A New Compounded Model Based on Fre'chet Distribution With Application in Failure Data

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Abstract. Recently the Extended Fre'chet distribution has appeard as the subject of so many research. This research aims to extend the Extended Fre'chet distribution as a three-parameter distribution to a four-parameter life-time distribution named Extended Frechet Power Series distributions. The Extended Frechet Power Series distribution happens to have decreasing, increasing, bathtub, and upside down bathtub hazard shapes for different values of its parameters. This proposed class of distributions can apply to modeling of hazard rate data in the Engineering, Medical, Economics and Insurance fildes. The new compounded distributions considered in this paper are seen to provide models for all of the different shaped hazard rates mentioned above. This flexibility permits the data to determine the nature of the hazard function without its being inadvertently imposed through the selection of an improper model. In this paper, some of the statistical properties such as the quantiles, moment generating functions and ordered statisticalls has been studied for this distributions. Also, the maximum likelihood estimation and capability of the quantile measures are discussed. The properties of the mean reversed residual life and failure rate functions have been discussed related to the order statistics. Finally, using two data sets leading to the numerical experiment, the functioning of the maximum likelihood estimators and their asymptotic results of Extended Frechet Power Series distribution are compared to several rival destributions. The results shown that this class of distributions have the better performance than other hazard rate distributions in order to hazard data modeling.

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1. Introduction

Statistical distributions are powerful tools for describing and modeling real life data. Adding one or more additional shape parameters to a particular probability distribution model can lead to extend of more robust and flexible distributions for predicting real data. This study focused on extending the Fréchet distribution which has been widely applied in extreme value theory. The family of Extreme Value distributions has been developed as the limiting distribution for minimum or maximum of several independent and identically distributed random variables with respect to the increase of the sample size. Combining the Gumbel, Fre'chet, and Weibull distribution that are also known as type I, II, and III extreme value distributions respectively lead to different extreme value distributions. The extreme value theory often deals with events with very small probabilities. These extreme value distributions and theire other forms are massively used in finance, economics, material sciences, telecommunications, etc. Market returns data, has been extensively modeled by Frechet (or type II extreme value) distributions that are often heavy-tailed in financial applications (Alves, 2010). The standard Frechet distribution with the CDF Has been generalized by Nadaraja and Kotz (2003), to a new distribution Called Extended Frechet (EF) distribution.

Consider a system of n components in series with independent and identically distributed lifetime EF distributions. It is concluded that the lifetime of the system is also distributed according to the EF distribution. Kotz and Nadaraja (2000), described the applications of the Frechet distribution such as accelerated life testing through earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speed, track records, etc. Recently, many distribution have been introduced for modeling lifetime data by compounding two or more life-time distribution. The two parameters Exponential Geometric (EG) distribution was introduced by Adamidis and Loukas (1998) compounding an exponential distribution with a geometric distribution. Min Wang (2012) proposed a three-parameter lifetime distribution and studied many of its properties. With the same method, Kus (2007) and Tahmasbi and Rezaei (2008) studied the Exponential Poisson (EP) and the Exponential Logarithmic (EL) distribution respectively. The weibull-Geometric (WG) and the Weibull-Poisson (WP) were introduced by Barreto-Souza et al. (2010)and Lu and Shi (2011), respectively. These were natural extended version of EG and EP respectively. In addition, Rodrigues (2011) proposed the Weibull-Negative Binomial (WNB) distribution which generally is a super model of WG and WP distributions. In the same way, any lifetime distribution could have been compounded with the power series distribution. The Exponential Power Series (EPS) family of distributions was introduced by Chahkandi and Ganjali (2009), which includes EP, EG and EL as sub-models. The Weibull Power Series (WPS) was studied by Morais and Barreto-Souza(2011) which contains the EPS distribution as a special case. The WPS distribution can potentially have a decreasing, increasing, bathtub, and upside-down bathtub failure rate function. Mahmudi and Jafari (2012) proposed the Generalized Exponential Power Series (GEPS) distribution following the same method used by Morais and Barreto-Souza (2011). The Extended Weibull Power Series (EWPS) was studied by Silva et al. (2013) which contains EPS and WPS distribution as sub-models. Bagheri et al.(2015) studied the family of Generalized Modified Weibull Power Series (GMWPS) distribution The Inverse Weibull Power Series (IWPS) distribution was introduced by Shafiei et al. (2015). Alizadeh et al. (2018) introduced the Exponentiated Power Lindly Power Series distribution. Akarawak et al. (2023) proposed a four-parameter continuous distribution known as the Inverted Gompertz Fréchet (IGoFre) distribution, which is an inverse transformation of the Gompertz Frechet distribution. Ramos et al. (2019) considered different methods of estimation of the unknown parameters both from frequentist and Bayesian viewpoint of Fr´echet distribution. Gomez et al. (2024) presented the Slash-Exponential-Fréchet distribution, which is an expanded version of the Fréchet distribution. Al-Jabouri, and Al-Taee (2023) compared several methods of estimating the parameter of Frechet distribution based on different Bayesian methods with (square loss, Linux and Unix) functions. In this research, several simulation experiments were conducted according to the difference in (sample size, value of distribution parameters and estimation methods) and the results were compared based on mean square error criteria, it is possible to use other estimation methods such as (moments and percentile), for other distributions such as (Gumbel and Lindley).

Consider a system of N components with positive continuous random life times $X_1, \cdots, X_N$, where N is a discrete positive integred random variable (rv). Clearly, the life time of such system is $X_{(1)} = min(X_1, \cdots, X_n)$, if the components are parallal and $X_{(n)} = max(X_1, \cdots, X_n)$ if they are in series. This leads to compounding life time distributions. Here, N can be as a power series disturbed rv. Now, let consider the parallel system where X_i's follow EF distributions. Then, we obtain a new family of Extended Frechet power series ($EFPS$) distributions. The Complementary Extended Frechet Geometric ($CEFG$), Extended Frechet Poisson (EFP), Extended Frechet Binomial (EFB), and the Extended Frechet Logarithmic (EFL) distributions are contained in the $EFPS$ model as special cases. The application of the new class of distributions in the industry and biological organisms studies made it a well-motivated model to study. For instance, consider the time to relapse of cancer under the first activation scheme. We denote the number of carcinogenic cells for an individual left active after the initial treatment as N, and the spent time for the ith carcinogenic cell to produce a detectable cancer mass by X_i for $i \geq 1$. Let N is distributed according to a power series distribution. Suppose that X_i for $i \geq 1$ is a sequence of independent and identically distributed EF rv's independent of N. Then one can model the time to relapse of cancer of a susceptible individual by the $EFPS$ class of distributions. As the second example, suppose that the hazard of an item happens according to the appearance of an unknown number of initial defects of the same kind, denoted by N, which can be identified only after the occurrence of the failure and repaired perfectly. The time to the failure of the device is represented by X_i due to the ith defect, for $i \geq 1$. Let $X_i, i = 1, \cdots, N$,be independent and identically distributed EF rv's and are independent of N, and N follows power series distribution Then the time to the first failure can be properly modeled by the $EFPS$ class of distributions. Also, there might be a question regarding the first activation scheme for certain diseaes. Suppose that the number of latent factors, N, that should all be activated by failure is distributed according to a power series distribution. Let X_i denote the resistant time to a disease manifestation due to the ith latent factor follow an EF distribution. In the last-activation scheme, the hazard happens after all N factors activation (see Cooner et al. (2007)). So, the new class of distributions could be able to model the time to the failure under the last-activation scheme.

We are motivated to extend the Frechet distribution to a more flexible generalized form based on the following:

(i) When standard probability distribution is developed by the addition of shape parameter(s), the new distributions performs better when applied to model extremely skewed data as compared to basic probability distribution.

(ii) The goodness of fit can be improved up on with the addition of shape parameter(s). This model is more flexible as for its skewness and kurtosis.

(iii) To further analyse extensively, the tail properties of a distribution one can extend the basic probability distribution by the addition of a shape parameter.

The rest of the paper is organized as follows. In Section 2, we introduce the new class EFPS distributions, and the CDF, survival and hazard rate functions. Some special cases and their properties are discussed. Mathematical and statistical properties of the class of EFPS distributions are the quantile, moments, etc. are obtained in Section 3. In Section 4, we present residual life and reversed failure rate functions and discuss the order statistics. Maximum likelihood estimates of the unknown parameters are presented in Section 5. In Section 6, real data applications are given to show the flexibility and potentiality of the EFPS distribution. Conclusions are presented in Section 7.

2. The new compound distribution

Let X be distributed according to a EF distribution. Then, X has the following CDF and PDF, respectively:

$$G(x) = 1 - \left[1 - e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu}, \quad x, \mu, \alpha, \lambda > 0. \tag{1}$$

And

$$g(x) = \lambda\mu\alpha^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu-1} . x > 0. \mu > 0. \alpha > 0. \lambda > 0 \cdot \quad (2)$$

Suppose that $\boldsymbol{X_1}, \cdots$.,, given N, are an i.i.d random sample of EF distribution. Let N be a discrete random variable having a power series probability mass function (PMF) (truncated at zero) as follows:

$$\boldsymbol{P^n = P(N = n) = \frac{a_n\theta^n}{C(\theta)}, n = 1, 2, \cdots}$$

(3)

Where $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n \; \theta^n < \infty$ and its first, second, and third derivatives respect to θ exist and are shown by $C'(\cdot), C''(\cdot)$ and $C^{-1}(\cdot)$. respectively.

