$. Since $\tau_o^* = (\sum_{j\in E_o}\hat{\lambda}_j^*)^{-1} \ge 1$ and $(\delta_o^* = (\sum_{j\in E_o}\hat{\lambda}_j^*)^{-1} \le 1)$, where $\hat{\lambda}_j^*$ $(j\in E_o)$ is the optimal solutions of the models (3) and (4) in which $\sum_{j\in E_o}\hat{\lambda}_j^* \le 1$ and $\sum_{j\in E_o}\hat{\lambda}_j^* \ge 1$. Then, the equation $\tau_o^*=\delta_o^*$ happens when $\sum_{j\in E_o}\hat{\lambda}_j^* = 1$, i.e., the DMU lies on the frontier, then we have $\tau_o^*=\delta_o^*=1$. Therefore, $Y_o = \underset{\sim}{Y}_o = \tilde{Y}_o$ and model (11) implies that $\beta_o^* = Y_o$. In this situation, by increasing the input to maintain the RTS constant, $\beta_o^* = Y_o$ should be held. So, according to the remark (1), (α_o,β_o^*) exhibits CRS.

Remark 3: If (X_o,Y_o) exhibits CRS and $\tau_o^* > 1$, then the new DMU$_o$ is considered as $\left(\frac{\theta_o^* X_o}{\sum_{j\in E_o}\hat{\lambda}_j^*},\frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}_j^*}\right)$ which $(\theta_o^*,\hat{\lambda}^*)$ is the optimal solution of the input-oriented CCR model. It should be shown that the new DMU$_o$ exhibits CRS. To do so, two following cases are investigated:

a) $\left(\frac{\theta_o^* X_o}{\sum_{j\in E_o}\hat{\lambda}_j^*},\frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}_j^*}\right)$ belongs to T_v.

b) $\theta_o^*=1$, i.e., DMU$_O$ is a CCR efficient DMU.

Proof: a) It is evident that $(\theta_o^* X_o,Y_o)\in T_c$. Let $\sum_{j\in E_o}\hat{\lambda}_j^* = K$, then:

$$\sum_{j\in E_o}\hat{\lambda}^*_j X_j \le \theta^*_o X_o \to \frac{\sum_{j\in E_o}\hat{\lambda}^*_j X_j}{K} \le \frac{\theta^*_o X_o}{K}$$

$$\sum_{j\in E_o}\hat{\lambda}^*_j Y_j \ge Y_o \qquad \to \frac{\sum_{j\in E_o}\hat{\lambda}^*_j Y_j}{K} \ge \frac{Y_o}{K}$$

$$\hat{\lambda}^*_j \ge 0 \ (j\in E_o) \to \frac{\hat{\lambda}^*_j}{K} \ge 0 \quad (j\in E_o)$$

Now, by setting $\frac{\theta^* X_o}{K} = \breve{X}_o$, $\frac{Y_o}{K} = \breve{Y}_o$ and $\frac{\hat{\lambda}^*_j}{K} = \mu_j$ we have:

$$\sum_{j\in E_o}\mu^*{}_j X_j \le \breve{X}_o$$
$$\sum_{j\in E_o}\mu^*{}_j Y_j \ge \breve{Y}_o$$
$$\mu^*{}_j \ge 0 \quad (j\in E_o)$$
$$\sum_{j\in E_o}\mu^*{}_j = 1$$

which means that $\left(\frac{\theta^*_o X_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}, \frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}\right)$ belongs to T_v.

Proof (b): Since the optimal projection of DMUO is in T_c, so for $\left(\theta^*_o X_o, Y_o\right) \in T_c$ we have $\theta^*_o = 1$. Also, $\left(\frac{\theta^*_o X_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}, \frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}\right)$ is a multiple of the $\left(\theta^*_o X_o, Y_o\right)$. It is clear that the efficiency of each point in T_c is equal to efficiency of its multiple. Therefore, for $\left(\frac{\theta^*_o X_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}, \frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}\right)$ we will have $\theta^*_o = 1$.

Moreover, if CCR efficiency score for any DMU in T_v is equal to 1, then the mentioned DMU exhibits CRS. Therefore, the new DMU, also exhibits CRS.