One can find detailed information on the power series class of distribution in Noack (1950). The binomial, Poisson, geometric and Logarithmic distributions are included in the class of power series distributions, (Jonson et al. (2005)). Table 1 shows useful quantities of some power series distributions (truncated at zero) defined by (3) such as Geometric, Poisson, Logarithmic and Binomial (with m being the number of replicas).

Table 1. Useful quantities of some power series distributions.

Distribution	a_n	$C(\theta)$	$C^{/}(\theta)$	$C^{//}(\theta)$	$C(\theta)^{-1}$	θ
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(1-\theta)^{-1}$	$\theta \in (0.1)$
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$log(\theta+1)$	$\theta \in (0.\infty)$
Logarithmic	n^{-1}	$-log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{\theta}$	$\theta \in (0.1)$
Binomial *	$\binom{m}{n}$	$(\theta+1)^m - 1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta-1)^{\frac{1}{m}} - 1$	$\theta \in (0.\infty)$

* θ is not the probability of succes here. The probability of succes of this binomial distribution is $\frac{\theta}{\theta+1}$.

Suppose that $X_{(N)} = max\{X_i\}_{i=1}^{N}$. Then the conditional CDF of $X_{(N)}|N = n$ is as follows:

$$G_{X_{(N)}|N=n}(x) = (G(x))^n = (1 - \left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu})^n, \qquad x, \mu, \alpha, \lambda > 0, n = 1,2, \cdots$$

Then we obtain

$$P(X_{(N)} \leq x, N = n) = \frac{a_n\theta^n}{C(\theta)}(G(x))^n, \; x, \alpha, \lambda > 0, n \geq 1$$

Now, the extended Fréchet power series represented by $EFPS(\alpha, \lambda, \mu, \theta)$ is defined by the marginal CDF of $X_{(N)}$ as follows:

$$F(x) = \frac{C\{\theta G(x)\}}{C(\theta)} = \frac{C\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right]\right\}}{C(\theta)}, \quad x, \mu, \alpha, \lambda, \theta > 0 \quad (4)$$

So the PDF hazard rate and the survival function of EFPS distribution are respectively as follows:

$$f(x) = \theta g(x)\frac{C'\{\theta G(x)\}}{C(\theta)} =$$

$$\theta\lambda\mu\alpha^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}} \times \left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu-1} \frac{C\prime\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right]\right\}}{C(\theta)},$$

(5)

$$h(x)=\lambda\mu\alpha^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu-1} \frac{C\prime\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right]\right\}}{C\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right]\right\}},$$

and

$$\overline{F}(x)=1-\frac{C\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right]\right\}}{C(\theta)}.$$

The $EFPS$ class of distributions includes several lifetime distributions that have been introduced and studied in the literature. Table 2 represents useful quantities and respective parameters for each distribution and shows a list of distributions that can be derived from the EFPS distribution.

Table 2. Some sub-models from the EFPS(α.μ.λ.θ) distribution.

Model	$C(\theta)$	α	λ	μ	References
FPS	$-$	$-$	$-$	1	Morais and Barreto-Souza (2011)
IE_XPS	$-$	$-$	1	1	Chahkandi and Ganjali (2009)
$IRPS$	$-$	$-$	2	1	Morais and Barreto-Souza(2011)
EFG	$\theta(1-\theta)^{-1}$	$-$	$-$	$-$	New
FG	$\theta(1-\theta)^{-1}$	$-$	$-$	1	Barreto-Souza et al . (2011)
IE_XG	$\theta(1-\theta)^{-1}$	$-$	1	1	Adamidis and Loukas (1998)
IRG	$\theta(1-\theta)^{-1}$	$-$	2	1	New
EFP_o	$e^{\theta}-1$	$-$	$-$	$-$	New
FP_o	$e^{\theta}-1$	$-$	$-$	1	Wanbo Lu and Daimin Shi (2012)
IE_XP_o	$e^{\theta}-1$	$-$	1	1	Wanbo Lu and Daimin Shi (2012)
IRP_o	$e^{\theta}-1$	$-$	2	1	New
EFB	$(\theta+1)^m-1$	$-$	$-$	$-$	New
FB	$(\theta+1)^m-1$	$-$	$-$	1	New
IE_XB	$(\theta+1)^m-1$	$-$	1	1	New
IRB	$(\theta+1)^m-1$	$-$	2	1	New
$EPLo$	$-log(1-\theta)$	$-$	$-$	$-$	New
FLo	$-log(1-\theta)$	$-$	$-$	1	New
IE_XLo	$-log(1-\theta)$		1	1	New
$IRLo$	$-log(1-\theta)$		2	1	New
G_eIWPS	-	$qc^{\frac{1}{\lambda}}$	$-$	1	New
G_eIRPS	-	$qc^{\frac{1}{2}}$	2	1	New
G_eIE_XPS	-	qc	1	1	New
					New

Where G = Geometric R = Rayleigh E_X = Exponential, Lo = Logarithmic, I= Inverse G_e = generalized and B = Binomial.

Proposition 2.1 Let $X_{(1)} = minX_{i=1}^{N}$. Then the CDF and PDF of $X_{(1)}$ are respectively as follows:

$$F_{X_{(1)}}(x) = 1 - \frac{C\{\theta[1-G(x)]\}}{C(\theta)} = 1 - \frac{C\left\{\theta\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu}\right\}}{C(\theta)}, x > 0,$$

and

$$f(x) = \theta\lambda\mu\alpha^{\lambda}x^{-(\lambda+1)}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu-1}\frac{C'\left\{\theta\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{\mu}\right\}}{C(\theta)}.$$

Proof.

$$F_{X_{(1)}}(x) = 1 - P(min(X_1,\dots,X_n) > x) = 1 - \prod_{i=1}^{n} P(X_i \geq x) = 1 - \prod_{i=1}^{n} 1 - (P(X_i \leqslant x) = 1 - (1-(1-G(x)))^n = \sum_{i=1}^{n}\frac{a_n\theta^n}{C(\theta)} - \sum_{i=1}^{n}\frac{a_n[\theta(1-G(x)]^n}{C(\theta)}$$
$$= 1 - \frac{C(\theta(1-G(x))}{C(\theta)}$$

By subsetting $G(x)$ in the above relation and taking derivative, we obtain PDF of X_1.

Proposition 2.2 The EF distribution is derived as a sub-model of the EFPS class of distributions when $\theta \longrightarrow 0^+$.

Proof. Using $C(\theta) = \sum_{n=1}^{\infty} a_n\ \theta^n$, we can write

$$\lim_{\theta\longrightarrow 0^+} F(x) = \lim_{\theta\longrightarrow 0^+}\left(1 - \frac{\sum_{n=1}^{\infty} a_n\left\{\theta\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right\}^{n}}{\sum_{n=1}^{\infty} a_n\theta^n}\right)$$
$$= \lim_{\theta\longrightarrow 0^+}\left(1 - \frac{a_1\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu} + \sum_{n=2}^{\infty} na_n\theta^{n-1}\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{n\mu}}{a_1 + \sum_{n=2}^{\infty} na_n\theta^{n-1}}\right)$$
$$= 1 - \left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}.$$

Proposition 2.3 The densitie of the EFPS family of distributions can be expressed as a linear combination of order statistics densities of EF with parameters α, λ and $n\mu$ i.e.,

$$f(x) = \sum_{n=1}^{\infty} P(N=n)g_{(1)}(x;n) = \sum_{n=1}^{\infty} P^n g(x;\alpha,\lambda,n\mu)\cdot \tag{6}$$

where $g_{(1)}(x;n)$ is the PDF of the extended Fréchet distribution with parameters α, λ and $n\mu$.

Therefore some mathematical properties of the EFPS class of distributions is obtained from those of the EF distribution.

Proof. Since $C'(\theta) = \sum_{n=1}^{\infty} na_n(C(\theta))^{n-1}$, we can write

$$f(x) = \theta g(x)\frac{C'(\theta-\theta G(x))}{C(\theta)} = \sum_{n=1}^{\infty}\theta g(x)\frac{na_n(\theta-\theta G(x))^{n-1}}{C(\theta)}$$

$$= \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} n g(x)[1 - G(x)]^{n-1} = \sum_{n=1}^{\infty} P(N = n) g_{(1)}(x; \alpha, n, \beta, \lambda)$$

in which $g_{(1)}(x; n)$ is the PDF of $X_{(1)} = \min(X_1, X_2, \cdots, X_n)$ given by

$$g_{X(1)}(x; n) = n g(x)[1 - G(x)]^{n-1} = n\lambda\mu\alpha^{\lambda} x^{-(\lambda+1)} e^{-\left(\frac{\alpha}{x}\right)^{\lambda}} \left[1 - e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{n\mu-1}$$

so $g_{(1)}(x; n)$ is the PDF of extended Frechet distribution with parameters α, λ and $n\mu$.

3. Extracting the sub-models

In this section we obtain four sub-models of EFPS such as EFP, EFG, EFB, and EFL distributions. By plotting the PDF CDF and hazard rate functions of these sub-model distributions for different values of parameters the flexibility of them are illustrated.

3.1 Extended Fréchet geometric distribution

The extended Fréchet geometric (EFG) distribution is derived by taking $a_n = 1$ and $C(\theta) = \theta(1-\theta)^{-1}(0 < \theta < 1)$. Therefore the CDF, PDF and hazard rate functions of EFG distribution are repectively as follows.

$$F(x) = \frac{(1-\theta)\left(1 - \left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)}{1 - \theta\left(1 - \left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)}, \qquad x, \mu, \alpha, \lambda, \theta > 0,$$

$$f(x) = \frac{\lambda\, \mu\, \alpha^{\lambda} x^{-\lambda-1} (1-\theta) \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}} \left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu-1}}{\theta\left(1 - \theta\left(1 - \left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)\right)}, \qquad x, \mu, \alpha, \lambda, \theta > 0,$$

and

$$h(x) = -\frac{\alpha^{\lambda} x^{-\lambda-1} (-1+\theta)\lambda\, \mu}{\theta\left\{\left(\mathrm{e}^{\left(\frac{\alpha}{x}\right)^{\lambda}} - 1\right)\left(\theta\left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu} - \theta + 1\right)\right\}}, \qquad x, \mu, \alpha, \lambda, \theta > 0 \cdot$$

Figure 1 illustrate the PDF, CDF and hazard rate functions of the EFG distribution for different values of parameters. Figure 1 indicates that the EFG distribution has proper PDF and that the PDF of the DFG model is non-monotonic. Also, the shape of the hazard function of EFG model can be increasing- decreasing and bathtub failure rates.

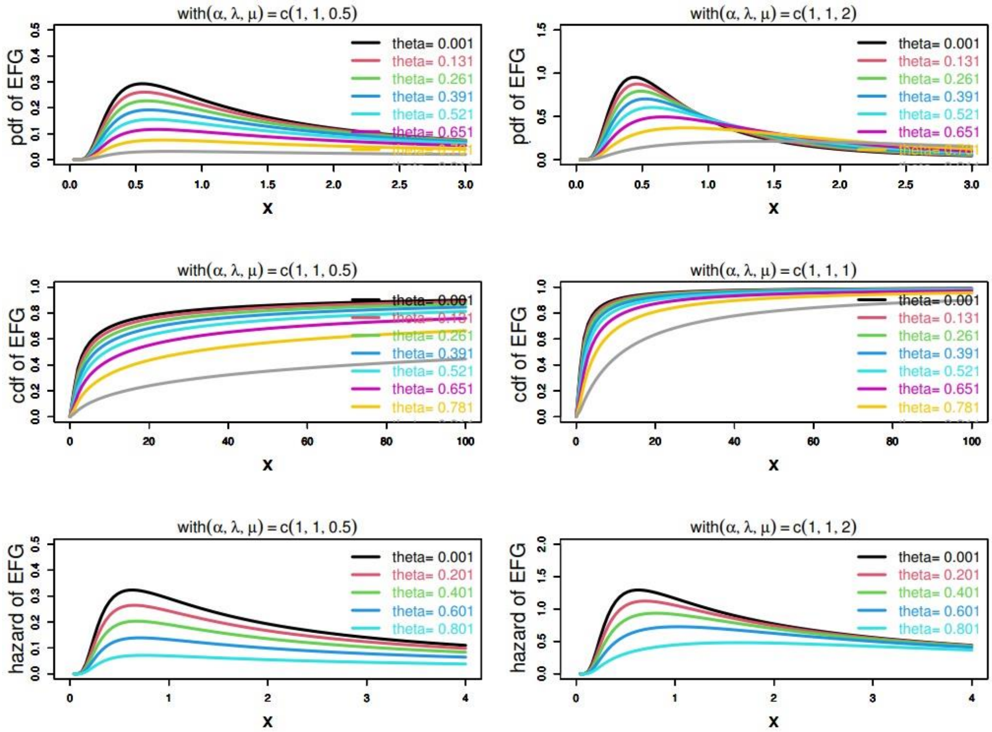

Figure 1. Plots of PDF , CDF and hazard rate function of the EFG distribution.

3.2 Extended Fréchet Poisson distribution

The extended Fréchet Poisson (EFP) distribution is derived by taking $a_n = (n!)^{-1}$and $C(\theta) = e^{\theta} - 1,\ \theta > 0$. The CDF PDF and the hazard rate functions of EFP distribution are respectively as follows.

$$F(x) = \frac{1 - \left(\mathrm{e}^{\theta\left(1-\left(1-\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)} \right)}{1 - e^{\theta}}, \qquad x, \mu, \alpha, \lambda, \theta > 0,$$

and

$$f(x) = \frac{\lambda\, \mu\, \alpha^{\lambda} x^{-\lambda-1}}{\mathrm{e}^{\theta} - 1} \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}} \left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu-1} \mathrm{e}^{\theta\left(1-\left(1-\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)}, \qquad x, \mu, \alpha, \lambda, \theta > 0,$$

$$h(x) = -\frac{\lambda\, \mu\, \alpha^{\lambda} x^{-\lambda-1} \mathrm{e}^{-\theta\left(1-\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu} - \left(\frac{\alpha}{x}\right)^{\lambda} + \theta} \left(1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu-1}}{\mathrm{e}^{-\theta\left(-1+\left(1-\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)} - \mathrm{e}^{\theta}}, \qquad x, \mu, \alpha, \lambda, \theta > 0 \cdot$$

Figure. 2 illustrate the PDF , CDF and the hazard rate function of the EFP distribution for different values of parameters.

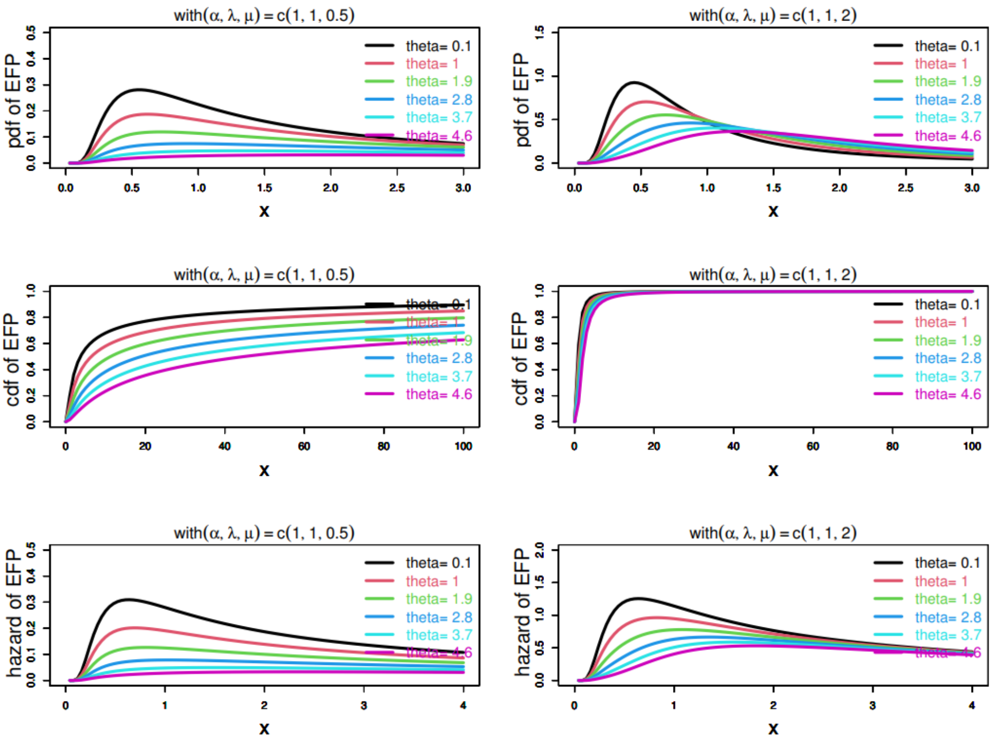

Figure 2. Plots of PDF , CDF and hazard rate function of the EFP distribution.

3.3 Extended Fréchet Binomial distribution

The extended Fréchet Binomial (EFB) distribution is obtained by taking $a_n = \binom{m}{n}$ and $C(\theta) = (\theta+1)^m - 1, (\theta > 0)$ where $m\,(m \geq n)$. The CDF, PDF and hazard rate functionn of EFB distribution are repectively as foloows.

$$F(x) = \frac{\left(\theta\left(1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)+1\right)^{m}-1}{(\theta+1)^m - 1},$$

$$f(x) = \frac{m\lambda\,\mu\,\alpha^{\lambda}x^{-\lambda-1}}{(\theta+1)^m-1}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu-1} \times \left(\theta\left(1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)+1\right)^{m-1},$$

and

$$h(x) = \frac{\alpha^{\lambda}x^{-\lambda-1}\lambda\,\mu\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\left(m\left(-\theta\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}+\theta+1\right)\right)^{m-1}}{\left((\theta+1)^m-\left(-\theta\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}+\theta+1\right)^{m}\right)^{-1}\left(e^{\left(\frac{\alpha}{x}\right)^{\lambda}}-1\right)}.$$

Figure 3 illustrate the PDF, CDF and the hazard rate function of the EFB distribution for different values of parameters.