Now, suppose that $\eta_o = \frac{1}{\sum_{j\in E_o}\hat{\lambda}^*_j}$. It was shown that $\left(\theta^*_o \eta^*_o X_o, \eta^*_o Y_o\right)$ exhibits CRS. Also, τ^*_o and δ^*_o defined in the models (5) and (6) are optimal solutions. Therefore, for each of these optimal solutions $\left(\theta^*_o \tau^*_o X_o, \tau^*_o Y_o\right)$ and $\left(\theta^*_o \delta^*_o X_o, \delta^*_o Y_o\right)$ also exhibit CRS.

So far it has been shown that $\left(\frac{\theta^*_o X_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}, \frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}\right)$ belongs to T_v and exhibits CRS. Seiford and Zhu [1] declared that $\hat{y}_{ro} = \chi y_{ro} \ ;(r = 1,2,...,s)$ in which $\min\{1, \delta^*_o\} \le \chi \le \max\{1, \tau^*_o\}$. Now considering that in model (11) $\underset{\sim}{Y}_o = \min\{1, \delta^*_o\} Y_o$, $\tilde{Y}_o = \max\{1, \tau^*_o\} Y_o$, and constriant $\underset{\sim}{Y}_o \le \beta_o \le \tilde{Y}_o$ is taken into account, if we show that $(\alpha_o, \tilde{Y}_o)$ and $\left(\alpha_o, \underset{\sim}{Y}_o\right)$ exhibit CRS, then for all output β^*_o, (α_o, β^*_o) also exhibits CRS.

According to the result (3), it is clear that for all optimal solution of CCR model for DMU_o i.e., $\left(\theta^*_o, \hat{\lambda}^*\right)$, $\left(\frac{\theta^*_o X_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}, \frac{Y_o}{\sum_{j\in E_o}\hat{\lambda}^*_j}\right)$ exhibits CRS. Therfore, according to the proposition (1) for all χ satisfying in $\min\{1, \delta^*_o\} \le \chi \le \max\{1, \tau^*_o\}$, $\left(X_o, \chi Y_o\right)$ also exhibit CRS. Thus,

$(X_o, \tilde{Y}_o)$ and $(X_o, \underset{\sim}{Y}_o)$ also exhibit CRS. Moreover, remark (1) states that a decrease or increase in an input does not change type of the RTS, so $(\alpha_o, \tilde{Y}_o)$ and $(\alpha_o, \underset{\sim}{Y}_o)$ exhibit CRS. Then, for all output Y including $Y = \beta_o^*$ that $\underset{\sim}{Y}_o \le Y \le \tilde{Y}_o$, (α_o, Y) exhibits CRS. Therefore, (α_o, β_o^*) exhibits CRS, which proves the theorem.

4.2 Increasing Returns to Scale (IRS)

Now, suppose that the under evaluation DMU i.e., DMU$_o$ exhibits IRS. In this case, two methods are suggested to tackle the output estimation problem in connection with preserving the type of RTS:

Case 1: Similar to what stated in section 3.2., the amounts obtained from proposition two are applied as the upper bound of β_{ro} in the inverse model (12). However, it is crucial to note that in the output estimation problem the goal is maintaining the primary type of RTS, so applying a minimal Archimedean number, $\varepsilon > 0$, in the model is necessary to ensure that the DMU is not placed in the CRS region. The presented model is as follows:

$$Max(\beta_{1o}, \beta_{2o}, ..., \beta_{so}) \qquad (12)$$

$$\begin{aligned} s.t. \sum_{j=1}^{n} \lambda_j x_{ij} &\le \alpha_{io}, & i &= 1,2,...,m \\ \sum_{j=1}^{n} \lambda_j y_{rj} &\ge \varphi_o^* \beta_{ro}, & r &= 1,2,...,s \\ y_{ro} \le \beta_{ro} &\le (\bar{y}_{ro} - \varepsilon), & r &= 1,2,...,s \\ \sum_{j=1}^{n} \lambda_j &= 1 \\ \lambda_j &\ge 0, & j &= 1,2,...,n \end{aligned}$$

which $\bar{Y}_o = \chi Y_o$, and α_o, φ_o^* are defined in section 3.2. Model (12) is a MOLP, so the Vector Maxima theorem [32] with weights equal to 1 is applied to tackle the presented model.