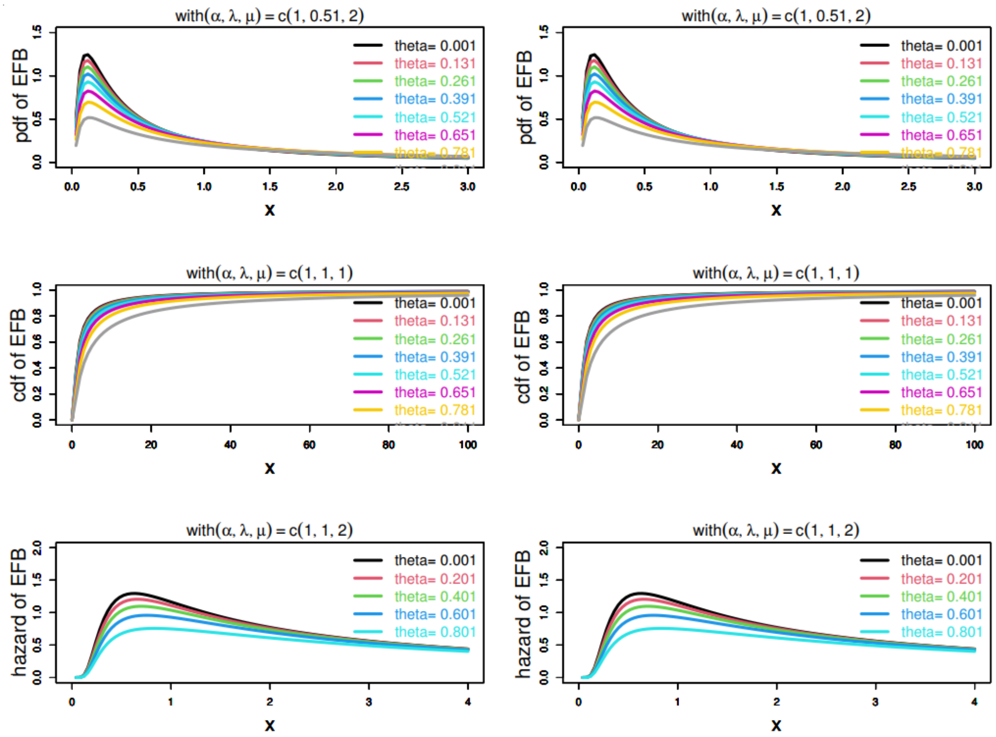

Figure 3. Plots of PDF , CDF and hazard rate function of the EFB distribution.

3.4 Extended Fréchet Logarithmic distribution

The xtended Frechet Logarithmic (EFL) distribution is obtained by taking $a_n = n^{-1}$ and $C(\theta) = -\log(1-\theta)$, $0 < \theta < 1$. The CDF PDF and the hazard rate functions of the EFL distribution are respectively as follows.

$$F(x) = \frac{log\left(1-\theta\left(1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)\right)}{log(1-\theta)}, \quad x,\alpha,\theta,\mu,\lambda > 0,$$

$$f(x) = -\frac{\lambda\,\mu\,\alpha^{\lambda}x^{-\lambda-1}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu-1}}{log(1-\theta)\left(1-\theta\left(1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)\right)}, \quad x,\alpha,\theta,\mu,\lambda > 0,$$

$$h(x) = \frac{-\lambda\,\mu\,\alpha^{\lambda}x^{-\lambda-1}e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu-1}}{\left(\theta\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}-\theta+1\right)\left(-\log(1-\theta)+\log\left(\theta\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}-\theta+1\right)\right)}.$$

figure 4 illustrate the PDF, CDF and the hazard rate function of the EFL distribution for different values of parameters.

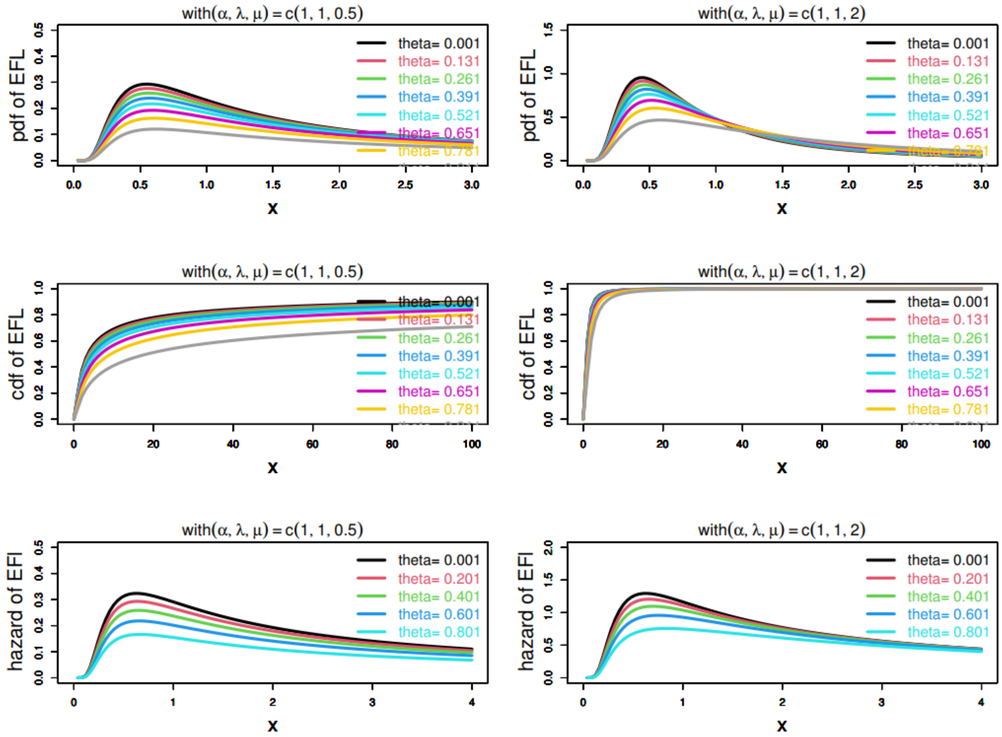

Figure 4. Plots of PDF , CDF and hazard rate function of the EFL distribution.

4. Evaluation of the properties of the EFPS distribution

In this section some of statistical properties of EFPS distribution such as incomplete and ordinary moments quantiles moment generating function mean deviation residual life and reversed failure rate functions and Bonferroni and Lorenz Curves.

4.1 The kewness and the kurtosis

Suppose that random variable X is distributed as (5). The quantile function say $Q(p)$ defined by $F(Q(p)) = p$ for $0 < p < 1$ is the root of

$$Q(p) = \alpha\left(-\log(1-[1-\frac{C^{-1}\{C(\theta)p\}}{\theta}]^{\frac{1}{\mu}})^{\frac{-1}{\lambda}}\right). \tag{7}$$

Also the Galton' skewness and the Moors' kurtosis defined by Galton (1883) abd Moors (1988) are respectively obtained as follows:

$$skewness = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{4}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

$$kurtosis = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Figure 5 illustarte the behavior of the Galton' skewness and Moors' kurtosis for EFP as functions of θ for representative values of $\alpha,\ \lambda$ and μ.

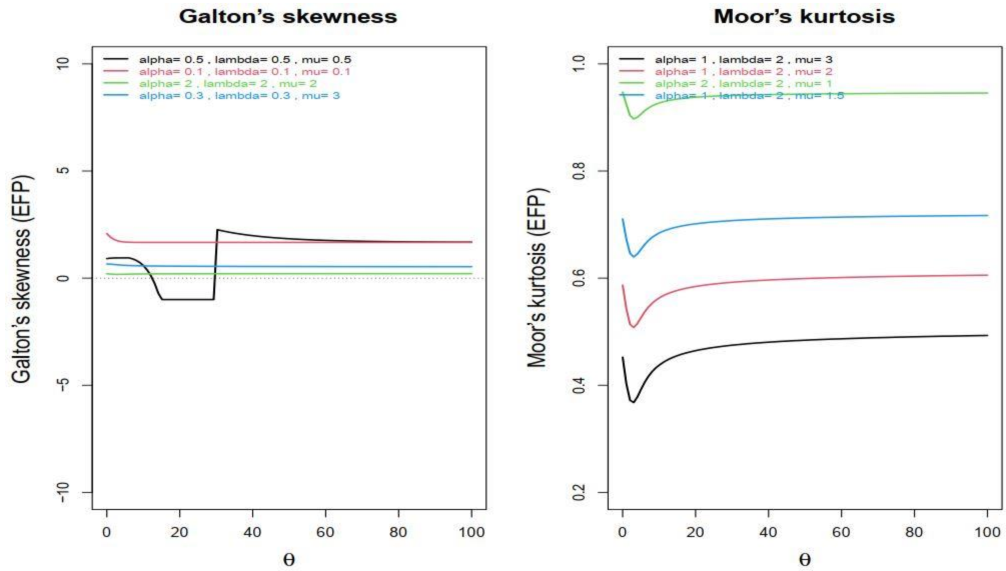

Figure 5. Galton's skewness and Moors' kurtosis for EFP for different values of α, λ and μ.

It is illustarted in the Figure 5 that for the EFP distribution both Galton' skewness and Moors' kurtosis decrease and stabilize when $\alpha.\lambda$ and μ are not getting very large and θ increases.

4.2 Moments and generator function

The r_{th} moment of X is derived from equation (6) as

$$\mu'_r = \int_0^{\infty} x^r f(x)dx$$

$$= \sum_{n=1}^{\infty} P^n n\lambda\mu\alpha^{\lambda} \int_0^{\infty} x^{(r-\lambda-1)} e^{-\left(\frac{\alpha}{x}\right)^{\lambda}} \left[1 - e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right]^{n\mu-1} dx \cdot \quad (8)$$

Using the series expansion

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} z^i, \ |z| < 1. \quad (9)$$

in which $b > 0$ is non-integer real Equation (8) becomes

$$\mu'_r = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^j \binom{b-1}{j} P^n n\lambda\mu\alpha^{\lambda} \int_0^{\infty} x^{r-\lambda-1} e^{-(j+1)\left(\frac{\alpha}{x}\right)^{\lambda}} dx$$

$$= \frac{n\mu\alpha^r}{(j+1)^{1-\frac{r}{\lambda}}} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \frac{a_n\theta^n}{C(\theta)} \Gamma\left(1 - \frac{r}{\lambda}\right) \cdot \quad (10)$$

The central moments μ_r and cumulants κ_r of the $EFPS$ distribution can be determined from equation (10) as $\mu_r = \sum_{m=0}^{r} \binom{r}{m} (-1)^m \mu'^{m}_1$, μ'_{r-m} and $\kappa_r = \mu'_r - \sum_{m=1}^{r-1} \binom{r-1}{m-1} \kappa_m \ \ \mu'_{r-m}$, respectively, where $\kappa_1 = \mu'_1$, $\kappa_2 = \mu'_2 - \ \mu'^2_1$, $\kappa_3 = \mu'^{-}_3 3\mu'_2 \ \mu'_1 + 2\mu'^3_1$ and $\kappa_4 = \mu'_4 - 4\mu'_1\mu'_3 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1$, etc. Moreover the skewness and kurtosis can be obtained from the third and fourth standardized cumulants in the forms $SK = \frac{\kappa_3}{\sqrt{\kappa_2^3}}$ and $KU = \frac{\kappa_4}{\kappa_2^2}$, respectively.

To derive the Bonferroni and Lorenz curves we need to obtaine the first incomplete moment. The Bonferroni and Lorenz curves are very useful in Reliability Economics Insurance Demography and Medicine. In economical studies the shape of the model is as important as the estimation of the parameters of the model. Obviously this is not only the econometrics but in other areas as well. For lifetime models the mean residual lifetime function and the conditional moments are also of interest. The conditional moments for $EFPS$ distribution are as follows:

$$\upsilon_s = E(X^s|X < t) = \int_0^t x^s f(x)dx = \sum_{n=1}^{\infty} P^n \int_0^t x^s g(x;\alpha,\lambda,n\mu)dx$$

$$= \frac{n\mu\alpha^s}{(j+1)^{1-\frac{s}{\lambda}}} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \frac{a_n\theta^n}{C(\theta)} \Gamma\left(1-\frac{s}{\lambda},(j+1)\left(\frac{\alpha}{t}\right)^{\lambda}\right). \quad (11)$$

in which $\Gamma(a,t) = \int_0^t z^{a-1}(1-z)^{b-1}dz$ is the lower incomplete gamma function.
The moment generating function (mgf) of EFPS distribution is given by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^/$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{n\mu\alpha^r}{(j+1)^{1-\frac{r}{\lambda}}} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \frac{a_n\theta^n}{C(\theta)} \Gamma\left(1-\frac{r}{\lambda}\right) \cdot \quad (12)$$

4.3 deviations

To measure the amount of scatter in the population we need to obtaine the mean deviation about the mean and mean deviation about the median. The mean deviation from the mean is a robust statistic being more resilient to outliers in a data set than standard deviation. Consider the variable X with probability distribution function $f(x)$, cumulative distribution function $F(x)$, mean $\mu = E(X)$ and $M =$ Median (X). Then the mean deviation about the mean and mean deviation about the median are defined respectively as follows:

$$\delta_1(x) = \int_0^{\infty} |x-\mu| f(x)dx = 2\mu F(\mu) - 2I(\mu)$$

and

$$\delta_2(x) = \int_0^{\infty} |x-M| f(x)dx = 2MF(M) - M + \mu - 2I(M).$$

Respectively, where

$$I(z) = \int_0^z x f(x)dx$$

$$= \frac{n\mu\alpha}{(j+1)^{1-\frac{1}{\lambda}}} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} \frac{a_n\theta^n}{C(\theta)} \Gamma\left(1-\frac{1}{\lambda},(j+1)\left(\frac{\alpha}{z}\right)^{\lambda}\right) \cdot \quad (13)$$

4.4 Applications in Economics

The main application of the Bonferroni and Lorenz curves and the Bonferroni and Gini indices are for studying income and poverty in Economics. Furthere more they are applicable in the fields such as Demography Reliability Medicine and Insurance. The Bonferroni and Lorenz curves of EFPS distribution are as follows:

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x)dx$$

$$= \frac{1}{p\mu}\left[\frac{n\mu\alpha}{(j+1)^{1-\frac{1}{\lambda}}}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{a_n\theta^n}{C(\theta)}\Gamma\left(1-\frac{1}{\lambda},(j+1)\left(\frac{\alpha}{q}\right)^{\lambda}\right)\right].$$

and

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx$$
$$= \frac{1}{\mu}\left[\frac{n\mu\alpha}{(j+1)^{1-\frac{1}{\lambda}}}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{a_n\theta^n}{C(\theta)}\Gamma\left(1-\frac{1}{\lambda},(j+1)\left(\frac{\alpha}{q}\right)^{\lambda}\right)\right]\cdot$$

5 Hazard rate

Reliability theory deals with the continious and discrete lifetime distributions. The failue rate function and the mean residual life (MRL) are the two common measures in the study of the lifetime of an item. Distributions with a decreasing increasing bathtub-shaped or upside-down bathtub-shaped MRL are mainly used to model different types of lifetime data that may arise from many areas such as survival analysis reliability actuarial sciences economics etc. For instance in biomedical sciences researchers analyze survivorship studies by using the MRL (see Gupta 1981). The hazard rate function also has various applications including modelling the lifetime of electronic mechanical and electro-mechanical products. For instance to improve the quality of products after production Mi (1996) discussed useful models when the hazard rate function of the products follows a bathtub shape. On the other hand Peck and Zerdt (1974) showed the influence of the upside-down bathtub shaped hazard rate functions for modelling lifetimes of mechanical parts and semiconductors.

Suppose that a component survives up to time $t \geq 0$. The residual lifetime of the component starts from time t to an unknown time represented by X which is denoted by $|X-t|X>t$. It is well known that the mean residual life function and ratio of two consecutive moments of residual life uniquely determine the distribution of the lifetime (see Gupta and Gupta 1983). Hence the r^{th}-order moment of the residual lifetime is obtained by the following formula

$$\mu_r(t) = E((X-t)^r|X>t) = \frac{1}{\overline{F}(t)}\int_t^{\infty}(x-t)^r f(x,\varphi)dx, \qquad r \geq 1\cdot$$

Applying the binomial expansion of $(x-t)^r$ into the above equation

$$\mu_r(t) = \frac{1}{\overline{F}(t)}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d=0}^{r}(-t)^d\binom{r}{d}(-1)^j\binom{b-1}{j}P^n\int_t^{\infty}x^{r-\lambda-d-1}e^{-(j+1)\left(\frac{\alpha}{x}\right)^{\lambda}}dx$$
$$= \frac{1}{\overline{F}(t)}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d=0}^{r}(-t)^d\binom{r}{d}(-1)^j\binom{b-1}{j}P^n\gamma\left(\frac{d-r}{\lambda}+1,(j+1)\left(\frac{\alpha}{t}\right)^{\lambda}\right)\cdot$$

in which $\gamma(a,t) = \int_t^{\infty} z^{a-1}(1-z)^{b-1}dz$ is the upper incomplete gamma function.