Theorem 3: Let (λ^*, β_o^*) is a Pareto optimal solution of the model (12). If (X_o, Y_o) exhibits IRS, then $(\alpha_o = X_o + \Delta X_o, \beta_o^*)$ exhibits IRS.

Proof: It is clear that $\delta_o^* > 1$. The following remarks are applied to show the correctness of the Theorem:

Remark 4: If (X_o, Y_o) exhibits IRS, then a decrease or increase of an input does not change the type of RTS. So, $(\bar{X}, Y_o)$ for all $\bar{X} \ge X_o$ also exhibit IRS (to elaborate more, see Fig.3.).

Remark 5: Let (X_o, Y_o) exhibits IRS, and δ_o^* is the optimal solution of the model (6) such that $\delta_o^* > 1$. Then, if $(\alpha_o = X_o + \Delta X_o, \beta_o^*)$ is the coordinates of the new DMU$_o$, Then DMU$_o$ exhibits IRS. According to proposition 2, $\chi \in R^{IRS} = \{\chi; 1 \le \chi \le \delta_o^*\}$ which χ shows the proportional changes in outputs, i.e., $\bar{Y}_o = \chi Y_o$. Moreover, in the model (12), the constraint $Y_o \le \beta_o \le (\bar{Y}_o - \varepsilon)$ is held. Therefore, (X_o, β_o^*) exhibits IRS.

According to remarks (4) and (5) DMU$_O$ with $(\alpha_o = X_o + \Delta X_o, \beta_o^*)$ also exhibits IRS.

In the second method, we use the closest MPSS to DMU$_O$ for finding the upper limit for outputs of DMU$_o$ and put this upper limit in the inverse model so that the IRS is preserved for DMU$_o$. Therefore, we put

y_{ro}^{MPSS} obtained form relation (8) as an upper limit in the following model:

$$Max(\beta_{1o}, \beta_{2o}, ..., \beta_{so}) \quad (13)$$

$$s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \le \alpha_{io}, \quad i = 1, 2, ..., m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge \varphi_o^* \beta_{ro}, \ r = 1, 2, ..., s$$

$$y_{ro} \le \beta_{ro} \le (y_{ro}^{MPSS} - \varepsilon), r = 1, 2, ..., s$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0, \quad j = 1, 2, ..., n$$

where α_o represents $X_o + \Delta X_o$ ($\Delta X_o \ge 0$ and $\Delta X_o \ne 0$) as before and φ_o^* is given as the optimal value of the model (10) and the ε is a very small non-Archimedean number.

Theorem 4: Model (13) is feasible and has a finite optimal solution.

Proof: Clearly $\beta_o = Y_o$, $\varphi_o^* = 1$, $\lambda_o = 1$, $\lambda_k = 0$, $(k \ne o)$ is a feasible solution for the model (13).

We know that $(\lambda, \beta_o) \in \mathbb{R}^n \times \mathbb{R}^s$ are variables, so the set of recession directions are as follows:

$$D = \left\{ \begin{array}{l} (d_1, d_2, ..., d_n, d_1^{'}, ..., d_s^{'}) \ne 0; \sum_k d_k x_k \le 0; \forall k, \\ \sum_k d_k y_{rk} - \varphi_o^* \ge 0, \ d_r^{'} = 0, \ r = 1, ..., s \end{array} \right\}$$

According to the constraint $\sum_k d_k x_k \le 0$, we would have two cases:

a) If all components of x are positive, then the feasible region would be bounded because the constraint $\sum_k d_k x_k \le 0$ implies that $(d_1, d_2, ..., d_n) = 0$. Also, according to the third constraint we have $d_r^{'} = 0, (r = 1, ..., s)$ and in this case there is no recession direction. Therefore, the feasible region is bound. Thus, the model has a finite optimal solution.