We also discuss the properties of the reversed residual life which is defined as the $t-X|X\leq t\cdot$ It denotes the time elapsed from the failure of a component given that its life is less than or equal to t. This random variable is also called the time since failure. See Kundu and Nanda(2010) and Nanda et al. (2003) for more details. Furthermore the mean reversed residual life and ratio of two consecutive moments of reversed residual life uniquely characterize the distribution of the lifetime. The r^{th}-order moment of the reversed residual life is obtained by the following formula.

$$m_r(t)=E((t-X)^r|X\le t)=\frac{1}{F(t)}\int_0^t (t-x)^r f(x,\varphi)dx,\qquad r\ge 1\cdot$$

Applying the binomial expansion of $(t-x)^r$ into the above formula gives

$$\mu_r(t)=\frac{1}{F(t)}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d=0}^{r}(-t)^d\binom{r}{d}(-1)^j\binom{b-1}{j}P^n\int_0^t x^{r-\lambda-d-1}e^{-(j+1)\left(\frac{\alpha}{x}\right)^{\lambda}}dx$$

$$=\frac{1}{F(t)}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\sum_{d=0}^{r}(-t)^d\binom{r}{d}(-1)^j\binom{b-1}{j}P^n\Gamma\left(\frac{d-r}{\lambda}+1,(j+1)\left(\frac{\alpha}{t}\right)^{\lambda}\right)\cdot$$

5.1 Order Statistics

Moments of order statistics is very important in reliability and quality control where the future failure of the products needs to be predicted based on the times of a few early failures. These predictions are often made by the moments of order statistics. Here we derive the PDFs of the i_{th} order statistic of the EFPS distribution in closed form expressions. Furthere more the measures of kurtosis and skewness of the distribution of the i_{th} order statistic for different values of sample size and i are presented here. Suppose that $X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$ be a sequence of n random sample where $X_i,\ i=1,\cdots,n$ is distributed according to the $EFPS$ distribution. The PDF of the i_{th} order statistic $X_{i:n}$ denoted by $f_{i:n}(x)\ \ i=1,2,\ldots,n$ is given by

$$f_{i:n}(x)=\frac{f(x)}{\beta(i,n-i+1)}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)$$

$$=\frac{f(x)}{\beta(i,n-i+1)}\left[\frac{C\left\{\theta\left(1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)\right\}}{C(\theta)}\right]^{i-1}\times\left[1-\frac{C\left\{\theta\left(1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^{\lambda}}\right)^{\mu}\right)\right\}}{C(\theta)}\right]^{n-i}\qquad(14)$$

in which $f(x)$ is the PDF given by (6). Using binomial expansion we can write (14) as

$$f_{i:n}(x)=\frac{1}{\beta(i,n-i+1)}f(x)\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{i+j-1}\qquad(15)$$

The corresponding CDF of $X_{i:n}$ is given by

$$F_{i:n}(x)=\sum_{k=i}^{n}\binom{n}{k}(F(x))^k(1-F(x))^{n-k}$$

$$=\sum_{k=i}^{n}\sum_{j=0}^{n-k}(-1)^j\binom{n}{k}\binom{n-k}{j}(F(x))^{k+j}.$$

Also we can write

$$f_{i:n}(x)=\frac{1}{\beta(i,n-i+1)}\sum_{n=1}^{\infty}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{i+j-1}P^n f_{EFPS}(x,\alpha,\lambda,n\mu)\cdot$$

Hence the s_{th} raw moment $X_{i;n}$ comes immediately from the above equation where

$$E(X_{i:n}^s)=\frac{1}{\beta(i,n-i+1)}\sum_{n=1}^{\infty}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}P^nE\left[Z^s(F(Z))^{i+j-1}\right]$$

where Z follows the EFPS$(x,\alpha,\lambda,n\mu)$ distribution.

6 Inferenece on real data

6.1 Statistical inference

The maximum likelihood method is the most common method employed in the statistical inference. The MLEs enjoy interesting properties and is appliable in constructing confidence intervals and also in test statistics. The estimations based on large samples delivers simple appliable approximations for finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either numerically or analytically. In this subsection the ML estimation of the parameters of the EFPS distribution is obtained based on complete samples. Suppose that the random sample $X_1,\cdots,X_n$ are distributed as the EFPS distribution given by (5). Let $\phi=(\alpha,\mu,\lambda,\theta)^T$ be $p\times 1$ vector of parameters. The log-likelihood function for ϕ is as follows:

$$L = nlog\lambda + nlog\mu + nlog\theta + n\lambda log\alpha - (\lambda+1)\sum_{i=1}^n logx_i - \sum_{i=1}^n (\frac{\alpha}{x_i})^\lambda - nlogC(\theta) + \sum_{i=1}^n logC'\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}. \quad (16)$$

The corresponding score function is given by $U_n(\phi)=\left(\frac{\partial L}{\partial\alpha},\frac{\partial L}{\partial\mu},\frac{\partial L}{\partial\lambda},\frac{\partial L}{\partial\theta}\right)^T$ where

$$\frac{\partial L}{\partial\alpha} = \frac{n\lambda}{\alpha} - \lambda\sum_{i=1}^n \frac{1}{x_i}(\frac{\alpha}{x_i})^{\lambda-1} + \theta\mu\lambda\sum_{i=1}^n \frac{1}{x_i}(\frac{\alpha}{x_i})^{\lambda-1}e^{-\left(\frac{\alpha}{x}\right)^\lambda} \times\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^{\mu-1}\left\{\frac{C''\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}{C'\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}\right\}, \quad (17)$$

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} + nlog\alpha - \sum_{i=1}^n logx_i - \sum_{i=1}^n (\frac{\alpha}{x_i})^\lambda log(\frac{\alpha}{x_i}) + \theta\mu\sum_{i=1}^n (\frac{\alpha}{x_i})^\lambda log(\frac{\alpha}{x_i})e^{-\left(\frac{\alpha}{x}\right)^\lambda} \times\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^{\mu-1}\left\{\frac{C''\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}{C'\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}\right\}, \quad (18)$$

$$\frac{\partial L}{\partial\mu} = \frac{n}{\mu} + \sum_{i=1}^n \theta\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu log\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)\times\left\{\frac{C''\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}{C'\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}\right\}, \quad (19)$$

and

$$\frac{\partial L}{\partial\theta} = \frac{n}{\theta} - n\frac{C'(\theta)}{C(\theta)} + \sum_{i=1}^n \frac{\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu C''\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}{C'\left\{\theta\left[1-\left(1-e^{-\left(\frac{\alpha}{x}\right)^\lambda}\right)^\mu\right]\right\}}. \quad (20)$$

The MLE of ϕ, represented as $\hat{\phi}$, can be secured by finding the solution to the equation

$U_n(\phi) = 0$ through numerical methodologies, for instance, the Newton-Rapson algorithm. In another way, the MLEs can be directly secured by optimizing the log-likelihood function as provided in (16) and using the "BFGS" method of the "optim" subroutine in R software (see R Core Team (2023)). The "BFGS" method is a limited-memory quasi-Newton method for approximating the Hessian matrix of the target distribution. It is worth mentioning that the parameter vector $\phi = (\alpha, \mu, \lambda, \theta)^T$ can be easily obtained, thanks to the properties of the PDF f . The smooth and continuous nature of the function f , along with the existence and finiteness of its first and second derivatives, ensure that the equation $U_n(\phi) = 0$ has roots. These roots correspond to the MLEs of the vector ϕ . By employing relevant calculus techniques, it is possible to verify that the solutions correspond to a maximum. We can estimate the asymptotic variance of the MLEs, denoted as $\hat{\phi}$, using the Fisher information matrix. The Fisher information matrix is needed for the interval estimation of the parameters of the model. The Fisher information matrix, denoted as $I_n(\phi)$, is calculated as the negative expectation of the second derivative of the log-likelihood function (16) respect to $\phi = (\alpha, \mu, \lambda, \theta)^T$. Under regularity conditions, the MLEs are asymptotically normal. The corresponding information matrix is as follows.

$$I_n(\phi) = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\mu} & I_{\alpha\lambda} & I_{\alpha\theta} \\ I_{\mu\alpha} & I_{\mu\mu} & I_{\mu\lambda} & I_{\mu\theta} \\ I_{\lambda\alpha} & I_{\lambda\mu} & I_{\lambda\lambda} & I_{\lambda\theta} \\ I_{\theta\alpha} & I_{\theta\mu} & I_{\theta\lambda} & I_{\theta\theta} \end{bmatrix}$$

where

$$I_{\lambda\lambda} = \frac{\partial^2 L}{\partial \lambda^2}, \quad I_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2}, \quad I_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, \quad I_{\mu\mu} = \frac{\partial^2 L}{\partial \mu^2}, \quad I_{\alpha\theta} = \frac{\partial^2 L}{\partial \alpha \partial \theta},$$
$$I_{\alpha\lambda} = \frac{\partial^2 L}{\partial \alpha \partial \lambda}, \quad I_{\mu\lambda} = \frac{\partial^2 L}{\partial \mu \partial \lambda}, \quad I_{\theta\lambda} = \frac{\partial^2 L}{\partial \theta \partial \lambda}, \quad I_{\mu\alpha} = \frac{\partial^2 L}{\partial \mu \partial \alpha}$$

Applying the usual large sample approximation MLE of ϕ , i.e. $\hat{\phi}$ can be treated as being approximately $N_4(\phi J_n(\phi)^{-1})$ where $J_n(\phi) = E[I_n(\phi)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary the asymptotic distribution of $\sqrt{n}(\hat{\phi} - \phi)$ is $N_4(0 J(\phi)^{-1})$ where $J(\phi) = \lim_{n \to \infty} n^{-1} I_n(\Phi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\phi)$ is replaced by the average sample information matrix evaluated at $\hat{\phi}$ say $n^{-1} I_n(\hat{\phi})$. The estimated asymptotic multivariate normal $N_4(\phi I_n(\hat{\phi})^{-1})$ distribution of $\hat{\phi}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1-\gamma)$ asymptotic confidence interval for each parameter ϕ_r is given by

$$ACI_r = \left(\hat{\phi}_r - Z_{\frac{\gamma}{2}} \sqrt{\widehat{I_{rr}}} \hat{\phi}_r + Z_{\frac{\gamma}{2}} \sqrt{\widehat{I_{rr}}} \right)$$

where $\widehat{I_{rr}}$ is the (rr) diagonal element of $I_n(\hat{\phi})^{-1}$ for $r = 1,2,3,4$ and $Z_{\frac{\gamma}{2}}$ is the quantile $1 - \frac{\gamma}{2}$ of the standard normal distribution.

Table 3: The PDF of F, EF, BF, BEF, W, EW, MW and GLFRG distrinutons

Distribution	PDF
Fréchet (F)	$f_F(x;\alpha,\lambda)=\frac{\lambda}{x}(\frac{\alpha}{x})^{\lambda}e^{-(\frac{\alpha}{x})^{\lambda}}$
EF	$f_{EF}(x;\alpha,\lambda,\mu)=\lambda\mu\alpha^{\lambda}x^{-\lambda-1}e^{-(\alpha/x)^{\lambda}}(1-e^{-(\alpha/x)^{\lambda}})^{\mu-1}$
Beta Fréchet (BF)	$f_{BF}(x;\alpha,\lambda,\theta,\gamma)=\frac{1}{B(\theta\gamma)}\lambda\alpha^{\lambda}x^{-\lambda-1}e^{-\theta(\alpha/x)^{\lambda}}(1-e^{-(\alpha/x)^{\lambda}})^{\gamma-1}$
Beta Exponentiated Fréchet (BEF)	$f_{BEF}(x;\alpha,\lambda,\mu,\theta,\gamma)=\frac{1}{B(\theta\gamma)}\lambda\mu\alpha^{\mu}x^{-\mu-1}e^{-(\alpha/x)^{\mu}}[1-e^{-(\alpha/x)^{\mu}}]^{\gamma\lambda-1}\times\{1-(1-e^{-(\alpha/x)^{\mu}})^{\lambda}\}^{\theta-1}$
Weibull (W)	$f_W(x;\alpha,\gamma)=\alpha\gamma x^{\gamma-1}e^{-\alpha x^{\gamma}}$
Exponentiated Weibull (EW)	$f_{EW}(x;\alpha,\mu,\gamma)=\alpha\gamma\mu^{\gamma}x^{\gamma-1}e^{-(\mu x)^{\gamma}}\{1-e^{-(\mu x)^{\gamma}}\}^{\alpha-1}$
Modified Weibull (MW)	$f_{MW}(x;\alpha,\lambda,\gamma)=\alpha x^{\gamma-1}(\gamma+\lambda x)e^{\lambda x}e^{-\alpha x^{\gamma}e^{\lambda x}}$
Generalized Linear Failure Rate-Geometric (GLFRG)	$f_{GLFRG}(x;\alpha,\theta,\lambda,\mu)=\frac{\alpha(\lambda+\mu x)(1-\theta)e^{-\lambda x-1/2\mu x^2}\{1-e^{-\lambda x-1/2\mu x^2}\}^{\alpha-1}}{[1-\theta\{1-(1-e^{-\lambda x-1/2\mu x^2}\}^{\alpha})]^2}$

6.2 Real data

In this section we compare the performance of the EFPS distribution with respect to some other distributions using two real data sets to show the efficiency of $EFPS$ class of distributions. All computations are performed using the R.
We compare EFLG distribution with some of distributions that theire PDF are expressed in Table 4.

6.2.1 Data set 1: Strength of glass fibers data

The tensile strength data of glass fiber, re-ported by Smith and Naylor (1987), is used to illustrate the methodologies developed in this paper. In the article of Smith and Naylor (1987), The experimental strength data set of glass fiber of 1.5 cm length, are provided,originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. The strength data are:

0.55 0.93 1.25 1.36 1.49 1.52 1.58 1.61 1.64 1.68 1.73 1.81 2.00 0.74 1.04 1.27 1.39 1.49 1.53 1.59 1.61 1.66 1.68 1.76 1.82 2.01 0.77 1.11 1.28 1.42 1.50 1.54 1.60 1.62 1.66 1.69 1.76 1.84 2.24 0.81 1.13 1.29 1.48 1.50 1.55 1.61 1.62 1.66 1.70 1.77 1.84 0.84 1.24 1.30 1.48 1.51 1.55 1.61 1.63 1.67 1.70 1.78 1.89.

Based on the results in the Table 4 the $EFPS$ distribution shows a better fit to the the strength of the glass fibers data than the others. We see that at least one of the $EFPS$ distributions have smaller AIC, BIC, CM and K-S statistics with respect to other distributinos. Furthere more, Figures 6 and 7 show the estimated survival function plot TTT plot and Kaplan–Meier curve of the fitted distributions for the strength of glass fibers data.

Table 4. Estimates of fitted distributions for strength of glass fibers data.

	Dist.	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\gamma}$	m2L	AIC	BIC	K-S	CM
1	EFP	1092.9	0.337	18138.7	4.083		33.38	41.38	49.95	0.16	7.76
2	EFG	123.08	0.449	3687.5	0.940		25.40	33.40	41.98	0.11	8.28
3	EFL	92.951	0.477	3714.2	0.995		29.93	37.93	46.51	0.20	7.66
4	EFB	1137.1	0.338	20299.2	0.224		33.73	41.73	50.30	0.16	7.70
5	F	1.264	2.887				93.71	97.71	101.99	0.24	6.34
6	EF	697.7	0.374	16532.1			48.48	54.48	60.91	0.23	6.93
7	BF	3.805	2.903		0.041	1.615	91.22	99.22	107.79	0.79	6.48
8	BEF	0.830	2.817	0.491	61.766	12.83	62.52	72.52	83.23	1.00	20.92
9	W	0.059	5.780				30.41	34.41	38.70	0.15	7.93
10	EW	0.671		0.582		7.284	29.35	35.35	41.78	0.15	7.80
11	MW	0.008	2.160			2.402	28.71	34.71	41.14	0.14	7.78

6.2.2 Data set 2: Strength of single carbon fiber

we consider the strength of single carbon fiber data, which was originally reported by Badar and priest (1982) and it represents the strength measured in Gpa for single carbon fibers tows. The data are presented as follow.
10 33 44 56 59 72 74 77 92 93 96 100 100 102 105 107 107 108 108 108 109 112 113 115 116 120 121 122 122 124 130 134 136 139 144 146 153 159 160 163 163 168 171 172 176 183 195 196 197 202 213 215 216 222 230 231 240 245 251 253 254 254 278 293 327 342 347 361 402 432 458 555.
Based on the results in the Table 5 the $EFPS$ distributions show a better fit to the strength of the carbon fibers data than the other distributions. We see that at least one of the $EFPS$ distributions have smaller AIC, BIC, CM and K-S statistics with respect to other distributinos. Furthere more Figures 8 and 9 show the estimated survival function plot TTT plot and Kaplan–Meier curve of the fitted distributions for the strength of carbon fibers data.

Table 5: Estimates of fitted distributions for strength of single carbon fibers data.

	Dist.	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\gamma}$	m2L	AIC	BIC	K-S	CM
1	EFP	55.160	0.455	147.695	20.908		113.02	121.02	129.59	0.09	0.07
2	EFG	4.214	2.415	6.608	0.001		112.70	120.70	129.28	0.08	0.06
3	EFL	2.461	6.127	2.056	0.995		111.69	119.69	128.26	0.07	0.04
4	EFB	49.861	0.498	167.024	1.515		112.95	120.95	129.53	0.08	0.07
5	F	2.721	5.433				117.80	121.80	126.09	0.10	0.10
6	EF	4.295	2.363	7.032			112.70	118.70	125.13	0.08	0.06
7	BF	6.327	4.656		0.020	0.860	119.90	127.90	136.47	0.11	0.13
8	BEF	4.262	2.346	1.862	1.969	4.108	112.70	122.70	133.42	0.29	2.32
9	W	0.002	5.049				123.91	127.91	132.20	0.09	0.12
10	EW	37.134		0.869		1.454	112.62	118.62	125.05	0.08	0.06
11	MW	0	3.4			4e-04	335.39	341.39	347.82	0.58	9.10
12	GLFRG	11.731	0.004	0.584	0.298		112.92	120.92	129.49	0.08	0.07