b) If at least one x_l is negative, then we assume $(d_1, ..., d_l, ..., d_n) = (0, ..., 1, ..., 0)$. So, the second and third constraints imply that $d_l y_{rl} \ge 0$. Then, we can find a recession direction like $(d_1, ..., d_l, ..., d_n, d_1^{'}, ..., d_s^{'}) = (0, ..., 1, ..., 0, 0, ..., 0)$ and in this case, the feasible region is unbounded. Now,

$$cd = (0, ..., 0, \beta_{1o}, ..., \beta_{so}).(0, ..., 1, ..., 0, 0, ..., 0) = 0$$

It shows that the objective function would not be infinite. Consequently, the given model is feasible, and the objective function is finite. In the following, we provide a numerical example.

5. Numerical Example

Assume that 20 DMUs are given as follows:

Table 1. Twenty DMUs, Including information of Iranian bank branches

DMUs	Personnel Score	Received Claims	Profit Received	Loan	Interest On Short-Term Deposits	Interest on Long-Term Deposits	Interest on Current Deposits
	i1	i2	i3	O1	O2	O3	O4
DMU1	12	8,820	39,766	1,704	159,268	205,772	24,725
DMU2	8	6,056	21,623	631	87,890	114,102	24,232
DMU3	8	6,029	23,806	1,328	109,849	139,903	28,173

DMU4	7	5,105	15,271	1,012	81,732	93,508	10,287
DMU5	6	4,750	14,902	841	102,673	57,815	12,855
DMU6	8	6,600	35,063	1,539	90,697	187,684	14,864
DMU7	6	4,259	10,096	845	59,062	59,296	19,572
DMU8	7	5,268	42,831	974	71,515	276,756	20,458
DMU9	8	5,551	28,920	910	108,341	169,019	53,406
DMU10	6	5,806	19,751	547	53,178	80,598	11,514
DMU11	9	4,990	24,042	628	101,074	112,216	8,927
DMU12	11	6,911	38,773	1,454	187,459	208,145	22,563
DMU13	7	4,680	14,561	876	83,393	86,538	13,593
DMU14	9	6,715	37,909	2,579	217,896	184,680	18,665
DMU15	10	6,937	21,439	1,200	88,195	109,366	33,684
DMU16	12	9,190	53,061	4,305	365,082	237,591	32,618
DMU17	8	5,218	32,107	545	95,324	234,707	18,724
DMU18	6	5,166	22,805	549	233,181	117,132	35,496
DMU19	8	5,727	30,078	1,254	80,771	133,154	45,644
DMU20	6	4,554	17,801	506	46,263	94,002	10,059

The following are the definitions of outputs and inputs in Table 1.

- Personnel score: The score is calculated by combining and adjusting the weighted values of many parameters, including the number of people, executive positions, work experience, compensation, and staff training hours.
- Received Claims: When the bank provides banking facilities to the customers, and the customer cannot pay his installments at the specified time, in this case, the customer's debt to the bank for the facilities is called the bank's claims. Thus, the amount of collection of these claims by the bank over a while is called received claims.
- Profit received: The interest the bank receives from customers for providing facilities.
- Loan: The amount of money that the bank provides to the customers, and the customers repay the mentioned facilities along with the interest based on the type of contract.
- Interest On Short-Term Deposits: The interest the bank pays to customers' short-term accounts. The higher this number is, the lower the performance of the bank.
- Interest On Long-Term Deposits: The interest the bank pays to customers' short-term accounts. The higher this number is, the lower the performance of the bank.
- Interest On Current Deposits: The interest the bank pays to customers' checking accounts. The higher this number is, the lower the performance of the bank.

As can be seen, the second left column of Table (2) shows the type of RTS for the DMUs, which are obtained using the models of sections 2-3. To show the practical application of the proposed models (11) and (12), we have considered all the DMUs that exhibit constant or increasing RTS. Respectively, from the output estimation perspective, by using the two proposed models (11) and (12), the new outputs have been estimated under the conditions that the previous classification of RTS (either constant or increasing) of the given DMUs has been maintained. The mentioned statement shows the practicality of the proposed models. For instance, if the model (11) is applied for DMU_{16} which exhibits CRS, the outputs are estimated so that the new classification of the RTS is maintained and remains constant.

Furthermore, suppose the model (12) is applied for which exhibits IRS. In that case, the outputs are estimated so that the new classification of the RTS for the new DMU is maintained and remains increasing. However, as seen in Figure 3, if the standard output estimation model proposed by Wei et al. (9) is used, the increase in outputs can continue until the classification of RTS of DMUs with CRS or IRS is changed. This means that model (9) does not preserve the primary classification of RTS of the DMU under evaluation.

Table 2. The type of RTS for the DMUs

DMUs	**RTS**	α_{1j}	α_{2j}	α_{3j}	β^*_{1j}	β^*_{2j}	β^*_{3j}	β^*_{4j}	RTS_N
DMU1	DRS								
DMU2	IRS	0.7333	0.7249	0.4483	0.186	0.306	0.412	0.577	IRS
DMU3	IRS	0.7333	0.7217	0.4935	0.360	0.351	0.517	0.615	IRS
DMU4	IRS	0.6417	0.6111	0.3166	0.293	0.273	0.338	0.295	IRS
DMU5	IRS	0.5500	0.5686	0.3089	0.217	0.357	0.315	0.474	IRS
DMU6	DRS								
DMU7	IRS	0.5500	0.5099	0.2093	0.196	0.196	0.231	0.386	IRS
DMU8	DRS								
DMU9	CRS	0.7333	0.6645	0.5995	0.211	0.297	0.611	1.000	CRS
DMU10	IRS	0.5500	0.6950	0.4095	0.201	0.230	0.385	0.341	IRS
DMU11	IRS	0.8250	0.5973	0.4984	0.193	0.366	0.427	0.221	IRS
DMU12	DRS								
DMU13	IRS	0.6417	0.5602	0.3019	0.220	0.319	0.313	0.406	IRS
DMUs	RTS	α_{1j}	α_{2j}	α_{3j}	β^*_{1j}	β^*_{2j}	β^*_{3j}	β^*_{4j}	RTS_N
DMU14	IRS	0.8250	0.8037	0.7859	0.640	0.637	0.713	0.373	IRS
DMU15	IRS	0.9167	0.8303	0.4445	0.279	0.294	0.473	0.705	IRS
DMU16	CRS	1.1000	1.1000	1.1000	1.000	1.000	0.858	0.611	CRS
DMU17	DRS								
DMU18	IRS	0.5500	0.6184	0.4728	0.137	0.661	0.453	0.665	IRS
DMU19	IRS	0.7333	0.6855	0.6235	0.307	0.233	0.506	0.900	IRS
DMU20	IRS	0.5500	0.5451	0.3690	0.164	0.177	0.467	0.263	IRS

6. Conclusion

The RTS is an economic concept that would play a crucial role regarding the expansion or limitation of the under evaluation DMUs in the field of DEA. Determining the classification of RTS for a DMU enables the decision-maker to decide on the DMU's expansion or limitation to have the best performance. Although the problem of stability and sensitivity of classification of RTS in DEA has been studied in the literature, the stated issue has not been presented in the inverse DEA area. So, in this paper, we considered the output estimation problem on T_v in which besides preserving the efficiency of the under-evaluation DMU, the classification of RTS remains unchanged. For this purpose, two cases are discussed in this research: when the DMU operates under CRS, and the other case considers DMUs that exhibit IRS. Regarding the DMUs exhibiting CRS, we provided two methods. In the first method efficient DMUs from the reference set were used to model the problem. In the other one, the upper bound obtained from the sensitivity analysis method presented by Seiford and Joe Zhu [17] was applied to determine the maximum output increase such that the primary type of RTS is maintained. Furthermore, for the DMUs that operate under IRS a method based on the MPSS was provided to address the problem. Also, we proved that the presented model is feasible and has a finite solution. Finally, a numerical example was provided for evaluating the models' results. The results showed that our models preserve the primary classification of RTS for the under evaluation DMUs, while previous models provided in the literature did not manage to maintain the classification of RTS in the output estimation problem. Therefore, our models help decision makers have enough information about how to invest to gain more profit or how should continue their activities to preserve the company from bankruptcy.

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