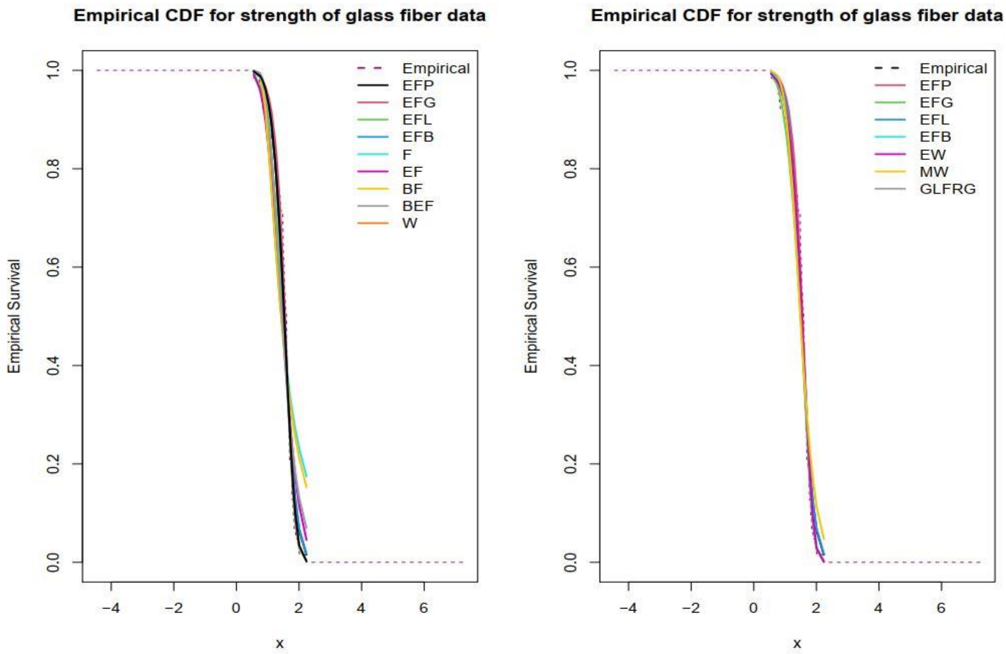

Figure 6: Estimated survival function and the empirical survival function for the strength of glass fibers data

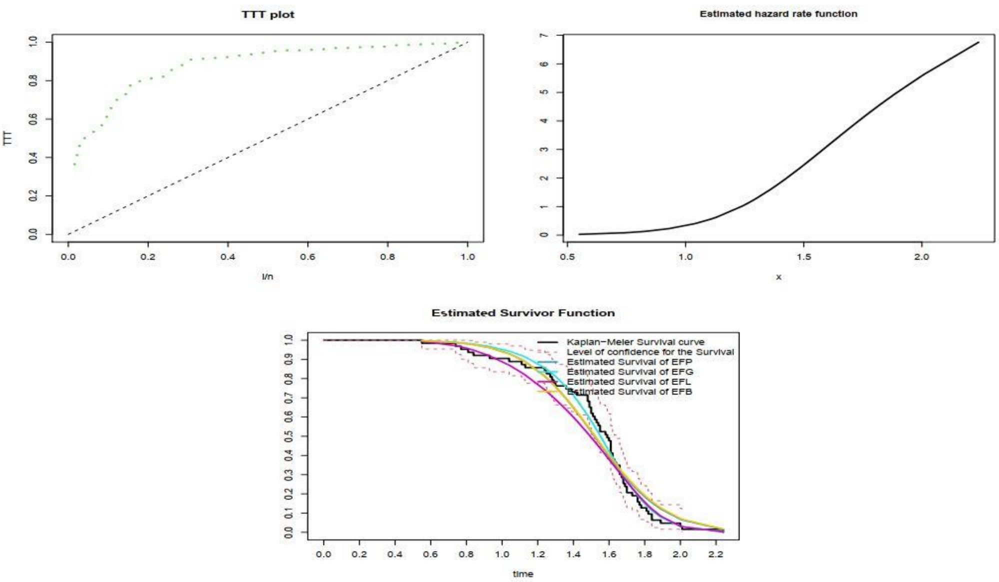

Figure 7: Empirical TTT-plot (top left) estimated hazard rate function (top right) estimated survival function (bottom) for the fitted distributions to the strength of glass fibers data.

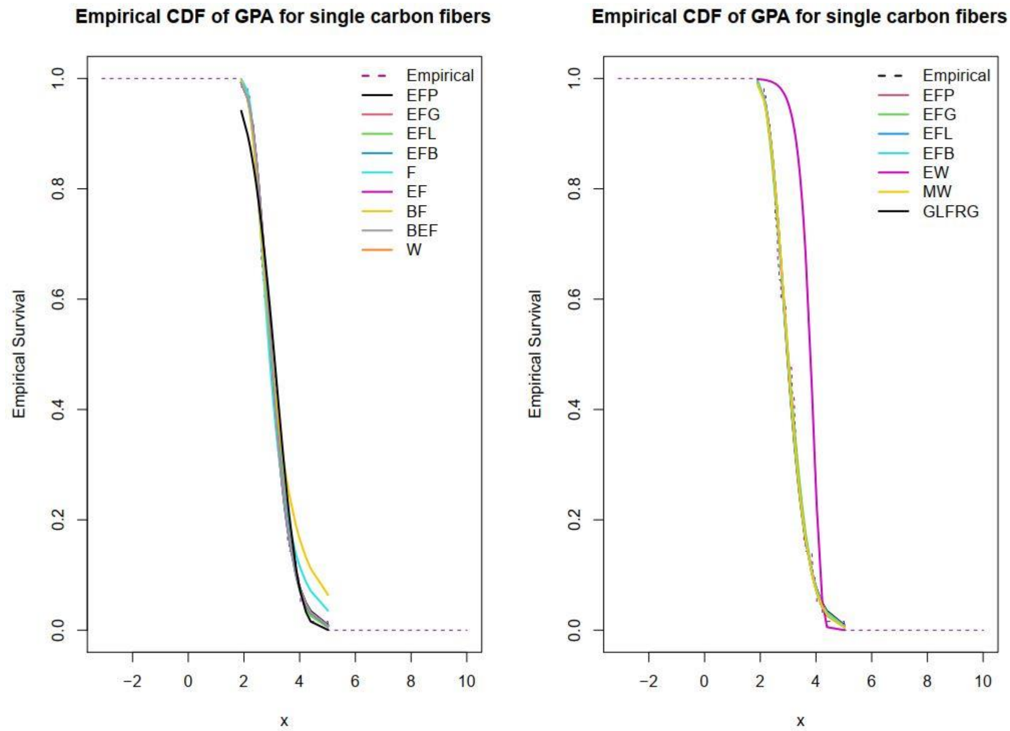

Figure 8: Estimated survival function and the empirical survival function for the single carbon fibers data

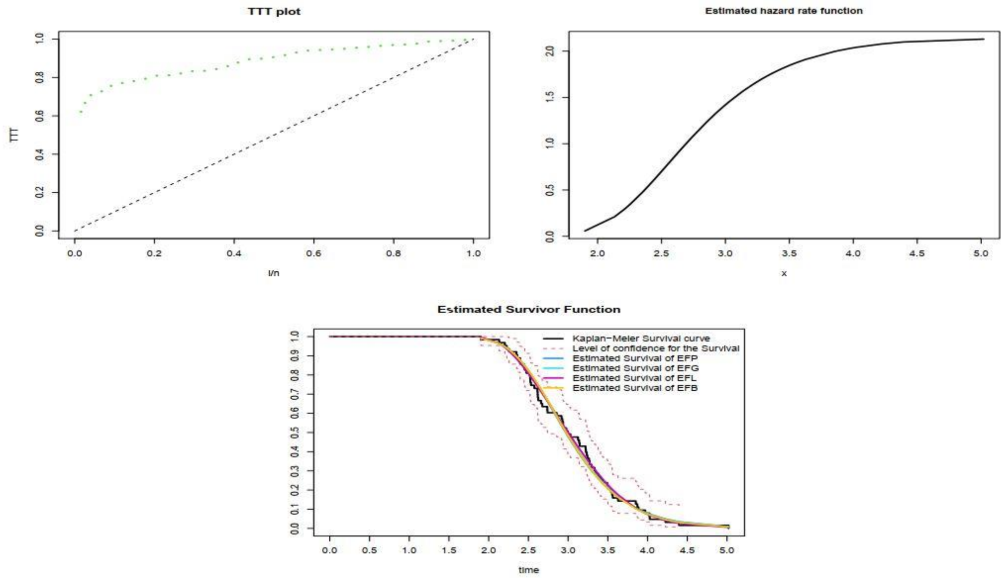

Figure 9: Empirical TTT-plot (top left) estimated hazard rate function (top right) estimated survival function (bottom) for the fitted distributions to the single carbon fibers data.

7 Conclusions

We introduced and studied the properties of new class of distributions called the $EFPS$ distribution as a compounding of the EF and PS distributions. The $EFPS$ distribution shows a good performance in modeling different types of hazard data with a bathtub-shaped hazard rates. The $EFPS$ class of distributions show more flexibility than the

Weibule EW Fréchet BEF, EF, MW, BF, EW and the $GLFRG$ distributions. The real data analysis prove that the $EFPS$ class of distributions perform reliable in the real data applications. We hope that the $EFPS$ distribution attract more attentions in survival analysis.

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